Three dimensional Lotka-Volterra Systems with 3:-1:2-resonance

Waleed Aziz[1], Colin Christopher[2], Jaume Llibre[3] and Chara Pantazi[4]

Abstract. We study the local integrability at the origin of a nine parametric family of a three dimensional Lotka–Volterra differential systems with 3:-1:2-resonance. We present the necessary and sufficient conditions on the parameters of the family that guarantee the existence of two independent local first integrals at the origin of coordinates. Additionally we check the cases where the origin is linearizable.

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1. Introduction and statement of the main results

Lotka–Volterra systems [30, 46] have been used in population biology to describe the evolution of conflicting species, see for example [37, 38]. During the years these systems have been used in many branches of sciences like laser physics [27], neural networks [40], plasma physics [28], chemical kinetics [39], etc. The dynamical behaviour of these models has been widely studied, see for instance [1, 14, 8, 45, 49, 34] among others. The integrability of some Lotka–Voltera families was considered by several authors, see for example [15, 16, 18, 31, 34, 36]. Similar technics applied to some more general families like the ones in [5, 23]. Some general properties of these systems are studied in the works [12, 13].

In general, for a three dimensional system, the existence of a first integral is very helpful because reduces the study of the dynamics from three dimensions to two. Moreover, the knowledge of two independent first integrals determines completely the trajectories of the system. Hence, the study of the existence of first integrals is an important issue in the qualitative theory of dynamical systems. There are many technics for the search of first integrals of a differential system. Some of them are: Lax pairs [3, 29], Lie symmetries [2, 41], Noether symmetries [17], Darboux theory of integrability [21, 22], Painlevé analysis [9], Differential Galois Theory [44, 48].

Here we use the same method which was employed in [4, 6] for studying the local integrability at the origin of all the Lotka–Voltera three dimensional differential systems of the form

$$\begin{aligned} \dot{x} &= x(3 + ax + by + cz) = P(x, y, z), \\ \dot{y} &= y(-1 + dx + ey + fz) = Q(x, y, z), \\ \dot{z} &= z(2 + gx + hy + kz) = R(x, y, z). \end{aligned} \tag{1}$$

We consider the case of (3:-1:2)-resonance in order to continue the work in [6]. Note that we are working in the Siegel domain with two independent resonances. Moreover, when restricted to y=0, the nodal behaviour has no resonance. Hence, using the behaviour associated to the resonance of the node we tried to find the generic behaviour of the integrability of these systems. It seems that the problem still is very complicated and more cases should be studied in order to find out the general mechanisms for obtaining the integrability. Our aim is to understand the integrability mechanisms for general resonances.

The main result of this work is Theorem 1 where we characterize all the Lotka–Voltera differential systems (1) having two independent local first integrals. We provide a complete classification for the integrability and linearizability conditions of systems (1) with $(3 : -1 : 2)$-resonance.

Theorem 1. *Consider the three dimensional Lotka-Volterra system with* $(3 : -1 : 2)$*-resonance. The origin of system* (1) *is integrable if and only if one of the following conditions hold.*

1) $r_1 = 2ef - ek - hk = 0,\ r_2 = 2cd + cg - 5dk + 3fg - gk = 0$
$r_3 = 2bk - 2ce + 3ek - 3hk = 0,\ r_4 = 2bf - bk - ch - 3fh + 3hk = 0$
$r_5 = 2bd + bg - 2de - 3dh + 2eg = 0,\ r_6 = 2af + ak - 2dk - gk = 0$
$r_7 = 2ae + ah - 2de - eg = 0,\ r_8 = 2ac - 5ak - 2cd - cg + 5dk - 3fg + 4gk = 0$
$r_9 = 2ab - 2ah - 2bd - bg + 3dh = 0$

2) $b + 2e = c = d = f = k = 0$

3) $a = e + h = f = d = g = 0$

4) $a = d = g = h = 0$

5) $b = c = f = k = 0$

6) $a - d = b - e = g = h = 0$

7) $b = c = e - h = f - k = 0$

8) $b = h = 0$

9) $b + e = c = f = k = 0$

10) $h = b + e = 0$

11) $b + 2e = d = h = 0$

12) $8a - 3g = b + 3e = 8d + g = h = 0$

13) $b = e + h = f = 0$

14) $b - h = e + h = f = 0$

15) $b - 2h = d = e + h = f = 0$

16) $a - 2d - g = e + h = f = b + h = d + g = 0$

17) $b = 2c - 9k = 2e + h = 2f + k = 0$

18) $2b - h = 2c - 7k = 2e + h = 2f + k = 0$

19) $b + h = c + k = e - h = f - k = 0$

20) $b + 2h = c + 2k = d = e - h = f - k = 0$.

Additionally, system (1) *is linearizable if and only if either one of the conditions* $2 - 11$, $13 - 16$, 19, 20 *or one of the following subcases of case 1) holds.*

1.1) $dh - eg = k = f = c = b - e = a - d$

1.2) $k = g = f = d = c = a$

1.3) $f - k = e - h = d - g = c - k = b - h = a - g = 0$

1.4) $bk - ch = g = f - k = e - h = d = a$.

The basic notions that we use for the proof of Theorem 1 are presented in Section 2. Then, in Section 3 we give the proof of Theorem 1.

2. Preliminaries

Consider the three dimensional system (1) and the corresponding vector field $\mathcal{X}$ given by

$$\mathcal{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}.$$

Let U be an open neighborhood of the origin of $\mathbb{C}^3$. We say that a non-locally constant analytic function $H : U \to \mathbb{C}$ is a *local first integral* of system (1) if it is constant on all solutions (trajectories) of the system contained in U. Hence, the first integral H of system (1) satisfies

$$\mathcal{X}H = P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} + R\frac{\partial H}{\partial z} = 0,$$

in all the points of U.

Two local first integrals H_1 and H_2 defined in U are *independent* if their gradients are linearly independent in $\mathbb{C}^3$ (except perhaps in a set of Lebesgue measure zero).

An *inverse Jacobi multiplier* of $\mathcal{X}$ is a function $M : U \to \mathbb{C}$ that satisfies the equation

$$\mathcal{X}(M) = M\text{div}\mathcal{X} \quad \text{in} \quad U,$$

or, equivalently, $\text{div}(\mathcal{X}/M) = 0$. Here $\text{div}\mathcal{X} = \partial P/\partial x + \partial Q/\partial y + \partial R/\partial z$. For general properties of the Darboux Jacobi multipliers see [7] and references therein.

One method for finding two local independent first integrals consists in finding an inverse Jacobi multiplier and a local first integral because then we can use the Jacobi multiplier to obtain the second first integral, see [26].

Another way of finding two local independent first integrals is to show that the system (1) is *integrable*, following the works of [10, 11, 43, 47]. That is, there is an analytic change of coordinates near to identity

$$(X, Y, Z) = \big(x + o(x, y, z),\ y + o(x, y, z),\ z + o(x, y, z)\big), \tag{2}$$

which transforms system (1) into the system

$$\dot{X} = 3X\zeta(x, y, z), \quad \dot{Y} = -Y\zeta(x, y, z), \quad \dot{Z} = 2Z\zeta(x, y, z), \tag{3}$$

where $\zeta = 1 + o(x, y, z)$.

Note that XY^3 and Y^2Z are first integrals of system (3) and we can pull them back to the first integrals of system (1) of the form

$$\phi_1 = xy^3(1 + O(x, y, z)), \qquad \phi_2 = y^2z(1 + O(x, y, z)). \tag{4}$$

Conversely, given two independent first integrals ϕ_1 and ϕ_2 of system (1) of the form (4), then there exits a change of coordinates X, Y, Z so that these first integrals can be written in the form XY^3 and Y^2Z and for some ζ the new system is of the form (3).

We say that system (1) is *linearizable at the origin* if and only if the change of coordinates (2) can be chosen such that $\zeta \equiv 1$. Hence, system (1) can be rewritten as

$$\dot{X} = 3X, \quad \dot{Y} = -Y, \quad \dot{Z} = 2Z.$$

Hence, we can choose the transformation (2) such that the coefficients of the non–linear terms in $\zeta(x, y, z)$ vanish, see also [19, 20, 25].

Let $\ell = \ell(x, y)$ be a polynomial. An *invariant algebraic surface* $\ell = 0$ of the polynomial differential system (1) is an algebraic surface which satisfies

$$\dot{\ell} = \mathcal{X}\ell = P\frac{\partial \ell}{\partial x} + Q\frac{\partial \ell}{\partial y} + R\frac{\partial \ell}{\partial z} = \ell L_\ell, \tag{5}$$

for some polynomial $L_\ell \in \mathbb{C}[x, y, z]$. The polynomial L is called a *cofactor* of the invariant algebraic surface $\ell = 0$. Note that from relation (5) we have that any cofactor has at most degree one because the polynomial vector field has degree two. There are other functions which satisfy (5), for example there are invariant analytic surfaces. Then, the degree of the cofactor L_ℓ is imposed rather than following from equation (5).

Exponential factors also satisfy (5) and are related to the multiplicity of the invariant surfaces, see [32, 33]. Exponential factors appear for those parameter values where two surfaces collapse or an invariant surface is coincident with the plane at infinity. Let

$$E(x, y, z) = \exp(f(x, y, z)/g(x, y, z)),$$

where $f, g \in \mathbb{C}[x, y, z]$ are relatively prime polynomials. We say that E in an *exponential factor* of (1) if

$$\mathcal{X}E = EL_E, \tag{6}$$

for some polynomial L_E of degree at most one, and L_E is called the *cofactor* of the exponential factor E.

Invariant algebraic surfaces and exponential factors have an important role in the explicit calculation of first integrals. More concretely, if there is a linear combination between their cofactors, then the existence of a first integral is immediate, as the following theorem shows, [21, 22].

Theorem 2. *Let $\ell_1, \cdots, \ell_p$ invariant algebraic surfaces of system (1) with cofactors $L_{\ell_1}, \cdots, L_{\ell_p}$ and let $E_1, \cdots, E_q$ be exponential factors of system (1) with cofactors $L_{E_1}, \cdots, L_{E_q}$. If there exist real numbers $\alpha_1, \cdots, \alpha_p$, $\beta_1, \cdots, \beta_q$, not all of them equal to zero, such that*

$$\alpha_1 L_{\ell_1} + \cdots + \alpha_p L_{\ell_p} + \beta_1 E_1 + \cdots + \beta_q E_q = 0,$$

then system (1) has a first integral (called a Darboux first integral)

$$H = \ell_1^{\alpha_1} \cdots \ell_p^{\alpha_p} E_1^{\beta_1} \cdots E_q^{\beta_q}.$$

The following two theorems are crucial in our study and they are proved in [6]. We use the multi–index notation $X^I = x^i y^j z^k$ in order to simplify the presentation of Theorem 3.

Theorem 3 (Theorem 1 of [6]). *Suppose the analytic vector field*

$$x\left(\lambda + \sum_{|I|>0} A_{xI} X^I\right)\frac{\partial}{\partial x} + y\left(\mu + \sum_{|I|>0} A_{yI} X^I\right)\frac{\partial}{\partial y} + z\left(\nu + \sum_{|I|>0} A_{zI} X^I\right)\frac{\partial}{\partial z},$$

has a first integral $\phi = x^\alpha y^\beta z^\gamma(1+O(x, y, z))$ with at least one of α, β, $\gamma \neq 0$ and an inverse Jacobi multiplier $M = x^r y^s z^t(1+O(x, y, z))$ and suppose that the cross product of $(r-i-1, s-j-1, t-k-1)$ and (α, β, γ) is bounded away from zero for any integers $i, j, k \geq 0$. Then the vector field has a second analytic first integral of the form $\psi = x^{1-r} y^{1-s} z^{1-t}(1+O(x, y, z))$ and hence system (1) is integrable.

Remark 1. Theorem 3 can be generalized to allow the cross product to be either zero or bounded away from zero. In this case, there are resonant terms which must be shown to be zero when the cross product is zero. We do not need this more detailed case here as alternative proofs are given in cases where the cross product can vanish.

The proof of the theorem follows directly from applying a change of coordinates, and a scaling, so that the quantities ϕ and M are just simple monomials, and then working term by term.

Remark 2. It can happen that the second first integral is also of the form $\psi = (x^\alpha y^\beta z^\gamma)^\lambda(1+O(x,y,z))$ for some λ. However, it follows from the proof of Theorem 3, that this second integral will be functionally independent of ϕ. Therefore, $\psi^{1/\lambda}/\phi - 1$ is an analytic first integral which must also be expressible in the form $x^\delta y^\epsilon z^\zeta(1+O(x,y,z))$ for some δ, ϵ, ζ. If (δ,ϵ,ζ) is a multiple of (α,β,γ) then we can repeat this process, eventually ending up with a second first integral of the form above with (δ,ϵ,ζ) independent of (α,β,γ), as required to give two first integrals in the form (4).

Hence for system (1), in normal situation, from the expression of a first integral and an inverse Jacobi multiplier we can obtain the explicit expression of the other first integral.

Theorem 4 (Theorem 2 of [6]**).** *Suppose that we have a three dimensional Lotka-Volterra system with $(\lambda : \mu : \nu)$-resonance, for which the origin is integrable, and there exists a function $\xi = x^\alpha y^\beta z^\gamma(1+O(x,y,z))$ such that $X(\xi) = k\xi$ for some constant $k = \alpha\lambda + \beta\mu + \gamma\nu \neq 0$, then the system is linearizable.*

Proof. From the hypothesis, we can bring the system to the form

$$\dot{x} = \lambda x\zeta, \quad \dot{y} = \mu y\zeta, \quad \dot{z} = \nu z\zeta.$$

Let $\xi = x^\alpha y^\beta z^\gamma g(x,y,z)$, then the linearizing change of coordinates is given by

$$(X,Y,Z) = (x\, g^{\lambda/k}, y\, g^{\mu/k}, z\, g^{\nu/k}). \tag{7}$$

□

An ordered tuple of the complex numbers $(\lambda_1,\lambda_2,\lambda_3)$ in $\mathbb{C}^3$ is called *resonant* if there exist non negative integers $\alpha = (\alpha_1,\alpha_2,\alpha_3) \in \mathbb{Z}^3_+$ such that

$$\alpha_1\lambda_1 + \alpha_2\lambda_2 + \alpha_3\lambda_3 = \lambda_j, \quad |\alpha| = \alpha_1+\alpha_2+\alpha_3 \geq 2. \tag{8}$$

The natural number $|\alpha| = \alpha_1+\alpha_2+\alpha_3$ is the *order of the resonance* and the monomial $X^n = x^{\alpha_1}y^{\alpha_2}z^{\alpha_3}e_j$ is called a *resonant monomial.* The coefficient of the monomial X^n in the system (1) is called a *resonant coefficient* and the corresponding term is called a *resonant term.*

The Poincaré domain is the collection of all tuples $(\lambda_1,\lambda_2,\lambda_3)$ such that the convex hull of the point set $\{\lambda_1,\lambda_2,\lambda_3\} \in \mathbb{C}$ does not contain the origin inside or on the boundary. It is well known that singular points whose eigenvalues lie in the Poicaré domain can be brought, via an analytic change of coordinates, to a normal form which contains only resonant terms.

Then, there are only a finite number of resonant terms that will need to be checked in order to show linearizability.

3. Proof of Theorem 1

After some calculations we see that systems (1) admit the invariant hyperplanes $x = 0,\ y = 0,\ z = 0$ with cofactors $L_x = 3 + ax + by + cz,\ L_y = -1 + dx + ey + fz,\ L_z = 2 + gx + hy + kz$ respectively.

Our aim is to find conditions that guarantees the existence of two independent analytic first integrals of system (1) of the form

$$\phi_1 = xy^3(1+O(x,y,z)) \quad \text{and} \quad \phi_2 = y^2z(1+O(x,y,z)). \tag{9}$$

So, we first express ϕ_1 and ϕ_2 as power series up to terms of order 18. Then we compute the obstructions that ϕ_1 and ϕ_2 form first integrals. A factorized Gröbner basis is obtained, giving the necessary conditions for integrability and linearizability described in Theorem 1. To achieve this, we have used both Maple and Reduce. Also we have use the minAssGTZ algorithm of Singular [42] in order to check that the conditions found were irreducible. The reader can find more details about the mechanism that we have applied in Section 3 of [6].

Proof of Theorem 1. Here, in order to complete the proof of Theorem 1 we shall show below that each of these conditions is also sufficient for the integrability of system (1).

Case 1 If $k \neq 0$ the system has the invariant plane $\ell = 1 + \frac{a}{3}x - ey + \frac{k}{2}z = 0$ with cofactor $L_\ell = ax + ey + kz$. Indeed, the relation (5) gives

$$\left(\left(\frac{ab}{3} - de + 2\frac{ae}{3}\right)y + \left(\frac{ac}{3} + \frac{gk}{2} - 5\frac{ak}{6}\right)z\right)x + \left(-ef + \frac{hk}{2} + \frac{ek}{2}\right)zy = 0.$$

If we set

$$c_1 = \frac{ba}{3} - de + 2\frac{ea}{3}, \qquad c_2 = \frac{ca}{3} + \frac{gk}{2} - 5\frac{ka}{6}, \qquad c_3 = -fe + \frac{hk}{2} + \frac{ke}{2},$$

and note that

$$c_1 = r_5/6 + r_7/3 + r_9/6, \qquad c_2 = r_2/6 + r_8/6, \qquad c_3 = -r_1/2,$$

then $c_1 = c_2 = c_3 = 0$ is a direct consequence of the conditions satisfied for this case.

Now, using the Darboux theory of integrability (see Theorem 2 and for more details chapter 8 of [24]), we obtain the two independent first integrals

$$\phi_1 = xy^3\ell^{-\frac{3f+c}{k}}, \qquad \phi_2 = y^2z\ell^{1-\frac{2f}{k}}.$$

If $k = 0$ we consider the following two subcases.

Subcase 1 $k = a = e = 2bd + bg - 3dh = 2bf - ch - 3fh = 2cd + cg + 3fg = 0.$

i) $b \neq 0$ and $h \neq 0$. We get the exponential factor $E = \exp(dhx - bhy + bfz)$ with cofactor $L_E = 3dhx + bhy + 2bfz$. Thus, by the Darboux theory of integrability, system (1) admits the two first integrals

$$\phi_1 = xy^3\ell^{-\frac{1}{h}}, \qquad \phi_2 = y^2z\ell^{-\frac{1}{b}}.$$

ii) $h = 0$ and $b \neq 0$. We distinguish the following subcases.

a) $a = 2d + g = e = f = h = k = 0$. System (1) admits the exponential factor $E = \exp(dx - by + cz/2)$ with cofactor $L_E = 3d + by + cz$. In this case we obtain the two independent analytic first integrals

$$\phi_1 = xy^3E^{-1}, \qquad \phi_2 = y^2z.$$

b) $2a - 2d - g = c = f = h = k = 2bd + bg - 2de + 2eg$.
If $e \neq 0$ we obtain the invariant hyperplane $\ell = 1 + \frac{a}{3}x - ey$ with cofactor $L_\ell = ax + ey$ and the two independent analytic first integrals

$$\phi_1 = x\,y^3\,\ell^{-3-\frac{b}{e}}, \qquad \phi_2 = y^2 z \ell^{-2}.$$

If $e = 0$ we have an exponential factor $E = \exp(dx - by)$ with cofactor $L_E = 3dx + by$. System (1) admits the two analytic independent first integrals

$$\phi_1 = x\,y^3\,E^{-1}, \qquad \phi_2 = y^2\,z.$$

(iii) $a = b = d = e = k = c + 3f = 0$. There is an exponential factor $E = \exp(gx/3 - hy + fz)$ with cofactor $L_E = gx + hy + 2fz$ and we obtain the two analytic first integrals

$$\phi_1 = xy^3 E, \qquad \phi_2 = y^2 z E^{-1}.$$

iv) If $b = h = 0$ then we fall in Case 8.

Subcase 2 $k = c = f = 2ab - 2ae - 3ah + 3eg = 2ae + ah - 2de - eg = 2bd + bg - 2de - 3dh + 2eg = 0$. When $e \neq 0$ the system has the invariant plane $\ell = 1 + \frac{a}{3}x - ey = 0$ with cofactor $L_\ell = ax + ey$ and we obtain the two independent first integrals

$$\phi_1 = x\,y^3\,\ell^{-3-\frac{b}{e}}, \qquad \phi_2 = y^2\,z\,\ell^{-2+\frac{h}{e}}.$$

When $e = 0$ the system (1) admits the exponential factor $E = \exp(dx - by)$ with cofactor $L_E = 3dx + by$ and then we can construct the two independent first integrals

$$\phi_1 = x\,y^3\,E^{-1}, \qquad \phi_2 = y^2\,z\,E^{-\frac{h}{b}},$$

if $b \neq 0$. When $b = 0$ we can reduce to Case 5.

Case 1.1 For $e \neq 0$ the system has the invariant plane $\ell = 1 + \frac{d}{3}x - ey = 0$ with cofactor $L_\ell = dx + ey$ and a linearizing change of coordinates is given by

$$(X, Y, Z) = \left(x\ell^{-1},\ y\ell^{-1},\ z\ell^{-\frac{h}{e}}\right).$$

For $e = 0$ we distinguish two subcases. First, for $h = 0$ and $d \neq 0$ we get a linearizing change of coordinates

$$(X, Y, Z) = \left(x\ell^{-1},\ y\ell^{-1},\ z\ell^{-\frac{g}{d}}\right).$$

If $d = 0$, a linearizing change of coordinates is given by

$$(X, Y, Z) = \left(x,\ y,\ z\exp\left(hy - \frac{g}{3}x\right)\right).$$

Case 1.2 When $e \neq 0$, the linearizing change of coordinates is

$$(X, Y, Z) = \left(x(1 - ey)^{-\frac{b}{e}},\ y(1 - ey)^{-1},\ z(1 - ey)^{-\frac{h}{e}}\right).$$

When $e=0$, the linearizing change of coordinates is

$$(X,Y,Z) = (x\exp(by),\ y,\ z\exp(hy)).$$

Case 1.3 In this case the linearizing change of coordinates is

$$(X,Y,Z) = \left(x\ell^{-1},\ y\ell^{-1},\ z\ell^{-1}\right),$$

where $\ell = 1 + \frac{a}{3}x - by + \frac{c}{2}z$.

Case 1.4 When $h \neq 0$, the linearizing change of coordinates is given by

$$(X,Y,Z) = \left(x\ell^{-\frac{b}{h}},\ y\ell^{-1},\ z\ell^{-1}\right),$$

where $\ell = 1 - hy + \frac{k}{2}z$. When $h=0$ we have two possibilities. First for $b=0$ and $k \neq 0$ then we have a linearizing change of coordinates

$$(X,Y,Z) = \left(x\ell^{-\frac{c}{k}},\ y\ell^{-1},\ z\ell^{-1}\right).$$

Second, if $k=0$, then the linearizing change of coordinates is

$$(X,Y,Z) = \left(x\exp\left(by - \frac{c}{2}z\right),\ y,\ z\right).$$

In order to facilitate the understanding of the proof we group the remaining cases according to the techniques used to prove integrability.

Group 1. In Cases 2 and 3 the variables x and z do not appear in the differential equation respect to the variable y. Moreover, we can obtain a sufficient number of invariant algebraic surfaces and exponential factors to construct two first integrals via Theorem 2.

Case 2 The system becomes

$$\dot{x} = x(3 + ax - 2ey), \qquad \dot{y} = y(-1 + ey), \qquad \dot{z} = z(2 + gx + hy).$$

For $ae \neq 0$ this system has the invariant plane $\ell_1 = 1 - ey = 0$ and the invariant algebraic surface $\ell_2 = 1 + (a/3)x + (ae/3)xy + (ae^2/3)xy^2 = 0$ with cofactors $L_{\ell_1} = ey$ and $L_{\ell_2} = ax$, respectively. Here we obtain the two independent first integrals

$$\phi_1 = x\,y^3\,\ell_1^2\,\ell_2^{-\frac{1}{3}} \quad \text{and} \quad \phi_2 = y^2\,z\,\ell_1^{-(2+\frac{h}{e})}\,\ell_2^{-\frac{g}{a}}.$$

When $e=0$ and $a \neq 0$ the system has the invariant plane $\ell = 1 + ax/3$ and the exponential factor $E = \exp(y)$ with cofactors $L_\ell = ax$ and $L_E = -y$ which yield to the two independent first integrals

$$\phi_1 = x\,y^3\,\ell^{-1} \quad \text{and} \quad \phi_2 = y^2\,z\,\ell^{-\frac{g}{a}}\,E^h.$$

While $a=0$ and $e \neq 0$ we have an invariant plane $\ell = 1 - ey$ and an exponential factor $E = \exp(x/(1-ey))$ with cofactors $L_\ell = ey$ and $L_E = 3x$ and we obtain the two independent first integrals

$$\phi_1 = x\,y^3\,\ell^{-1} \quad \text{and} \quad \phi_2 = y^2\,z\,\ell^{-2-\frac{h}{e}}\,E^{-\frac{g}{3}}.$$

Finally when $a = e = 0$ there exist exponential factors $E_1 = \exp(x)$ and $E_2 = \exp(y)$ with cofactors $L_{E_1} = 3x$ and $L_{E_2} = -y$ and we have the two independent first integrals

$$\phi_1 = x\,y^3 \quad \text{and} \quad \phi_2 = y^2\,z\,E_1^{-\frac{g}{3}}\,E_2^{h}.$$

Theorem 4 guarantees the linearizability of the system in this case as $\xi = y/(1-ay/3)$ satisfies $\dot{\xi} = -\xi$.

Case 3 The system takes the form

$$\dot{x} = x(3 + by + cz), \qquad \dot{y} = y(-1 - hy), \qquad \dot{z} = z(2 + hy + kz).$$

The system has the invariant plane $\ell_1 = 1+hy = 0$ and the invariant algebraic surface $\ell_2 = 1+kz/2-(kh/2)yz = 0$ with cofactors $L_{\ell_1} = -hy$ and $L_{\ell_2} = kz$.

When $h \neq 0$ and $k \neq 0$ then we have a linearizing change of coordiates

$$(X, Y, Z) = \left(x\ell_1^{\frac{b}{h}}\ell_2^{-\frac{c}{k}},\ y\ell_1^{-1},\ z\ell_1\ell_2^{-1}\right).$$

When $h = 0$ we replace $\ell_1^{\frac{b}{h}}$ by the exponential factor $\exp(by)$, and when $k = 0$ we replace $\ell_2^{-\frac{c}{k}}$ by $\exp(cz(1-hz)/2)$.

Group 2. Similarly to the previous group, in Cases 4 and 5 there is also a differential equation which involves only one variable. But in this group we only obtain one invariant algebraic surface from which we are able to construct a first integral and an inverse Jacobi Multiplier. From Theorem 3 we can construct another first integral. Additionally, a linearizing transformation can be constructed.

Case 4 We provide two different proofs. In the first we prove the existence of two independent local first integrals. In the second we provide, for generic parameter values, one first integral and an Inverse Jacobi Multiplier. The system is

$$\dot{x} = x(3 + by + cz), \qquad \dot{y} = y(-1 + ey + fz), \qquad \dot{z} = z(2 + kz).$$

The system for $k \neq 0$ has an algebraic invariant surface $\ell = 1 + \frac{k}{2}z = 0$ with cofactor $L_\ell = kz$.

Using the change of coordinates $Z = z\ell^{-1}$ we can linearize the last equation and bring it into the form $\dot{Z} = 2Z$. To linearize the second equation we seek an invariant surface of the form $\tilde{\ell} = A + By = 0$ with $A = A(Z)$ and $A(0) = 1$ and $B = B(Z)$ with cofactor $L_{\tilde{\ell}} = ey + fz$. To find A and B we need to solve the system of the two differential equations

$$\dot{A} = Afz, \qquad \dot{B} - B = Ae.$$

For $k = 0$ we have $A = \exp\left(fZ/2\right)$ whereas for $k \neq 0$ we have $A = (1 - kZ/2)^{-f/k}$.

Now write $A = \sum_{i \geq 0} a_i Z^i$ and $B = \sum_{i \geq 0} b_i Z^i$. From the relation $\dot{B} - B = Ae$ we have $b_i = e\,a_i/(2i-1)$ and so B is a convergent series. Hence the second equation is linearized and taking $Y = y/\tilde{\ell}$ we get $\dot{Y} = -Y$. In order to linearize the first equation we set $X = x\exp(-\gamma(Y,Z))$ with γ an analytic function satisfying $\dot\gamma = b\,y(Y,Z) + c\,z(Z)$. We set

$$\gamma(Y,Z) = \sum_{i+j>0} \gamma_{ij} Y^i Z^j, \quad y(Y,Z) = \sum_{i+j>0} \eta_{ij} Y^i Z^j, \quad z(Z) = \sum_{j>0} \zeta_j Z^j,$$

and hence, we need to solve

$$\sum_{i+j>0} (2j-i)\gamma_{ij} Y^i Z^j = b \sum_{i+j>0} \eta_{ij} Y^i Z^j + c \sum_{j>0} \zeta_j Z^j.$$

If $y(Y,Z)$ contains no terms of the form $(Y^2 Z)^n$ we obtain

$$\gamma_{0j} = \frac{b\,\eta_{0j} + c\,\zeta_j}{2j}, \qquad \gamma_{ij} = \frac{b\,\eta_{ij}}{2j-i}, \; i > 0.$$

Clearly γ is a convergent series. Now we should show that the inverse transformation $y = AY/(1-BY)$ contains no term like $(Y^2Z)^n$. Note that

$$y = \frac{AY}{1-BY} = \sum A\,B^k\,Y^{k+1}.$$

We assume that $k+1 = 2n$ for some n. We should show that $AB^k = AB^{2n-1}$ contains no term like Z^n. Note that

$$\dot B = \frac{dB}{dt} = 2Z\frac{dB}{dZ}.$$

From equation $\dot B - B = Ae$ we obtain that

$$\Big(\frac{2Z}{2n}\frac{dB^{2n}}{dZ} - B^{2n}\Big) = e\,A\,B^{2n-1}.$$

Now on the left hand side the coefficient in Z^n in B^{2n} vanishes. So either $e = 0$, that means $B \equiv 0$, or the coefficient of Z^n in AB^{2n-1} vanishes. Therefore y given by $Y = y/(A+By)$ contains no term like $(Y^2Z)^n$. Thus we have set up the existence of a linearizing transformation and this completes the first proof.

Now we will provide the second proof. When $bk \neq 0$ the system has the first integral

$$\phi = x^{-2ek} y^{2bk} z^{(b+3e)k} \left(1 + \frac{1}{2}kz\right)^{2ce-3ek-2be-bk},$$

and the inverse Jacobi multiplier

$$IJM = x^{1+\frac{2e}{b}} z^{\frac{1}{2}-\frac{3e}{b}} (1 + \frac{1}{2}kz)^{\frac{3}{2}+\frac{f}{k}+\frac{3e}{b}-\frac{2ce}{bk}}.$$

Theorem 3 guarantees the existence of the second first integral

$$\psi = x^{-\frac{2e}{b}} y z^{\frac{1}{2}+\frac{3e}{b}} (1 + \ldots).$$

The desired first integrals of the initial system are

$$\phi_1 = \phi^{\frac{1}{2}\frac{b+6e}{bke}} \psi^{-3-\frac{b}{e}} = xy^3(1+\ldots) \quad \text{and} \quad \phi_2 = \phi^{\frac{2}{bk}} \psi^{-2} = y^2 z(1+\ldots).$$

For the remaining cases, we can work in a similar way.

Since $\xi = z\ell^{-1}$ satisfies $\dot\xi = 2\xi$, the system is linearizable by Theorem 4.

Case 5 The system becomes

$$\dot{x} = x(3 + ax), \qquad \dot{y} = y(-1 + dx + ey), \qquad \dot{z} = z(2 + gx + hy).$$

We will provide two different proofs as we did in Case 4.

In the first one the system has the invariant plane $\ell = 1 + (a/3)x = 0$ with cofactor $L_\ell = ax$. The change of coordinates $X = x\,\ell^{-1}$ will linearize the first equation and taking $Y = y/(A(X) + B(X)y)$ will then linearize the second equation provided $\tilde{\ell} = A+By = 0$ is an invariant surface with cofactor $L_{\tilde{\ell}} = d\,x + e\,y$. Therefore it suffices to find $A(X)$ and $B(X)$ satisfying the equations

$$\dot{A} = d\,A\,x, \qquad \dot{B} - B = e\,A, \qquad A(0) = 1. \tag{10}$$

From the first equation of (10) we can find $A = (1 + ax/3)^{\frac{d}{a}}$ (when $a = 0$ we take $A = \exp(dx/3)$). Write $A = \sum_{i\geq 0} a_i X^i$ and $B = \sum_{i\geq 0} b_i X^i$. Clearly B satisfies the second equation if we set $b_i = ea_i/(3i-1)$, and the covergence of B follows immediately from the convergence of A. The third equation would be linearizable by the transformation $Z = z\,\exp(-\gamma(X,Y))$ where $\dot{\gamma} = g\,x(X) + h\,y(X,Y)$. To find such γ we write

$$\gamma(X,Y) = \sum_{i+j>0} \gamma_{ij} X^i Y^j, \qquad x(X) = \sum_{i>0} \xi_i X^i, \qquad y(X,Y) = \sum_{i+j>0} \eta_{ij} X^i Y^j.$$

It is easy to see that from relation $\dot{\gamma} = g\,x(X) + h\,y(X,Y)$ we obtain

$$\sum (3i-j)\gamma_{ij} X^i Y^j = g \sum \xi_i X^i + h \sum \eta_{ij} X^i Y^j.$$

If $y(X,Y)$ contains no terms of the form $(XY^3)^k$, we can set

$$\gamma_{ij} = \frac{g\,\xi_i + h\,\eta_{ij}}{3i-j},$$

to give a convergent expansion of γ. Now it remains to show that the inverse transformation $y = AY/(1-BY)$ contains no term like $(XY^3)^k$. Note that

$$y = \frac{A\,Y}{1-BY} = \sum A\,B^k\,Y^{k+1}.$$

Assume that $k+1 = 3n$ for some n. We need to show that $AB^k = AB^{3n-1}$ contains no term like X^n. Note that

$$\dot{B} = \frac{dB}{dt} = 3X\frac{dB}{dX}.$$

From Equation (10) we obtain that

$$e\,A\,B^{3n-1} = \Big(\frac{3X}{3n}\frac{dB^{3n}}{dX} - B^{3n}\Big). \tag{11}$$

Clearly, the coefficient in X^n in B^{3n} vanishes on the right hand side so that either $e = 0$, that means $B \equiv 0$, or the coefficient of X^n in AB^{3n-1} vanishes. Therefore y in $Y = y/(A+By)$ contains no term like $(XY^3)^k$ and we have a linearizing transformation. This completes the first proof.

Now in the second proof we will just give the form of the first integrals and the inverse Jacobi multiplier since this procedure is exactly the same as

Case 4. We do not tackle the cases where the cross product is zero since we have already given a complete proof above. In cases (iii) and (iv), the integral obtained from Theorem 3 is of the same form and so we must rely on the arguments given in Remark 2 after Theorem 3.

i) When $ae \neq 0$ we have

$$\begin{aligned}
\phi &= x^{-a(2e+h)} y^{-3ah} z^{3ae} \left(1 + \frac{1}{3}ax\right)^{2ae-3eg+ah+3dh}_{,} \\
IJM &= x^{2+\frac{h}{3e}} y^{2+h/e} \left(1 + \frac{1}{3}ax\right)^{\frac{3eg-3ed-ah-3dh}{3ae})}_{,} \\
\psi &= x^{-1-\frac{h}{3e}} y^{-1-\frac{h}{e}} z(1+\ldots), \\
\phi_1 &= \phi^{\frac{1}{ae}} \psi^{-3} = xy^3(1+\ldots), \\
\phi_2 &= \phi^{\frac{3e+h}{3ae^2}} \psi^{-2-\frac{h}{e}} = y^2 z(1+\ldots).
\end{aligned}$$

ii) When $a = 0$ and $e \neq 0$ we obtain

$$\begin{aligned}
\phi &= x^{-2e-h} y^{-3h} z^{3e} \exp\left((dh - eg)x\right), \\
IJM &= x^{2+\frac{h}{3e}} y^{2+h/e} \exp\left(\frac{ge - de - dh}{3e} x\right), \\
\psi &= x^{-1-\frac{h}{3e}} y^{-1-\frac{h}{e}} z(1+\ldots), \\
\phi_1 &= \phi^{\frac{1}{e}} \psi^{-3} = xy^3(1+\ldots), \\
\phi_2 &= \phi^{\frac{3e+h}{3e^2}} \psi^{-2-\frac{h}{e}} = y^2 z(1+\ldots).
\end{aligned}$$

iii) When $e = 0$ and $a \neq 0$ we get

$$\phi = x\, y^3 (1 + \frac{a}{3}x)^{-1-\frac{3d}{a}}, \qquad IJM = x^{\frac{2}{3}} z (1 + \frac{a}{3}x)^{\frac{4}{3}+\frac{d}{a}}.$$

iv) When $a = e = 0$ we have

$$\phi = x\, y^3 \exp(-dx), \qquad IJM = x^{\frac{2}{3}} z \exp\left(\frac{d}{3}x\right).$$

Finally, since $\xi = x\ell^{-1}$ satisfies $\dot{\xi} = 3\xi$, the system is linearizable according to Theorem 4.

Group 3. This group is formed by Cases 6 and 7. Here, we also have a differential equation which involves only one variable and we can obtain one invariant algebraic surface. Thus we can construct one first integral and one inverse Jacobi Multiplier. Then by Theorem 3 we obtain the other first integral. In this group we use Theorem 4 to obtain a linearizing transformation.

Case 6 The system in this case is of the form

$$\dot{x} = x(3 + ax + by + cz), \qquad \dot{y} = y(-1 + ax + by + fz), \qquad \dot{z} = z(2 + kz).$$

For $k \neq 0$ the system has the invariant plane $\ell = 1 + \frac{k}{2}z = 0$ with cofactor $L_\ell = kz$. In this case we obtain the fist integral

$$\phi = x^{-1}\, y\, z^2\, \ell^{\frac{c-f-2k}{k}},$$

and the inverse Jacobi multiplier

$$IJM = x^3\, z^{\frac{-5}{2}}\, \ell^{\frac{9k+2f+4c}{2k}}.$$

For $k = 0$ the system admits the exponential factor $E = \exp(z/2)$ with cofactor $L_E = z$. In this case we obtain the first integral $\phi = x^{-1}\, y\, z^2\, E^{c-f}$ and the inverse Jacobi multiplier $IJM = x^3\, z^{-\frac{5}{2}}\, E^{f-2c}$. In both cases, Theorem 3 guarantees the existence of the second first integral $\psi = x^{-2} y\, z^{\frac{7}{2}}(1 + o(1))$. Thus we can construct two independent first integrals of the desired form

$$\phi_1 = \phi^7\psi^{-4} = xy^3(1 + o(1)) \quad \text{and} \quad \phi_2 = \phi^4\psi^{-2} = y^2 z(1 + o(1)).$$

Since $\xi = z\ell^{-1}$ satisfies $\dot{\xi} = 2\xi$ then by Theorem 4 the system is linearizable.

Case 7 We have the system

$$\dot{x} = x(3 + ax), \qquad \dot{y} = y(-1 + dx + ey + fz), \qquad \dot{z} = z(2 + gx + ey + fz).$$

For $a \neq 0$ the system has the invariant plane $\ell = 1 + ax/3 = 0$ with cofactor $L_\ell = ax$. Additionally, the system has the first integral

$$\phi = x\, y\, z^{-1} \ell^{\frac{g-d-a}{a}},$$

and the inverse Jacobi multiplier

$$IJM = x^{\frac{7}{3}} y^3 \ell^{\frac{3g-a-6d}{3a}}.$$

For $a = 0$ we obtain the first integral

$$\phi = x\, y\, z^{-1} E^{\frac{g-d}{3}},$$

and the inverse Jacobi multiplier

$$IJM = x^{\frac{7}{3}} y^3 \ell^{\frac{g-2d}{3}},$$

where $E = \exp(x)$. Then, by Theorem 3, there is a second first integral of the form $\psi = x^{-\frac{4}{3}} y^{-2} z(1 + \cdots)$ and hence we obtain two independent analytic first integrals of the desired form

$$\phi_1 = (\phi\psi)^{-3} = xy^3(1 + \cdots), \qquad \phi_2 = \phi^{-4}\psi^{-3} = yz^2(1 + \cdots).$$

From Theorem 4, since $\xi = x\ell^{-1}$ satisfies $\dot{\xi} = 3\xi$, we have that the system is linearizable.

Group 4. Cases 8 and 9 both have two equations which involve only two variables and so there is a linearizable node. The third equation can be linearized via a power series argument.

Case 8 The system can be written into the form

$$\dot{x} = x(3 + ax + cz), \qquad \dot{y} = y(-1 + dx + ey + fz), \qquad \dot{z} = z(2 + gx + kz).$$

Note that the first and third equation yields a linearizable node. Hence, there exists an analytic transformation of the form $X = x(1 + O(x, z))$ and $Z =$

$z(1+O(x,z))$ such that the two equations can be written as $\dot{X} = 3X$ and $\dot{Z} = 2Z$. Now in order to linearize the second equation we seek an invariant surface of the form $\ell = A + By = 0$ with cofactor $L_\ell = dx + ey + fz$ and $B = B(X,Z)$ and $A = A(X,Z)$ such that $A(0,0) = 1$. The change of variable $Y = y/(A+By)$ will then linearize the second equation. In order to find such A and B we have to solve the following system of differential equations

$$\dot{B} - B = e\,A, \qquad \dot{A} = (d\,x\, +\, f\,z)A. \tag{12}$$

We write $A = \exp(\alpha(X,Z))$ and from the second equation of (12) we obtain $\dot{\alpha}(X,Z) = d\,x(X,Z) + f\,z(X,Z)$. Suppose that $\alpha = \sum_{i+j>0} \alpha_{ij} X^i Z^j$. Then

$$\dot{\alpha}(X,Z) = \sum_{i+j>0} (3i+2j)\alpha_{ij} X^i Z^j = d\,x(X,Z) + f\,z(X,Z) =: \sum_{i+j>0} c_{ij} X^i Z^j.$$

Thus $\alpha_{ij} = \frac{c_{ij}}{3i+2j}$ for $i+j > 0$ and the convergence of $\sum_{i+j>0} c_{ij} X^i Z^j$ guarantees the convergence of α and hence of A. Now, we write $A = 1 + \sum_{i+j>0} a_{ij} X^i Z^j$ and $B = -e + \sum_{i+j>0} b_{ij} X^i Z^j$. Then from the first equation of (12) we find that $B = e + \sum_{i+j>0} \frac{e a_{ij}}{3i+2j-1} X^i Z^j$ which is clearly convergent. Hence, the system is linearizable.

Case 9 The system is

$$\dot{x} = x(3 + ax + by), \qquad \dot{y} = y(-1 + dx - by), \qquad \dot{z} = z(2 + gx + hy).$$

The change of variables $(X,Y) = (x, xy)$ transforms the first two equations into

$$\dot{X} = X(3 + aX) + bY, \qquad \dot{Y} = Y(2 + (a+d)X),$$

which is a linearizable node and hence there is a linearizing change of coordinates of the form $(\tilde{X},\tilde{Y}) = (X + bY + o(X,Y), \quad Y(1+o(1)))$ such that $\dot{\tilde{X}} = 3\tilde{X}$ and $\dot{\tilde{Y}} = 2\tilde{Y}$.

To linearize the third equation, assume that we have a function $\alpha = \alpha(\tilde{X},\tilde{Y})$ such that $\dot{\alpha} = g\,x(\tilde{X},\tilde{Y}) + h\,y(\tilde{X},\tilde{Y})$. Then the change of variable $\tilde{Z} = z\,\exp(-\alpha)$ will give $\dot{\tilde{Z}} = 2\,\tilde{Z}$. Writing

$$g\,x(\tilde{X},\tilde{Y}) + h\,y(\tilde{X},\tilde{Y}) = \sum a_{ij} \tilde{X}^i \tilde{Y}^j,$$

it is easy to see that $\alpha(\tilde{X},\tilde{Y}) = \sum \frac{a_{ij}}{3i+2j} \tilde{X}^i \tilde{Y}^j$. The convergence of α is then clear. Thus, the first integrals of the linear system $\tilde{X}^{-2}\tilde{Y}^3$ and $\tilde{X}^{-2}\tilde{Y}^2\tilde{Z}$ can be pulled back to get

$$\phi_1 = x\,y^3(1+\ldots) \quad \text{and} \quad \phi_2 = y^2 z(1+\ldots),$$

and linearizability follows from Theorem 4, since $\tilde{X} = x(1+\ldots)$ satisfies $\dot{\tilde{X}} = 3\tilde{X}$.

Group 5. The rest of the cases belong to this group. Here, the three differential equations involve all the variables. Each case is treated in a different manner.

Case 10 The system can be written as

$$\dot{x} = x(3+ax+by+cz), \qquad \dot{y} = y(-1+dx-by+fz), \qquad \dot{z} = z(2+gx+kz).$$

The change of variable $Y = x\,y$ transforms the system into the form

$$\dot{x} = x(3+ax+cz)+bY, \qquad \dot{Y} = Y(2+(a+d)x+(c+f)z), \qquad \dot{z} = z(2+gx+kz). \tag{13}$$

The singular point at the origin of system (13) is then in the Poincaré domain and it is easy to checked that there are no resonances. Therefore there exists a change of coordinates of the form

$$\left(X, \tilde{Y}, Z\right) = (x + bY + o(x, Y, z),\ Y\,(1 + o(1))\,,\ z\,(1 + o(1)))\,,$$

which linearizes system (13).

Furthermore, when $Y = 0$, the expression for X must be divisible by x. The first integrals of the linear system are then $\psi_1 = X^{-2}\tilde{Y}^3$ and $\psi_2 = \tilde{Y}Z^{-1}$. Pulling back these two first integrals we see that the initial system admits two independent analytic fist integrals of the desired form $\phi_1 = \psi_1 = x\,y^3(1+\ldots)$ and $\phi_2 = \psi_1\,{\psi_2}^{-1} = y^2\,z(1+\ldots)$. Since $\dot{Z} = 2Z$, Theorem 4 shows that the original system is in fact linearizable.

Case 11 The system is

$$\dot{x} = x(3 + ax - 2ey + cz), \qquad \dot{y} = y(-1 + ey + fz), \qquad \dot{z} = z(2 + gx + kz).$$

After the change of coordinates $(X, Y, Z) = (x^{\frac{1}{2}},\ x^{\frac{1}{2}}y,\ z)$ the system becomes

$$\begin{aligned} \dot{X} &= X\left(\frac{3}{2} + \frac{a}{2}X^2 + \frac{c}{2}z\right) - eY, \\ \dot{Y} &= Y\left(\frac{1}{2} + \frac{a}{2}X^2 + \left(f + \frac{c}{2}\right)z\right), \\ \dot{z} &= z(2 + gX^2 + kz). \end{aligned} \tag{14}$$

The singular point at the origin of system (14) is in the Poincaré domain. Since $z = 0$ is invariant, the two resonant monomials Y^4 and XY of the third equation must vanish. Additionally, for $z = 0$, a small calculation of the normal form of the two dimensional system shows that the resonant monomial Y^3 of the first equation must vanish. So system (14) is linearizable via an analytic change of coordinates which can be chosen as

$$(\tilde{X},\ \tilde{Y},\ \tilde{Z}) = (X - eY + o(X, Y, z),\ Y(1 + o(1)),\ z(1 + o(1)))\,.$$

Furthermore, from the $X \leftrightarrow -X$ symmetry of (14), the $o(1)$ terms in $\tilde{Y}$ and $\tilde{Z}$ are even in X and the $O(X, Y, z)$ term is odd in X. Also, when $Y = 0$, the expression for $\tilde{X}$ must be divisible by X. Thus, we can pull back the two first integrals $\psi_1 = \tilde{X}^{-1}\,\tilde{Y}^3$ and $\psi_2 = \tilde{X}^{-2}\,\tilde{Y}^2\,\tilde{Z}$ of the linear system to obtain the two first integrals of the initial system

$$\phi_1 = x\,y^3(1 + o(1)) \quad \text{and} \quad \phi_2 = y^2 z(1 + o(1)).$$

The expression for $\tilde{Z}$ in the original coordinates satisfies $\dot{Z} = 2Z$, and hence the system is also linearizable by Theorem 4.

Case 12 The system is

$$\dot{x} = x(3-3dx-3ey+cz), \quad \dot{y} = y(-1+dx+ey+fz), \quad \dot{z} = z(2-8dx+kz). \tag{15}$$

We let $\ell = 1 - dx - ex$ and perform the change of variables $(X,Y,Z) = (x, yz\ell^{-2}, z\ell^{-2})$. After a scaling by a factor of ℓ, this brings the system to the form

$$\begin{aligned}\dot{X} &= X(3 - ceY + cZ - cdXZ),\\ \dot{Y} &= Y(1 + (k+f)Z + e(f-k)Y + d(2c-f-k)XZ),\\ \dot{Z} &= Z(2 - e(2f-k)Y + kZ + d(2c-k)XZ).\end{aligned} \tag{16}$$

The origin of this system is in the Poincaré domain. Since $X = 0$ is invariant for system (16) the two resonant monomials Y^3 and YZ must vanish in the first equation. Additionally, $Z = 0$ is also invariant for system (16) and so the resonant monomial Y^2 in the last equation must vanish. Hence, for system (16) there is a linearising transformation of the form $(\tilde{X}, \tilde{Y}, \tilde{Z}) = (X(1+o(1)),\ Y(1+o(1)),\ Z(1+o(1)))$. The first integrals $\phi_1 = \tilde{X}\tilde{Y}^3\tilde{Z}^{-3}$ and $\phi_2 = \tilde{Y}^2\tilde{Z}^{-1}$ pull back to first integrals of the original system of the form $\phi_1 = xy^3(1+\dots)$ and $\phi_2 = y^2z(1+\dots)$.

Case 13 The system is

$$\dot{x} = x(3+ax+cz), \qquad \dot{y} = y(-1+dx+ey), \qquad \dot{z} = z(2+gx-ey+kz).$$

The change of coordinates $Y = yz$ transform the system above into

$$\dot{x} = x(3+ax+cz), \quad \dot{Y} = Y\big(1+(d+g)x+kz\big), \quad \dot{z} = z(2+gx+kz)-eY. \tag{17}$$

Note that the origin of system (17) is in the Poincaré domain. Since $x = 0$ is invariant for system (17), the resonant monomials YZ and Y^3 in the first equation should vanish. Moreover, for $x = 0$ a small calculation of the normal form for the two dimensional system shows that the resonant monomial, Y^2, should vanish in the last equation. Thus system (17) is linearizable by an analytic change of coordinates

$$(X, \tilde{Y}, Z) = (x(1+o(1)),\ Y(1+o(1)),\ z - eY + o(x,Y,z)).$$

We can now construct two first integrals for the linear system $\phi = X^{-1}\tilde{Y}^3$ and $\psi = X^{-1}\tilde{Y}Z$. Pulling back these two first integrals, we get two independent first integrals of the initial system and are of the desired form

$$\phi_1 = \phi^2\psi^{-3} = x\,y^3(1+o(1)) \quad \text{and} \quad \phi_2 = \phi\psi^{-1} = y^2z(1+o(1)).$$

Theorem 4 can then be applied to show the linearizability of the system from $\dot{X} = 3X$.

Case 14 The system is

$$\dot{x} = x(3+ax+by+cz), \qquad \dot{y} = y(-1+dx-by), \qquad \dot{z} = z(2+gx+by+kz).$$

The change of coordinates $(X, Z) = (x/(1+by)^2,\ z/(1+by))$ brings the system into the form

$$\begin{aligned} \dot{X} &= X(3 + aX(1+by) + cZ - 2bdXy)(1+by), \\ \dot{y} &= y(-1 + dX(1+by))(1+by), \\ \dot{Z} &= Z(2 + gX(1+by) + kZ - bdXy)(1+by). \end{aligned} \tag{18}$$

After rescaling the system by $(1 + bY)$ and applying the change of variable $Y = Xy$, system (18) becomes

$$\begin{aligned} \dot{X} &= X(3 + aX + (ab - 2bd)Y + cZ), \\ \dot{Y} &= Y(2 + (a+d)X + (ab + bd)Y + cZ), \\ \dot{Z} &= Z(2 + gX + (gb - bd)Y + kZ). \end{aligned} \tag{19}$$

Now the origin of system(19) is in the Poincaré domain and it is easy to see that there are no resonant terms. Hence, system (19) is linearizable via an analytic change of coordinates, $(\tilde{X}, \tilde{Y}, \tilde{Z}) = (X(1+o(1)),\ Y(1+o(1)),\ Z(1+o(1)))$. The desired first integrals are

$$\phi_1 = \tilde{X}^{-2}\tilde{Y}^3 = xy^3(1+\ldots) \quad \text{and} \quad \phi_2 = \tilde{X}^{-2}\tilde{Y}^2\tilde{Z} = y^2z(1+\ldots),$$

and the linearizability follows from Theorem 4 since $\dot{X} = 3X$, with $X = x(1+\ldots)$.

Case 15 The system is

$$\dot{x} = x(3 + ax + 2hy + cz), \qquad \dot{y} = y(-1 - hy), \qquad \dot{z} = z(2 + gx + hy + kz).$$

We use the change of variable $(X, Z) = (x/(3(1+by)),\ z/(2(1+by)))$ and rescale the resulting system by $1 + hy$ we get

$$\dot{X} = X\left(3 + \frac{a}{3}X + 2cZ\right), \quad \dot{y} = -y, \quad \dot{Z} = Z\left(2 + 3gX + \frac{k}{2}Z\right).$$

Clearly the first and third equation give a linearizable node and therefore the system is linearizable, and the original system must be integrable. Since $Y = y/(1+hy)$ satisfies $\dot{Y} = -Y$, the original system is linearizable by Theorem 4.

Case 16 The system is

$$\dot{x} = x(3 + ax + by + cz), \qquad \dot{y} = y(-1 + ax + by), \qquad \dot{z} = z(2 - ax - by + kz).$$

Performing the change of coordinates $(X, Y, Z) = (xz,\ yz,\ z)$ the system becomes

$$\dot{X} = X(5 + (c+k)Z), \quad \dot{Y} = Y(1 + kZ), \quad \dot{Z} = Z(2 + kZ) - aX - bY. \tag{20}$$

The origin of system (20) is in the Poincaré domain. Since $X = 0$ is invariant for (20), the resonant monomials YZ^2, Y^3Z and Y^5 in the first equation should vanish. For $X = 0$, a small calculation of the normal form for the two dimensional system shows that the resonant monomial, Y^2, in the last equation vanishes. Thus, system (20) is linearizable via a transformation

$(\tilde{X}, \tilde{Y}, \tilde{Z}) = (X(1+o(1)), Y(1+o(1)), Z + \frac{a}{3}X - bY + o(X, Y, Z))$. The first integrals of the linearized system, $\phi_1 = \tilde{X}\tilde{Y}^3\tilde{Z}^{-4}$ and $\phi_2 = \tilde{Y}^2\tilde{Z}^{-1}$, then pull back to integrals of the desired form $\phi_1 = x\,y^3(1+\ldots)$ and $\phi_2 = y^2\,z(1+\ldots)$. Theorem 4 can then be used to prove linearizability from the fact that $\tilde{Z} = z(1+\ldots)$ satisfies $\dot{\tilde{Z}} = \tilde{Z}$.

Case 17

$$\dot{x} = x(3+ax-9fz), \quad \dot{y} = y(-1+dx+ey+fz), \quad \dot{z} = z(2+gx-2ey-2fz).$$

The change of variables $(X, Y) = (x^{\frac{1}{2}}\,\ell^{-\frac{3}{2}}, y\,x^{\frac{1}{2}}\,\ell^{-\frac{3}{2}})$, where $\ell = 1 - ey - fz$, brings the system, after a scaling by ℓ to the form

$$\begin{aligned}
\dot{X} &= X\left(\frac{3}{2} + \frac{3}{2}\tilde{\ell}(deY + fgzX) + \frac{1}{2}a\tilde{\ell}^2\right),\\
\dot{Y} &= Y\left(\frac{1}{2} + \frac{3}{2}\tilde{\ell}(deY + fgzX) + (\frac{1}{2}a + d)\tilde{\ell}^2\right),\\
\dot{z} &= z\left(2 + g\tilde{\ell}^2\right),
\end{aligned} \tag{21}$$

where $\tilde{\ell} = X - eY - fXz$. The origin of the transformed system (21) is in the Poincaré domain. Since $X = 0$ is invariant for the system (21), we conclude that the resonant monomial Y^3 in the first equation should vanish. Moreover, $z = 0$ is invariant for system (21), and so the resonant monomials Y^4 and XY must also vanish in the last equation. So for system (21) there is a linearizing change of coordinates $(\tilde{X}, \tilde{Y}, \tilde{Z}) = (X(1+o(1), Y(1+o(1)), z(1+o(1)))$. Furthermore, from the form of the system (21), the terms of the series $\tilde{X}$ (resp. $\tilde{Y}$ and $\tilde{Z}$) are odd (resp. even) in the combined degree of X and Y. Thus the first integrals of (21), $\phi_1 = \tilde{X}^{-1}\tilde{Y}^3$ and $\phi_2 = \tilde{X}^{-2}\tilde{Y}^2\tilde{Z}$, pull back to first integrals of the original system of the required form: $\phi_1 = xy^3(1+o(1))$ and $\phi_2 = y^2z(1+o(1))$.

Case 18 We obtain the system

$$\dot{x} = x(3+ax-ey-7fz), \quad \dot{y} = y(-1+dx+ey+fz), \quad \dot{z} = z(2+gx-2ey-2fz).$$

A change of coordinate $X = x\,\ell^{-2}$ where $\ell = 1 - ey - fz$ transforms the above system into

$$\dot{X} = X(3+aX\ell+2X(edy+fgz))\,\ell, \quad \dot{y} = y(-1+dX\ell)\,\ell, \quad \dot{z} = Z(2+gX\ell)\ell. \tag{22}$$

We rescale the system by ℓ and perform the change of variables $Y = Xy$ to give

$$\begin{aligned}
\dot{X} &= X(3 + aX - (2d-a)eY + (2g-a)fXz),\\
\dot{Y} &= Y(2 + (a+d)X + (d-a)eY + (2g-d-a)fXz),\\
\dot{z} &= z(2 + gX - geY - gfXz).
\end{aligned} \tag{23}$$

The origin of (23) is in the Poincaré domain and it is easy to see that there are no resonant monomials. Hence system (23) is linearizable by a transformation

$(\tilde{X},\tilde{Y},\tilde{Z}) = (X(1+o(1)),\quad Y(1+o(1)),\quad z(1+o(1)))$. The first integrals in the desired forms are

$$\phi_1 = \tilde{X}^{-2}\tilde{Y}^3 = xy^3(1+\ldots) \quad \text{and} \quad \phi_2 = \tilde{X}^{-2}\tilde{Y}^2\tilde{Z} = y^2z(1+\ldots).$$

Case 19 In this case system (1) can be written as

$$\dot{x} = x(3+ax+by+cz), \quad \dot{y} = y(-1+dx-by-cz), \quad \dot{z} = z(2+gx-by-cz).$$

Under the change of coordinates $(X,Y,Z) = (x,\, xy,\, xz)$ we obtain the system

$$\dot{X} = X(3+aX)+bY+cZ, \qquad \dot{Y} = Y(2+(a+d)X), \qquad \dot{Z} = Z(5+(a+g)X), \tag{24}$$

and the singular point is in the Poincaré domain. Since $Z=0$ is invariant for system (24), the resonant monomial XY in the last equation must vanish. Thus, a change of coordinates $(\tilde{X},\tilde{Y},\tilde{Z}) = (X+bY-\frac{c}{2}Z+o(X,Y,Z), Y(1+o(1)), Z(1+o(1)))$ linearizes system (24).

The first integrals of the linear system are then $\phi_1 = \tilde{X}^{-2}\tilde{Y}^3$ and $\psi_2 = \tilde{X}^{-3}\tilde{Y}^2\tilde{Z}$, which pull back to first integrals in the desired form $\phi_1 = \phi = x\,y^3(1+\ldots)$ and $\phi_2 = \phi^{\frac{3}{2}}\,\psi = y^2\,z(1+\ldots)$. Since $\tilde{Z} = z(1+\ldots)$ satisfies $\dot{\tilde{Z}} = \tilde{Z}$, the system is linearizable from Theorem 4.

Case 20 The system in this case becomes

$$\dot{x} = x(3+ax-2hy-2kz), \quad \dot{y} = y(-1+hy+kz), \quad \dot{z} = z(2+gx+hy+kz).$$

After the change of coordinates $(X,Y,Z) = (x^{\frac{1}{2}},\; x^{\frac{1}{2}}y,\; x^{\frac{1}{2}}z)$, we get the system

$$\dot{X} = X\left(\frac{3}{2}+\frac{a}{2}X^2\right)-hY-kZ, \quad \dot{Y} = Y\left(\frac{1}{2}+\frac{a}{2}X^2\right), \quad \dot{Z} = Z\left(\frac{7}{2}+\left(\frac{a}{2}+g\right)X^2\right). \tag{25}$$

The origin of the resulting system is in the Poincaré domain. Since $Z=0$ is invariant for system (25), we see that the resonant monomials Y^7, XY^4 and X^2Y, must vanish in the last equation. Also, for $Z=0$, a small calculation of the normal form for the two dimensional system shows that the resonant monomial Y^3 in the first equation must also vanish. Thus, there is a transformation $(\tilde{X},\tilde{Y},\tilde{Z}) = (X-hY+\frac{k}{2}Z+o(X,Y,Z),\; Y(1+o(1)),\; Z(1+o(1)))$ which linearizes system (25).

Furthemore, since the system is symmetric under $X\mapsto -X$, the expressions for $\tilde{X}$ (resp. $\tilde{Y}$ and $\tilde{Z}$) are odd (resp. even) in X. Thus the first integrals of the linear system $\phi_1 = \tilde{X}^{-1}\tilde{Y}^3$ and $\phi_2 = \tilde{X}^{-2}\tilde{Y}^2\tilde{Z}$ pull back to first integrals $\phi_1 = x\,y^3(1+\ldots)$ and $\phi_2 = y^2\,z(1+\ldots)$ showing the system is integrable. Taking $\xi = \tilde{X}^2 = x(1+\ldots)$ we can apply Theorem 4 to show that the original system is linearizable. □

Remark 3. *According to Theorem* 4 *of* [6], *when the cofactors of x, y and z and the divergence of* (1) *are linearly independent, then the origin is linearizable if and only if it is integrable. However, since this condition only holds* *on an open set in general, we have preferred to use alternative methods here to establish the linearizability.*

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Waleed Aziz[1], Colin Christopher[2], Jaume Llibre[3] and Chara Pantazi[4]
[1] Department of Mathematics, College of Science, Salahaddin University-Erbil, Kurdistan Region-Iraq
Information Technology Department, Faculty of Science, Tishk International University, Erbil, KRG, Iraq
e-mail: `waleed.aziz@plymouth.ac.uk`
[2] School of Engineering, Computing and Mathematics, Plymouth University, Plymouth UK, PL4 8AA
e-mail: `C.Christopher@plymouth.ac.uk`
[3] Departament de Matemàtiques, Universitat Autònoma de Barcelona, Edifici C, 08193 Bellaterra, Barcelona, Catalonia, Spain
e-mail: `jllibre@mat.uab.cat`
[4] Departament de Matemàtiques, Universitat Politècnica de Catalunya, (EPSEB), Av. Doctor Marañón, 44–50, 08028 Barcelona, Spain
e-mail: `chara.pantazi@upc.edu`