TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 286, Number 2, December 1984

MINIMAL PERIODIC ORBITS FOR CONTINOUS MAPS OF THE INTERVAL

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ABSTRACT. For continuous maps of the interval into itself, Sarkovskii's Theorem gives the notion of minimal periodic orbit. We complete the characterization of the behavior of minimal periodic orbits. Also, we show for unimodal maps that the min-max essentially describes the behavior of minimal periodic orbits.

1. Introduction. We say that a periodic orbit of period n of a continuous map f of the interval is minimal, if n is the minimal period of f in Sarkovskii's ordering (see Definition 2.2). The aim of this paper is to characterize the behavior of a minimal periodic orbit relative to its natural ordering as a finite subset of the real line.

A periodic orbit P of a continuous map f of the interval will be called simple if f has a particular behavior on P, given in Definitions 2.3, 2.8, 2.15 and Proposition 4.6. The definitions of simple periodic orbit of period odd and a power of two were given by Stefan and Block, respectively. Also, for the above two cases they proved that a minimal periodic orbit has simple behavior.

The main result of this paper is to complete the characterization of the behavior of the minimal periodic orbits of continuous mappings of the interval. In fact we prove that every minimal periodic orbit has simple behavior. Moreover, for each simple behavior, that is for each simple periodic orbit P, we show there is a continuous map of the interval having P as a minimal periodic orbit (see Theorem 2.17 and Propositions 2.7 and 2.13).

For unimodal maps it is known that there is a strong relation between Sarkovskii's ordering and the min-max (see Theorem II.2.8 of [CE]). So, for unimodal maps, the notions of minimal periodic orbit and min-max are related to Sarkovskii's ordering. This implies the existence of some relation between them. The purpose of this paper on unimodal maps is to show the equivalence between the behavior described by simple periodic orbits and the "min-max itinerary" (see Theorem 3.4). Moreover, we prove that if an unimodal map has period n, then it has (at least) one simple periodic orbit of period n (see Theorem 3.5). Also, in Proposition 5.8, we characterize the shape of simple periodic orbits restricted to unimodal maps.

Theorem 3.4 was presented without proof in [AS].

Received by the editors May 12, 1983 and, in revised form, December 12, 1983.

1980 *Mathematics Subject Classification*. Primary 54H20.

Key words and phrases. Periodic point, minimal periodic orbit, simple periodic orbit, unimodal map, min-max.