MINIMAL PERIODIC ORBITS FOR CONTINOUS MAPS OF THE INTERVAL*

by

Lluís Alsedà Jaume Llibre Rafel Serra

Secció de Matemàtiques, Fac. de Ciències
Universitat Autònoma de Barcelona
Bellaterra, Barcelona, SPAIN.

Let I denote a closed interval on the real line and C(I) the set of continous maps from I to itself. A point $p \in I$ is a periodic point of a map $f \in C(I)$ if $f^n(p) = p$ for some positive integer n. The period of p is the least such integer n, and the orbit of p is the set $P = Orb(p) = \{f^k(p): k=1,2,\ldots,n\}$. We refer to such an orbit as a periodic orbit of period n. Let P(f) denote the set of positive integers n such that f has a periodic orbit of period n.

Let N denote the set of positive integers and $\rightarrow$ denote the following ordering of N:

$3 \rightarrow 5 \rightarrow 7 \rightarrow \ldots \rightarrow 2.3 \rightarrow 2.5 \rightarrow 2.7 \rightarrow \ldots \rightarrow 4.3 \rightarrow 4.5 \rightarrow 4.7 \rightarrow \ldots \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1.$

In the $\rightarrow$ ordering, called Sarkovskii's ordering, the smallest element is 3 and the greatest one is 1.

THEOREM 1 (Sarkovskii, [Sa] , [St] , [BGMY]). Let $f \in C(I)$ and suppose that $n \in P(f)$ and $n \rightarrow k$. Then $k \in P(f)$.

Definition 2 . Let $f \in C(I)$. Suppose that $P(f) \neq \{1,2,4,8,16,\ldots\}$ and let $n > 1$ the smallest element of P(f) in the $\rightarrow$ ordering. We say that a periodic orbit is minimal if its

* The complete version will appear in Trans. Amer. Math. Soc.

period is n and we will refer to such an orbit as MPO. Note that if $P(f) = \{1,2,4,8,16,\ldots\}$ then the smallest element of P(f) in the $\rightarrow$ ordering does not exist.

Definition 3 . Let $P = \{p_1, p_2,\ldots,p_n\}$ be a periodic orbit of $f \in C(I)$, with $p_1 < p_2 < \ldots < p_n$, of period $n = 2^m q$ where either $m \geq 0$ and $q = 1$ or $m = 0$ and $q \geq 3$. Suppose that $m = 0$ and let $t = (q+1)/2$. We say that P is a simple periodic orbit of type +, or equivalently SPO^+, if

$f(p_{t-k}) = p_{t+k+1}$ for $k = 0,1,2,\ldots,t-2$

$f(p_{t+k}) = p_{t-k}$ for $k = 1,2,3,\ldots,t-1$, and

$f(p_1) = p_t$

Similarly we say that P is a simple periodic orbit of type −, or equivalently SPO^-, if

$f(p_{t-k}) = p_{t+k}$ for $k = 1,2,3,\ldots,t-1$,

$f(p_{t+k}) = p_{t-k-1}$ for $k = 0,1,2,\ldots,t-2$, and

$f(p_q) = p_t$

For the case q=1 we define a simple periodic orbit, SPO, inductively. If m=1 then P is simple. Suppose $m > 1$, then we say P is simple if the two subsets $\{p_1,p_2,\ldots,p_{n/2}\}$ and $\{p_{(n/2)+1},\ldots,p_n\}$ of P are simple periodic orbits of period n/2 of f^2. Then we have $f(\{p_1,p_2,\ldots,p_{n/2}\}) = \{p_{(n/2)+1},\ldots,p_n\}$

This definition was given by Stefan [St] and Block [Bl1] for the cases m=0 and q=1, respectively. In [Bl1] the definition of simple periodic orbit is discussed for a periodic orbit of period 8 and some examples are given.

Definition 4 . Let $f \in C(I)$ and let $P = \{p_1,p_2,\ldots,p_n\}$ be a periodic orbit of f where $p_1 < p_2 < \ldots < p_n$. We denote by $\bar{f}$ the map such that:

(1) $\bar{f} \in C(I)$

(2) $\bar{f}(x) = f(p_1)$ for $x \leq p_1$

(3) $\bar{f}(x) = f(p_n)$ for $x \geq p_n$