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PERIODIC ORBITS OF MAPS OF Y

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ABSTRACT. We introduce some notions that are useful for studying the behavior of periodic orbits of maps of one-dimensional spaces. We use them to characterize the set of periods of periodic orbits for continuous maps of $Y = \{z \in \mathbf{C}: z^3 \in [0,1]\}$ into itself having zero as a fixed point. We also obtain new proofs of some known results for maps of an interval into itself.

INTRODUCTION

In recent years, there has been growing interest in studying the periodic orbits of maps of one-dimensional spaces. One of the first results, and the most spectacular one, is the Šarkovskiĭ theorem. We shall recall it to the reader.

Let $\mathbf{I}$ be the family of all continuous maps of the interval $[0,1]$ into itself (one can take any closed interval, but we choose $[0,1]$ to fix notation). For $f \in \mathbf{I}$, if $f^n(x) = x$, then we call the set $\{x, f(x), \dots, f^{n-1}(x)\}$ a *periodic orbit* of f, and its *period* is the smallest positive integer m such that $f^m(x) = x$. We denote by $\operatorname{Per}(f)$ the set of periods of all periodic orbits of f.

Let $\mathbf{N}$ be the set of all positive integers. The *Šarkovskiĭ ordering* of $\mathbf{N}$ is

$$3, 5, 7, 9, \dots, 2.3, 2.5, 2.7, 2.9, \dots,$$
$$2^2.3, 2^2.5, 2^2.7, 2^2.9, \dots, \dots, 2^3, 2^2, 2, 1.$$

If k stands to the right of n in the above ordering, we shall write $k >_s n$. If $k = 2^p.k'$ and $n = 2^q.n'$, where k' and n' are odd, then we have $k >_s n$ if and only if one of the following cases occurs:

(i) $k' > 1$, $n' > 1$, $p > q$.
(ii) $k' > 1$, $n' > 1$, $p = q$, $k' > n'$.
(iii) $k' = 1$, $n' > 1$.
(iv) $k' = 1$, $n' = 1$, $p < q$.

We denote $S(n) = \{n\} \cup \{k: k >_s n\}$ for $n \in \mathbf{N}$, $S(2^\infty) = \{2^i: i = 0, 1, \dots\}$, and $\mathbf{N}_s = \mathbf{N} \cup \{2^\infty\}$.

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