ORBITALLY UNIVERSAL CENTERS

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Abstract. In this paper we define when a polynomial differential system is orbitally universal and we show the relevance of this notion in the classical center problem, i.e. in the problem of distinguishing between a focus and a center.

1. Introduction and statement of the main results

In this work we consider differential systems in $\mathbb{R}^2$ of the form

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

with P and Q polynomials having at the origin an isolated singular point. As usual the dot denotes derivative with respect to the time t. Along this paper we also consider the associated *vector field* $\mathcal{X} = P(x, y)\partial/\partial x + Q(x, y)\partial/\partial y$ to the differential system (1).

One of the main open problem in the qualitative theory of dynamical systems is to characterize when a singular point of system (1) has a center. This problem is known as the *center problem* and it consists in distinguishing between a center and a focus. A center is a singular point for which there exists a punctured neighborhood filled of periodic orbits, and a focus has a punctured neighborhood filled of spiraling orbits. We note that the center problem goes back to Poincaré [31] and Dulac [16].

There exist different algorithms to determine the necessary conditions to have a center when the linear part has purely imaginary eigenvalues, or it has zero eigenvalues but the linear part is not identically zero, see [4, 8, 18, 28, 31]. The characterization when the linear part is identically zero is a hard problem still open, see [20, 21, 22, 26, 29] for some partial results.

Another problem is to provide sufficient conditions in order that a singular point of the differential system (1) be center. Several mechanism are known, and some conjectures are established, see [27]. In this work we define one mechanism that provides the sufficiency of the center problem. This mechanism joint with the condition of the existence of a local analytic first integral contain all the known mechanisms for detecting centers up to now. Before define it we need to introduce some definitions and results.

For a monodromic singular point (i.e. for a focus or a center) the most important mechanism to have a center is to have a smooth first integral defined in a neighborhood of the singular point, see [30]. However to check the existence of this first

2010 *Mathematics Subject Classification.* 34C05, 34A34, 34C14.
Key words and phrases. Center problem, universal center, orbitally reversible.

integral in general is a difficult problem. The main theory to find explicit expressions of first integrals of system (1) is the Liouville theory of integrability based on the existence of invariant algebraic curves and exponential factors.

An *invariant algebraic curve* of system (1) is a curve $f = 0$ with $f \in \mathbb{C}[x, y]$ invariant by the flow of (1), i.e. the orbital derivative $\dot{f} = \mathcal{X} f = P\partial f/\partial x + Q\partial f/\partial y$ vanishes on $f = 0$. As system (1) is polynomial, this condition implies that there exists a polynomial $K(x, y) \in \mathbb{C}[x, y]$ of degree less than or equal to $m-1$ such that $\mathcal{X} f = P\partial f/\partial x + Q\partial f/\partial y = Kf$. This polynomial K is called the *cofactor* of the invariant algebraic curve $f(x, y) = 0$.

A function of the form $e^{f/g}$ with f and g polynomials is called an *exponential factor* if there is a polynomial L of degree at most $m - 1$ such that $\mathcal{X}(e^{f/g}) = P\,\partial e^{f/g}/\partial x + Q\,\partial e^{f/g}/\partial y = L\,e^{f/g}$. The polynomial L is called the *cofactor* of the exponential factor $e^{f/g}$.

A non-locally constant function $H : U \subset \mathbb{R}^2 \to \mathbb{R}$ is a *first integral* of system (1) in the open set U if this function is constant in each solution $(x(t), y(t))$ of system (1) contained in U. In fact if $H \in C^1(U)$ is a first integral of system (1) on U if and only if $\mathcal{X} H = P\partial H/\partial x + Q\partial H/\partial y \equiv 0$ on U. A non-locally constant function $M : U \subset \mathbb{R}^2 \to \mathbb{R}$ is an *integrating factor* in U if

$$P\frac{\partial M}{\partial x} + Q\frac{\partial M}{\partial y} = -\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) M = -\mathrm{div}(\mathcal{X})M.$$

This integrating factor M is associated to the first integral H when $MP = -\partial H/\partial y$ and $MQ = \partial H/\partial x$. Of course $V = 1/M$ is an *inverse integrating factor* in $U \setminus \{M = 0\}$.

A polynomial differential system (1) has a *Liouville first integral* H if its associated inverse integrating factor is of the form

$$V = \exp\left(\frac{D}{E}\right)\prod_i C_i^{\alpha_i}, \tag{2}$$

where D, E and the C_i's are polynomials in $\mathbb{C}[x, y]$ and $\alpha_i \in \mathbb{C}$, see [13, 23, 32, 33]. The curves $C_i = 0$ are invariant algebraic curves of the differential system (1), and the exponential $\exp(D/E)$ is a product of some exponential factors associated to the multiple invariant algebraic curves of system (1), or to the invariant straight line at infinity, see for instance [12, 14, 15] or Chapter 8 of [17]. However the Liouville integrability does not give the sufficiency. In order to have a center we must add the condition that the Liouville first integral will be well-defined in a neighborhood of the origin, see [11], because for instance a linear focus has a Liouville first integral. Generalizations of the Liouville integrability (as the Weierstrass integrability) allow to use analytic invariant curves and analytic exponential factors but we must also add the same condition to have a center, see [24]. Consequently these integrability theories are not mechanisms to give the sufficiency of the center problem in stricto sensum.

For stating ours next results we need to recall some preliminary definitions. As usual we define the set of natural numbers $\mathbb{N} = \{1, 2, \ldots\}$. A polynomial f is *quasi-homogeneous of type* $\mathbf{t} = (t_1, t_2) \in \mathbb{N}^2$ *and of degree* k if $f(\varepsilon^{t_1}x, \varepsilon^{t_2}y) = \varepsilon^k f(x, y)$. The vector space of quasi-homogeneous polynomials of type $\mathbf{t}$ and degree k is denoted by $\mathcal{P}_k^{\mathbf{t}}$. A *polynomial vector field* $\mathbf{F} = (P, Q)^T$ *is quasi-homogeneous of type* $\mathbf{t}$ *and degree* k if $P \in \mathcal{P}_{k+t_1}^{\mathbf{t}}$ and $Q \in \mathcal{P}_{k+t_2}^{\mathbf{t}}$. The vector space of quasi-homogeneous polynomial vector fields of type $\mathbf{t}$ and degree k is denoted by $\mathcal{Q}_k^{\mathbf{t}}$. Given an analytic vector field $\mathbf{F}$, we write it as a quasi-homogeneous expansion corresponding to a fixed type $\mathbf{t}$:

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}_r(\mathbf{x}) + \mathbf{F}_{r+1}(\mathbf{x}) + \cdots = \sum_{j \geq r} \mathbf{F}_j,$$

where $\mathbf{x} \in \mathbb{R}^2$, $r \in \mathbb{N}$ and $\mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}$, i.e. each term $\mathbf{F}_j$ is a quasi-homogeneous vector field of type $\mathbf{t}$ and degree j. Any $\mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}$ can be uniquely written as

$$\mathbf{F}_j = \mathbf{X}_{h_j} + \mu_j \mathbf{D}_0, \tag{3}$$

where $\mu_j = \dfrac{1}{r + |\mathbf{t}|} \operatorname{div}(\mathbf{F}_j) \in \mathcal{P}_j^{\mathbf{t}}$, $h_j = \dfrac{1}{r + |\mathbf{t}|} \mathbf{D}_0 \wedge \mathbf{F}_j \in \mathcal{P}_{j+|\mathbf{t}|}^{\mathbf{t}}$, $\mathbf{D}_0 = (t_1 x, t_2 y)^T$, and $\mathbf{X}_{h_j} = (-\partial h_j/\partial y, \partial h_j/\partial x)^T$ is the Hamiltonian vector field with Hamiltonian function h_j, see [1, Prop.2.7] for more details of this decomposition.

Our work will focus with the generic case, that is, we will consider the center problem for the differential systems of the form:

$$\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x}) + \sum_{j > r} \mathbf{F}_j(\mathbf{x}), \tag{4}$$

where $\mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}$, $\mathbf{F}_r = \mathbf{X}_h + \mu \mathbf{D}_0$, $r \in \mathbb{N} \cup \{0\}$, $\mathbf{D}_0 = (t_1 x, t_2 y)^T$, $\mu_r \in \mathcal{P}_r^{\mathbf{t}}$, $h \in \mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$ and $h(x, y) \neq 0$ for all $(x, y) \in \mathcal{U} \setminus \{(0, 0)\}$ where $\mathcal{U} \subset \mathbb{R}^2$ is a neighborhood of the origin.

The following results about the system (4) are proved in [2].

Theorem 1. [2, Theorem 2] *The origin of system* (4) *is monodromic.*

Remark 2. *Theorem* 1 *provides us with a characterization of systems of type* (4)*, i.e. they are systems whose origin is monodromic and its Newton diagram consists of a single compact edge. Otherwise if* $\mathbf{F}_r = (P_r, Q_r)^T$ *then* $P_r(0, y) = 0$ *or* $Q_r(x, 0) = 0$*, i.e.* $x = 0$ *or* $y = 0$ *are invariant lines of* $\mathbf{F}_r$ *and then* x *or* y *are irreducible factors of* $h(x, y)$*, which contradicts the fact that* $h(x, y) \neq 0$ *in a neighborhood of the origin.*

For systems of type (4) *we can ensure that they can be transformed into a generalized Abel equation. These systems contain as a particular case the perturbations of linear-type, nilpotent monodromic systems and some generalized nilpotent monodromic systems namely systems* (4) *where* $\mathbf{F}_r = (-y^{2m-1}, x^{2n-1})^T$.

Theorem 3. [2, Theorem 5] *If the origin of* $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$ *is a focus, then the origin of system* (4) *is also a focus with the same stability.*

By Theorem 3 a necessary condition in order that the origin of system (4) be a center is that the origin of $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$ be a center. In [6, Theorem 3.3] necessary and sufficient conditions are given in order that a quasi-homogeneous system has a center at the origin.

We start from this condition, that is, we assume that system $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$ has a center at the origin. Under this condition we enunciate the following proposition.

Proposition 4. *Let* $(\mathrm{Cs}(\theta), \mathrm{Sn}(\theta))$ *be the solution of the initial value problem* $(\dot{x}, \dot{y})^T = \mathbf{F}_r(x, y)$, *with* $x(0) = 1$, *and* $y(0) = 0$, *then* $(\mathrm{Cs}(\theta), \mathrm{Sn}(\theta))$ *are* T*–periodic functions and if we apply the change of variable* $x = \rho^{t_1}\mathrm{Cs}(\theta)$, $y = \rho^{t_2}\mathrm{Sn}(\theta)$ *and the rescaling of time* $dt = d\tau/\rho^r$, *system (4) is transformed into*

$$\rho' = \sum_{i\geq 1} \alpha_i(\theta)\rho^{i+1}, \qquad \theta' = 1 + \sum_{i\geq 1} \beta_i(\theta)\rho^i, \tag{5}$$

where $' = d/d\tau$,

$$\begin{aligned} \alpha_i(\theta) &= \frac{1}{(r+|\mathbf{t}|)h(\theta)}\left[P_{r+i}(\theta)Q_r(\theta) - Q_{r+i}(\theta)P_r(\theta)\right], \\ \beta_i(\theta) &= \frac{r+i+|\mathbf{t}|}{(r+|\mathbf{t}|)h(\theta)} h_{r+i+|\mathbf{t}|}(\theta), \end{aligned}$$

$(P_{r+i}, Q_{r+i}) := \mathbf{F}_{r+i}$, $h_{r+i+|\mathbf{t}|}(x, y) = \frac{1}{r+i+|\mathbf{t}|}\left[t_1 x Q_{r+i}(x, y) - t_2 y P_{r+i}(x, y)\right]$ *and* $\alpha_i(\theta)$, $\beta_i(\theta)$ *are bounded rational functions in* $[0, T]$.

Proof. We should note that the point $(1, 0)$ belongs to the periodic ring, since $\mathbf{F}_r$ is a quasi-homogeneous global center. Therefore $\mathrm{Cs}(\theta)$ and $\mathrm{Sn}(\theta)$ are T–periodic functions. Differentiating with respect to the time we obtain $\dot{\mathbf{x}} = \frac{1}{\rho}\mathbf{D}_0\dot{\rho} + \frac{1}{\rho^r}\mathbf{F}_r(\mathbf{x})\dot{\theta}$. Therefore, $\dot{\mathbf{x}} \wedge \mathbf{F}_r(\mathbf{x}) = \frac{1}{\rho}\mathbf{D}_0 \wedge \mathbf{F}_r(\mathbf{x})\dot{\rho}$ and $\mathbf{D}_0 \wedge \dot{\mathbf{x}} = \frac{1}{\rho^r}\mathbf{D}_0 \wedge \mathbf{F}_r(\mathbf{x})\dot{\theta}$, where

$$\begin{aligned} \mathbf{D}_0 \wedge \mathbf{F}_r(\mathbf{x}) &= \mathbf{D}_0 \wedge (\mathbf{X}_h(\mathbf{x}) + \mu\mathbf{D}_0) = \mathbf{D}_0 \wedge \mathbf{X}_h = \nabla h \cdot \mathbf{D}_0 = (r+|\mathbf{t}|)h(\mathbf{x}) \\ &= (r+|\mathbf{t}|)\rho^{r+|\mathbf{t}|}h(\theta). \end{aligned}$$

In addition, using (3) conservative-dissipative decomposition of each quasi-homogeneous terms and denoting $(P_j, Q_j) := \mathbf{F}_j$, with $P_j \in \mathcal{P}^{\mathbf{t}}_{j+t_1}$ and $Q_j \in \mathcal{P}^{\mathbf{t}}_{j+t_2}$ we obtain

$$\begin{aligned} \dot{\mathbf{x}} \wedge \mathbf{F}_r(\mathbf{x}) &= \sum_{j\geq r} \mathbf{F}_j(\mathbf{x}) \wedge \mathbf{F}_r(\mathbf{x}) = \sum_{j>r} P_j(x, y)Q_r(x, y) - Q_j(x, y)P_r(x, y) \\ &= \sum_{j>r} \rho^{r+j+|\mathbf{t}|}\left[P_j(\theta)Q_r(\theta) - Q_j(\theta)P_r(\theta)\right], \\ \mathbf{D}_0 \wedge \dot{\mathbf{x}} &= \sum_{j\geq r} \mathbf{D}_0 \wedge \mathbf{F}_j(\mathbf{x}) = \sum_{j\geq r} \mathbf{D}_0 \wedge \left[\mathbf{X}_{h_{j+|\mathbf{t}|}}(\mathbf{x}) + \mu_j(\mathbf{x})\mathbf{D}_0\right] \\ &= \sum_{j\geq r} \mathbf{D}_0 \wedge \mathbf{X}_{h_{j+|\mathbf{t}|}}(\mathbf{x}) = \sum_{j\geq r} \nabla h_{j+|\mathbf{t}|}(\mathbf{x}) \cdot \mathbf{D}_0 \\ &= \sum_{j\geq r} (j+|\mathbf{t}|)h_{j+|\mathbf{t}|}(\mathbf{x}) = \sum_{j\geq r} (j+|\mathbf{t}|)\rho^{j+|\mathbf{t}|}h_{j+|\mathbf{t}|}(\theta). \end{aligned}$$

Denoting $h := h_{r+|t|}$ we obtain, after applying the reparametrization of the time $dt = d\tau/\rho^r$, that system (4) is transformed into the system of the statement.

Notice that $\alpha_i(\theta)$ and $\beta_i(\theta)$ are bounded rational functions since $h(\theta) \neq 0$ for all $\theta \in [0, T]$, otherwise there exists θ^* such that $h(\theta^*) = 0$ and if we take $x^* = \rho^{t_1}\mathrm{Cs}(\theta^*)$, $y^* = \rho^{t_2}\mathrm{Sn}(\theta^*)$ we get $h(x^*, y^*) = \rho^{r+|\mathbf{t}}h(\theta^*) = 0$, which contradicts the hypothesis $h(x, y) \neq 0$ for all (x, y) in a neighborhood of the origin. $\square$

Taking θ as the new independent variable, and developing in a neighborhood of the origin the right hand side of the differential equation (5) in powers series of the variable ρ we get the differential equation

$$\frac{d\rho}{d\theta}=\sum_{i=1}^{\infty}a_i(\theta)\rho^{i+1}, \tag{6}$$

on the cylinder $(\rho,\theta)\in\mathbb{R}\times[0,T]$ in a neighborhood of $\rho=0$ and where $a_i(\theta)$ are bounded rational functions in $[0,T]$ in the variables $\mathrm{Cs}(\theta)$ and $\mathrm{Sn}(\theta)$. Using the uniqueness theorem on the solutions of a differential equation, there is a unique solution of equation (6) with the initial value $\rho(0)=\rho_0$ for $|\rho_0|$ small enough. Equation (6) determines a *center* if for any sufficiently small initial value $\rho(0)$ the solution of (6) satisfies $\rho(0)=\rho(T)$. The *center problem* for the differential equation (6) is to find conditions on the functions $a_i(\theta)$ under which the equation has a center at $\rho=0$.

Below we show a generalization of [7, Theorem 3.1] which gives us a sufficient center condition for the system (4) called the composition conjecture.

Theorem 5. *Suppose that there are a differentiable function σ with $\sigma(0)=\sigma(T)$ and continuous functions f_i defined on $I=\sigma([0,T])$ such that in equation (6) $a_i(\theta)=f_i(\sigma(\theta))\sigma'(\theta)$ for all $i\geq 1$. Then the origin of equation (6) is a center.*

The center problem and an explicit expression for the first return map of the differential equation (6) for perturbation of linear centers have been studied by Brudnyi in [9, 10]. It is possible to extend these concepts for the case of system (4). The expression of the first return map is given in terms of the following iterated integrals of order k

$$I_{i_1\dots i_k}(a) := \int\cdots\int_{0\leq s_1\leq\cdots\leq s_k\leq T} a_{i_k}(s_k)\cdots a_{i_1}(s_1)\,ds_k\cdots ds_1, \tag{7}$$

where, by convention we assume that $I_{\emptyset}=1$. Let $\rho(\theta;\rho_0;a)$ with $\theta\in[0,T]$ be the solution of equation (6) with $a=(a_1(\theta),a_2(\theta),\dots)$ such that $\rho(0;\rho_0;a)=\rho_0$. Then $\mathcal{P}(a)(\rho_0):=\rho(T;\rho_0;a)$ is the first return map of the differential equation (6). The following is proved in [9, 10].

Theorem 6. *For sufficiently small initial values ρ_0 the first return map $P(a)$ is an absolutely convergent power series $\mathcal{P}(a)(\rho_0)=\rho_0+\sum_{n=1}^{\infty}c_n(a)\rho_0^{n+1}$, where*

$$c_n(a)=\sum_{i_1+\cdots+i_k=n} c_{i_1\dots i_k}I_{i_1\dots i_k}(a), \quad \textit{and}$$

$$c_{i_1\dots i_k}=(n-i_1+1)\cdot(n-i_1-i_2+1)\cdot(n-i_1-i_2-i_3+1)\cdots 1.$$

By Theorem 6 the center set $\mathcal{C}$ of the differential equations (4) is determined by the system of equations $c_n(a)=0$, for $n=1,2,\dots$. The coefficient $c_n(a)$ is a polynomial whose variables are the coefficients of the functions $a_i(\theta)$ which appear in the differential equation (6) for $i=1,2,\dots,n$.

The following definition is given in [10]. Differential equation (6) has a *universal center* if for all positive integers $i_1, \ldots, i_k$ with $k \geq 1$ the iterated integral $I_{i_1 \ldots i_k}(a) = 0$.

Definition 1. *We say that the origin of system* (4) *is a universal center if there exist coordinates* $(\rho, \theta) \in \mathbb{R} \times [0, T]$ *such that system* (4) *in these coordinates can be expressed as* (6) *with* $a_i(\theta)$ *rational smooth functions in* $[0, T]$*, and for all positive integers* $i_1, \ldots, i_k$ *with* $k \geq 1$ *the iterated integral* $I_{i_1 \ldots i_k}(a) = 0$.

In [25] it is proved that the property of being universal center is not invariant under changes of variables also it is shown the following result that relates the universal centers with those who verify the composition conjecture.

Theorem 7. *The origin of system* (4) *is a universal center if and only if there exist coordinates* (ρ, θ) *such that the generalized Abel equation* (6) *satisfies the composition conjecture, that is, verifies the hypothesis of Theorem* 5.

Definition 2. *The polynomial differential system* (4) *is orbitally universal if after an analytic change of coordinates and a reparametrization of time equation* (6) *associated to it, satisfies the conditions to have a universal center at the origin.*

The relevance of the notion of orbitally universal with respect to the classical center problem is given in the following remark.

Remark 8. *Assume that the polynomial differential system* (4) *has a focus or a center at the origin. If this system has an analytic first integral defined in a neighborhood of the origin, or it is orbitally universal, then it has a center at the origin.*

The claim of this remark follows easily, because if the system has a local analytic first integral at the singular point localized at the origin, being this a focus or a center, the proof that it is a center is straightforward because this condition is incompatible with the existence of a focus. On the other hand, if the system is orbitally universal from its definition it has a center at the origin.

We recall that a system is *orbitally reversible* if after an analytic change of coordinates and a reparametrization of time the corresponding system is symmetric respect to a straight line passing through the origin (also called *time–reversible*). Doing a rotation the axis of symmetry always can be the x-axis and then we say in this case that it is R_x-reversible. Now we see that Definition 2 contains all the orbitally reversible systems.

Theorem 9. *All orbitally reversible systems* (4) *are orbitally universal systems.*

Theorem 9 is proved in section 2.

Remark 10. *From the proof of Theorem 9 if the origin of system* (4) *is a center* R_x*-reversible, then is a universal center.*

Theorem 11. *Consider* $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$*, with* $\mathbf{F}_r \in \mathcal{Q}_r^{\mathbf{t}}$*, and with a center at the origin. Let* $(\mathrm{Cs}(\theta), \mathrm{Sn}(\theta))$ *be the periodic solution of period* T *of the initial value problem* $(\dot{x}, \dot{y})^T = \mathbf{F}_r(x, y)$*, with* $x(0) = 1$*, and* $y(0) = 0$*. Then the origin of the system* $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x}) + \mu_k(\mathbf{x})\mathbf{D}_0$*, with* $\mu_k \in \mathcal{P}_k^{\mathbf{t}}$*,* $k > r$*, is a center if and only if* $\int_0^T \mu_k(\mathrm{Cs}(\theta), \mathrm{Sn}(\theta))d\theta = 0$*. Moreover this center is universal.*

Theorem 11 is proved in section 2.

According with all the known results on centers we make the following conjecture which would characterize all the centers of the polynomial and also analytic differential systems of the form (4).

Conjecture 1. *The origin of system* (4) *is a center if the system has a local analytic first integral, or it is orbitally universal.*

A strong support to this conjecture is provided by the next result.

Theorem 12. *Any center with purely imaginary eigenvalues, or any nilpotent center satisfies Conjecture* 1.

Theorem 12 is proved in section 2.

Finally in Section 3 we provide examples of centers satisfying the results of our theorems.

2. Proof of the theorems

Proof of Theorem 9. For singular points with purely imaginary eigenvalues we can use the result established in [25, Theorem 3], which shows that any time-reversible system with a singular point with purely imaginary eigenvalues is universal. The proof for a general monodromic singular point is as follows.

If system (1) is orbitally reversible then there exists a type $\mathbf{t} = (t_1, t_2)$, a change of variables and a rescaling of time such that system (1) is transformed into a system of the form $(\dot{x}, \dot{y})^T = \sum_{j \geq r} \mathbf{G}_j$, $\mathbf{G}_j \in \mathcal{Q}_j^{\mathbf{t}}$, which is R_x-reversible. Hence $\mathbf{G}_j = (P_j(x,y), Q_j(x,y))^T$, with $P_j \in \mathcal{P}_{j+t_1}^{\mathbf{t}}$ even in x, $Q_j \in \mathcal{P}_{j+t_2}^{\mathbf{t}}$ odd in x. Now we apply the change of variables $x = u^{t_1} \cos\theta$, $y = u^{t_2} \sin\theta$, and the rescaling of time $dt = (t_1 \cos^2\theta + t_2 \sin^2\theta) dT/u^r$, and the system becomes

$$u' = u \sum_{i \geq 0} u^i \alpha_i(\cos\theta, \sin\theta), \qquad \theta' = \sum_{i \geq 0} u^i \beta_i(\cos\theta, \sin\theta),$$

where

$$\begin{aligned}
\alpha_i(\cos\theta, \sin\theta) = &\quad \cos\theta \, P_{r+i}(\cos\theta, \sin\theta) + \sin\theta \, Q_{r+i}(\cos\theta, \sin\theta), \\
\beta_i(\cos\theta, \sin\theta) = &\quad t_1 \cos\theta Q_{r+i}(\cos\theta, \sin\theta) - t_2 \sin\theta P_{r+i}(\cos\theta, \sin\theta).
\end{aligned}$$

Consequently α_i is an odd function in $\cos\theta$, and β_i is an even function in $\cos\theta$. Moreover $\beta_0(\cos(\theta), \sin(\theta)) \neq 0$, $\forall \theta \in [0, 2\pi]$, otherwise there exists $\theta^* \in [0, 2\pi]$ such that

$$\begin{aligned}
0 &= u^{r+|\mathbf{t}|} \beta_0(\cos\theta^*, \sin\theta^*) \\
&= t_1 u^{t_1} \cos(\theta^*) u^{r+t_2} Q_r(\cos\theta^*, \sin\theta^*) - t_2 u^{t_2} \sin\theta^* u^{r+t_1} P_r(\cos\theta^*, \sin\theta^*) \\
&= t_1 x^* Q_r(x^*, y^*) - t_2 y^* P_r(x^*, y^*) = h_{r+|\mathbf{t}|}(x^*, y^*),
\end{aligned}$$

where $x^* = u^{t_1} \cos\theta^*$ and $y^* = u^{t_2} \sin\theta^*$. This is a contradiction with the hypothesis $h(x,y) \neq 0$ in a neighborhood of the origin. Consequently, taking into account that

$\beta_0(\cos(\theta), \sin(\theta)) \neq 0$, we can write the differential equation

$$\frac{du}{d\theta} = \frac{u \sum_{i\geq 0} u^i \dfrac{\alpha_i(\theta)}{\beta_0(\theta)}}{1 + \sum_{i>0} u^i \dfrac{\beta_i(\theta)}{\beta_0(\theta)}} = \sum_{i\geq 0} u^{i+1} A_i(\theta),$$

where for shortness we have defined $g(\cos\theta, \sin\theta) := g(\theta)$ and $A_i(\theta)$ is an odd function of $\cos\theta$ for all $i \geq 0$.

If now we apply the change $\rho = u \exp\left(-\int_0^\theta A_0(s)ds\right)$ we get

$$\frac{d\rho}{d\theta} = \sum_{i\geq 1} \rho^{i+1} a_i(\theta), \quad \text{where} \quad a_i(\theta) = A_i(\theta) \exp\left(i \int_0^\theta A_0(s)ds\right).$$

As the functions $A_i(\theta)$ are odd in the variable $\cos(\theta)$ for $i \geq 0$,there exist functions g_i such that $A_i(\theta) = g_i(\sigma(\theta))\sigma'(\theta)$ where $\sigma(\theta) = \sin(\theta)$. If we define $G(s) = \int_0^s g_0(t)dt$ then $\int_0^\theta A_0(s)ds = \int_0^\theta g_0(\sigma(s))\sigma'(s)ds = G(\sigma(\theta)$, and consequently for $i \geq 1$ $a_i(\theta) = f_i(\sigma(\theta))\sigma'(\theta)$ where $f_i(s) = g_i(s)\exp(iG(s))$.

In short the previous equation can be written as

$$\frac{d\rho}{d\theta} = \sum_{i\geq 1} \rho^{i+1} f_i(\sigma(\theta))\sigma'(\theta), \quad \text{with} \quad \sigma(\theta) = \sin\theta.$$

Therefore by Theorem 5 this differential equation satisfies the composition condition, and by Theorem 7 it is a universal center. Consequently any orbitally reversible system is orbitally universal. This completes the proof of the theorem. □

Proof of Theorem 11. The associated Abel equation (6) to the system $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x}) + \mu_k(\mathbf{x})\mathbf{D}_0$, using the generalized polar coordinates given in Proposition 4, is

$$\frac{d\rho}{d\theta} = \sum_{i\geq 1} \rho^{i+1} a_i(\theta) = \mu_k(\text{Cs}(\theta), \text{Sn}(\theta))\rho^{k+1-r}.$$

Hence we have that $a_{k-r}(\theta) = \mu_k(\text{Cs}(\theta, \text{Sn}(\theta))$ and $a_i(\theta) = 0$ for $i \neq k - r$. Consequently we have that if $\int_0^T \mu_k(\text{Cs}(\theta, \text{Sn}(\theta))d\theta \neq 0$, then the origin is a focus and if $\int_0^T \mu_k(\text{Cs}(\theta), \text{Sn}(\theta))d\theta = 0$, then we define $\sigma(\theta) := \int_0^\theta \mu_k(\text{Cs}(\alpha), \text{Sn}(\alpha))d\alpha$ and we have that $\sigma(0) = \sigma(T) = 0$, that is, $\sigma(\theta)$ is a periodic function of period T. Now we take $f_{k-r}(\sigma(\theta)) = 1$, and $f_i(\theta) = 0$ if $i \neq k - r$, and we have $a_i(\theta) = f_i(\theta)\sigma'(\theta)$ for all $i \geq 1$. Applying Theorem 5 the differential equation satisfies the composition condition, and by Theorem 7 the center is universal. □

Proof of Theorem 12. The nondegenerate centers by the Poincaré theorem [31] are orbitally equivalent to $(-y, x)^T$ therefore are analytically integrable and orbitally universal. The nilpotent center by the Berthier-Moussu theorem [8] are orbitally reversible, therefore by Theorem 9 are orbitally universal. □

3. Examples

Example 1. Consider the polynomial differential system

$$\dot{x} = y + x^4, \qquad \dot{y} = -x, \tag{8}$$

that has neither an invariant algebraic curve, nor an integrating factor of the form (2), and consequently is not Liouvillian integrable, see [19]. However system (8) is analytic integrable because it has a nondegenerate center at the origin. In fact it has a center at the origin because system (8) is invariant under the symmetry $(x, y, t) \to (-x, y, -t)$. Therefore by the Poincaré theorem (see [31]) the system has an analytic first integral in a neighborhood of the origin. Moreover this system has a universal center at the origin because it has a R_x-reversible center.

Example 2. Consider the polynomial differential system

$$\dot{x} = y + x^2, \qquad \dot{y} = -x^3, \tag{9}$$

that is not analytically integrable, see [20]. In fact it has a C^∞ first integral which is Liouville, see [20]. This first integral can be obtained doing the change of variables $(u, v) \to (x^2, y)$ passing to a linear focus which has a Liouville first integral not well-defined at the origin. System (9) has the inverse integrating factor $V = x^4 + 2x^2y + 2y^2$, and using the Singer theorem [33] the system is Liouville integrable. Moreover system (9) is invariant by the symmetry $(x, y, t) \to (-x, y, -t)$, hence it has a center at the origin. Using Theorem 9 it is also an orbitally universal system.

Example 3. Consider the polynomial differential system

$$\begin{aligned} \dot{x} &= y^3 + 2ax^3y + 2x(b\,x^4 + c\,xy^2), \\ \dot{y} &= -x^5 - 3ax^2y^2 + 3y(b\,x^4 + c\,xy^2), \end{aligned} \tag{10}$$

with $bc \neq 0$. System (10) was studied in [3] and it is neither formally orbitally reversible, nor analytically integrable. Nevertheless, there exist $b, c \in \mathbb{R} \setminus \{0\}$ and $a \in \left(-1/\sqrt{6}, 0\right) \cup \left(0, 1/\sqrt{6}\right)$ such that the origin of system (10) becomes a center, see [3]. Moreover system (10) in the generalized polar coordinates $x = u^2 \cos\theta$, $y = u^3 \text{sen}\theta$ turns into a Bernoulli differential equation (see [3]), and consequently it has a Liouville first integral in these new coordinates. We can deduce that is Liouville integrable because it has the inverse integrating factor $V = (2x^6+12ax^3y^2+3y^4)^{13/12}$. The system (10) is of the form $\dot{\mathbf{x}} = \mathbf{F}_7 + \mathbf{F}_8$ with $\mathbf{F}_i \in \mathcal{Q}_i^{(2,3)}$, $\mathbf{F}_7 = \mathbf{X}_h$, $h = -\dfrac{1}{4}y^4 - ax^3y^2 - \dfrac{1}{6}x^6$. From [3] we deduce that the origin of system $\dot{\mathbf{x}} = \mathbf{F}_7(\mathbf{x})$ is a center and $\mathbf{F}_8 = \mu_8 \mathbf{D}_0$, with $\mu_8 = b\,x^4 + c\,xy^2$ and $\mathbf{D}_0 = (2x, 3y)^T$. Applying Theorem 11 this system has a universal center at the origin.

Acknowledgements

The first and second authors are partially supported by a MINECO/FEDER grant number MTM2014-56272-C2-2 and by the *Consejería de Educación y Ciencia de la Junta de Andalucía* (projects P12-FQM-1658, FQM-276). The third author is partially supported by a MINECO/FEDER grant number MTM2017-84383-P and by an AGAUR (Generalitat de Catalunya) grant number 2017SGR-1276. The fourth author is partially supported by the Ministerio de Economía, Industria y Competitividad, Agencia Estatal de Investigación grant MTM2016-77278-P (FEDER) and grant MDM-2014-0445, the Agència de Gestió d'Ajuts Universitaris i de Recerca grant 2017SGR1617, and the H2020 European Research Council grant MSCA-RISE-2017-777911.

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