NEW LOWER BOUNDS FOR THE HILBERT NUMBERS USING REVERSIBLE CENTERS

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Abstract. In this paper we provide the best lower bounds, that are known up to now, for the Hilbert numbers of polynomial vector fields of degree N, $H(N)$, for small values of N. These limit cycles appear bifurcating from new symmetric Darboux reversible centers with very high simultaneous cyclicity. The considered systems have, at least, three centers, one on the reversibility straight line and two symmetric about it. More concretely, the limit cycles are in a three nests configuration and the total number of limit cycles is at least $2n+m$, for some values of n and m. The new lower bounds are obtained using simultaneous degenerate Hopf bifurcations. In particular, $H(4) \geq 28$, $H(5) \geq 37$, $H(6) \geq 53$, $H(7) \geq 74$, $H(8) \geq 96$, $H(9) \geq 120$, and $H(10) \geq 142$.

1. Introduction

We consider two-dimensional differential systems

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

in which P and Q are polynomials of degree N. The maximum possible number, $H(N)$, of limit cycles of system (1) is known as the Hilbert number. As usual we define limit cycle as every isolated periodic solution.

As it follows from former definition, the Hilbert number refers to the total amount of limit cycles that system (1) exhibits and, in this sense, it is a global concept. For instance, Shi (see [24]) and Chen and Wang (see [5]) proved that $H(2) \geq 4$. For cubics, Li, Liu, and Yang (see [18]) and Li and Liu (see [19]) proved that $H(3) \geq 13$. In our work we prove, among other new best lower bounds for the Hilbert numbers, that $H(4) \geq 28$.

The aim of this work is to obtain new lower bounds values for $H(N)$. This goal is attained by simultaneously perturbing some reversible centers. We proceed studying simultaneous degenerate Hopf bifurcations of reversible centers and, to overcome heavy computations, we use an efficient way (parallelization method) for the Lyapunov quantities calculation.

This work strongly relies on the results of Christopher, [7], and Han, [12] and [13]. The idea is to estimate the generic cyclicity of a family of simultaneous centers from the series expansion of the Lyapunov quantities at a point on the center variety. In particular, if the first r linear terms of the Lyapunov quantities are independent, then the cyclicity is bigger or equal than r, if we consider also the trace as another independent parameter. The germ of this linearization idea can be glimpsed at the work of Chicone and Jacobs, see [6], where, for higher order limit cycles bifurcations,

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