ON THE SINGULARITIES OF THE PLANAR CUBIC POLYNOMIAL DIFFERENTIAL SYSTEMS AND THE EULER JACOBI FORMULA

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Abstract. Using the Euler-Jacobi formula we obtain an algebraic relation between the singular points of a polynomial vector field and their topological indices. Using this formula we obtain the configuration of the singular points together with their topological indices for the planar cubic polynomial differential systems when these systems have nine finite singular points.

1. Introduction and statement of the main results

Consider in $\mathbb{R}^2$ the polynomial differential system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

where $P(x, y)$ and $Q(x, y)$ are real polynomials of degree 3, called a planar cubic polynomial differential system, or simply *cubic system*.

The motivation of our paper comes from the fact that for the planar quadratic polynomial differential systems the characterization of all configurations of the indices of the singular points of all quadratic differential systems that have four singular points is the well-known Berlinskii's Theorem proved in [2, 6], and reproved in [4] using the Euler-Jacobi formula. We say that a quadrilateral is *convex* if any vertex of it is contained in the convex hull of the other three vertices, otherwise the quadrilateral is called *concave*. Then the Berlinskii's Theorem can be stated as follows. *Assume that a real quadratic differential system has exactly four real singular points. In this case if the quadrilateral formed by these points is convex, then two opposite singular points are anti-saddles (i.e. nodes, foci or centers) and the other two are saddles. If this quadrilateral is concave, then either the three exterior vertices are saddles and the interior vertex is an anti-saddle, or the exterior vertices are anti-saddles and the interior vertex is a saddle.*

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We want to extend the Berlinskii's Theorem to the case of cubic systems and obtain all configurations of the singular points together with their topological indices when the cubic systems have the maximum number of finite singular points, i.e. nine singular points.

By Bézout's Theorem (see [7] for a proof of this theorem) the maximum number of singular points of a cubic polynomial differential system is nine. To these cubic polynomial differential systems (1) having nine singular points we can apply the Euler-Jacobi formula (see [1] for a proof of such formula). For such systems the Jacobian determinant

$$J = \begin{vmatrix} \partial P/\partial x & \partial P/\partial y \\ \partial Q/\partial x & \partial Q/\partial y \end{vmatrix}$$

evaluated at each singular point does not vanish, and for any polynomial R of degree less than or equal to 3 we have

$$\sum_{a\in A} \frac{R(a)}{J(a)} = 0, \tag{2}$$

where A is the set of finite singular points of system (1). Given a finite subset B of $\mathbb{R}^2$, we denote by $\hat{B}$ its convex hull, by $\partial\hat{B}$ the boundary of $\hat{B}$, and by $\#B$ the cardinal of the set B.

Set $A_0 = A$ and for $i \geq 1$ $A_i = A \cap \partial(A \setminus (A_0 \widehat{\cup \cdots \cup} A_{i-1})]$. There is an integer q such that $A_{q+1} = \emptyset$ and $A_q \neq \emptyset$.

We say that A has the *configuration* $(K_0; K_1; K_2; \ldots; K_q)$ where $K_i = \#(A_i \cap \partial\hat{A}_i)$. We say that the singular points of system (1) which belong to $A_i \cap \partial\hat{A}_i$ are on the *i-th level*.

We are also interested in the study of the (topological) indices of the singular points of system (1).

We recall that if we assume that $\#A = 9$ then the Jacobian determinant J is non-zero at any singular point of system (1), and that the topological indices of the singular points are 1 (respectively -1) if $J > 0$ (respectively $J < 0$), see for more details [11] or [8], and then the number K_i of the i-th level is substituted by the vector $(n_i^1+, n_i^2-, \ldots, n_i^{m_i-1}+, n_i^{m_i}-)$, where n_i^j are positive integers such that $\sum_j n_i^j = K_i$. More precisely, since $A_i \cap \partial\hat{A}_i$ is a convex polygon, these numbers take into account the number of consecutive points with positive and negative indices, viewing the ith level oriented counterclockwise: n_i^1 corresponds to the string with largest number of consecutive points with positive indices. If there are several strings with the same number of points we choose one of them such that the next string (that has points with negative indices) is as large as possible, and successively. We continue the process with n_i^2 and so on. Furthermore, when $A_i \cap \partial\hat{A}_i$ is formed by two points then clearly we only have the three following possible strings $2+$; $2-$ and $+, -$. If $A_i \cap \partial\hat{A}_i$ is a segment of more than two points, since the system is cubic the segment has exactly three singularities (see Lemma 4), then the possible strings are $3+$; $2+,-$; $+,2-$ and $3-$.

We denote by $i(a)$ the topological index of a singular point a of a polynomial differential system.

With the notation introduced the Berlinskii's Theorem can be stated as follows.

Theorem 1 (Berlinskii's Theorem)**.** *For planar quadratic polynomial differential systems such that $\#A = 4$ the following statements hold:*

(a) $\sum_{a\in A} i(a) = 0$ *or* $|\sum_{a\in A} i(a)| = 2$.
(b) *If* $\sum_{a\in A} i(a) = 0$ *there is only the configuration* $(4) = (+,-,+,-)$.
(c) *If* $|\sum_{a\in A} i(a)| = 2$ *there are only the two configurations* $(3;1)$ *with either* $(3+,-)$ *or* $(3-,+)$.

There exist examples of quadratic polynomial differential systems with such configurations.

Berlinskii's Theorem has been extended to polynomial differential systems (1) with degrees of the polynomials P and Q equal to 2 and 3 in [4], and to degrees 2 and 4 in [10]. But the cubic case is much more difficult and its complete solution is presented in this paper.

It was proved in [9, 10] that in the case of cubic polynomial polynomial differential systems $|\sum_{a\in A} i(a)| = 1$, or $|\sum_{a\in A} i(a)| = 3$. In the next theorem, which is our first main theorem we characterize all possible configurations for cubic differential systems when $|\sum_{a\in A} i(a)| = 3$.

Theorem 2. *For planar cubic polynomial differential systems such that $\#A = 9$ and $|\sum_{a\in A} i(a)| = 3$ only the following configurations are possible:*

(5;3;1) *with* $(5+;3-;+)$, $(5-;3+;-)$;
(4;4;1) *with* $(4+;+,3-;+)$, $(4-;3+,-;-)$;
(4;3;2) *with* $(4+;2+,-;2-)$, $(4+;3-;2+)$, $(4+;+,2-;+,-)$, $(4-;+,2-;2+)$, $(4-;3+;2-)$, $(4-;2+,-;+,-)$;
(3;6) *with* $(3+;+,-,+,-,+,-)$, $(3-;+,-,+,-,+,-)$;
(3;5;1) *with* $(3+;+,2-,+,-;+)$, $(3-;2+,-,+,-;-)$;
(3;4;2) *with* $(3+;2+,2-;+,-)$, $(3-;2+,2-;+,-)$;
(3;3;3) *with* $(3+;2+,-;+,2-)$, $(3-;+,2-;2+,-)$;

and there exist examples of cubic polynomial differential systems with such configurations.

Theorem 2 is proved in section 3. To finish the cubic case it is necessary to characterize all possible configurations when $|\sum_{a\in A} i(a)| = 1$, and it is done in the following theorem.

Theorem 3. *For planar cubic polynomial differential systems such that $\#A = 9$ and $|\sum_{a\in A} i(a)| = 1$ only the following configurations are possible:*

(9) *with* $(2+,-,+,-,+,-,+,-)$, $(+,2-,+,-,+,-,+,-)$;
(8;1) *with* $(+,-,+,-,+,-,+,-;+)$, $(3+,-,+,-,+,-;-)$, $(+,-,+,-,+,-,+,-;-)$, $(+,3-,+,-,+,-;+)$;
(7;2) *with* $(2+,-,+,-,+,-,+;+,-)$, $(+,2-,+,-,+,-,+;+,-)$;
(6;3) *with* $(2+,-,2+,-;+,2-)$, $(3+,-,+,-;+,2-)$, $(+,2-,+,2-;2+,-)$, $(+,2-,+,2-;2+,-)$;
(5;4) *with* $(2+,-,+,-;+,-,+,-)$, $(+,2-,+,-;+,-,+,-)$;
(5;3;1) *with* $(4+,-:3-;+)$, $(+,4-;3+;-)$;
(4;5) *with* $(3+,-;+,2-,+,-)$, $(+,-,+,-;2+,-,+,-)$, $(+,3-;2+,-,+,-)$, $(+,-,+,-;+,2-,+,-)$;
(4;4;1) *with* $(4+;4-,+)$, $(3+,-;+,3-;+)$, $(+,-,+,-;+,-,+,-;+)$, $(+,-,+,-;3+,-;-)$, $(4-;4+,-)$, $(+,3-;3+,-;-)$, $(+,-,+,-;+,-,+,-;-)$, $(+,-,+,-;+,3-;-)$;
(4;3;2) *with* $(4+;3-;+,-)$, $(3+,-;+,2-;+,-)$, $(+,-,+,-;2+,-;+,-)$, $(4-;3+;+,-)$, $(+,3-;2+,-;+,-)$, $(+,-,+,-;+,2-;+,-)$;
(3;6) *with* $(2+,-;+,-,+,-,+,-)$, $(3+;+,3-,+,-)$, $(+,2-;+,-,+,-,+,-)$, $(3-;3+,-,+,+)$;
(3;5;1) *with* $(2+,-;2+,3-;+)$, $(3+;+,4-;+)$, $(3+;+,2-,+,-;-)$, $(2+,-;2+,-,+,-;-)$, $(+,2-;3+,2-;-)$, $(3-;4+,-;-)$, $(3-;2+,-,+,-;+)$, $(+,2-;+,2-,+,-;+)$;
(3;4;2) *with* $(2+,-;+,3-;2+)$, $(3+;+,3-;+,-)$, $(2+,-;2+,2-;+,-)$, $(+,2-;3+,-;2-)$, $(3-;3+,-;+,-)$, $(+,2-;2+,2-;+,-)$;
(3;3;3) *with* $(3+;3-;2+,-)$, $(2+,-;+,2-;2+,-)$, $(2+,-;2+,-;+,2-)$, $(3+;+,2-;+,2-)$, $(3-;3+;+,2-)$, $(+,2-;2+,-;+,2-)$, $(+,2-;+,2-;2+,-)$, $(3-;2+,-;2+,-)$.

and there exist examples of cubic polynomial differential systems with such configurations.

The existence of some configurations $(K_0;K_1)$ and $(K_0;k_1;K_2)$ without the possible distributions of the topological indices already are mentioned in [4].

The proof of Theorem 3 is given in section 4. As mentioned above with Theorems 2 and 3 we provide the classification of the configurations of the singular points and their topological indices for all planar cubic polynomial differential systems with nine real singular points.

2. Preliminaries

We observe that if a configuration exists for a cubic polynomial vector field X with $\#A = 9$ then it is possible to construct the same configuration but interchanging the points with index $+1$ with the points with index -1.

To do so, it is sufficient to take $Y = (-P, Q)$ instead of $X = (P, Q)$. So we can restrict to the case in which $\sum_{a\in A} i_X(a) \geq 0$. Therefore changing X by Y if necessary we only need to study the cases $\sum_{a\in A} i_X(a) = 1$ and $\sum_{a\in A} i_X(a) = 3$.

In the proof of Theorems 2 and 3 we will denote by $\{p_1, \ldots, p_9\}$ the set of points of A if there is no information about their indices, by $\{p^+_{j_1}, \ldots, p^+_{i_l}\}$ the set of points of A with positive index, and by $\{p^-_{i_1}, \ldots, p^-_{i_k}\}$ the set of points of A with negative index. Also we will denote by L_{ij} the straight line $L_{ij}(x, y) = 0$ through the points p_i and p_j where $i \in \{1, \ldots, k\}, j \in \{1, \ldots, l\}$, and by L_i a straight line through a point $p_i \in \partial\hat{A}$ such that for all $q \in A$ we have $L_i(A) \neq 0$ and it is zero only at q.

We will also use the following auxiliary result proved in [3].

Lemma 4. *Let $X = (P, Q)$ be a polynomial vector field such that the maximum of the degrees of P and Q is n, i.e.* $\max(deg\,P, deg\,Q) = n$. *If X has n singular points on a straight line $L(x, y) = 0$, this line is an isocline. If X has $n + 1$ singular points on $L(x, y) = 0$ then this line is full of singular points.*

We observe that configurations of the form $(2+; *)$ cannot occur because the eight singular points would be on a straight line, and by Lemma 4 this straight line will be full of singular points, a contradiction. Moreover, configurations of the form $(1+; *)$ have no meaning.

3. Proof of Theorem 2

In this case we have 6 points with positive index and 3 points with negative one.

We first show that there are no singular points with index -1 in $\partial\hat{A}$. Assume that $p^-_1 \in \partial\hat{A}$. Take $C(x, y) = L_1(x, y)(L_{23}(x, y))^2$. Since $C(P^+_i) \geq 0$ for $i = 1, \ldots, 6$, applying the Euler-Jacobi formula to C we reach to a contradiction. Hence $\#(A \cap \partial\hat{A}) \leq 6$ and the configuration of A must be $(K+, *)$ with $K \leq 6$. However, if $K = 6$, then applying the Euler-Jacobi formula to $C(x, y) = L_{12}(x, y)L_{34}(x, y)L_{56}(x, y)$ we have a contradiction. So, $K \leq 5$. As explained section 2 we also have $K \geq 3$. We consider each value of K separately.

Case $K = 5$. Write $\{p^+_1, p^+_2, p^+_3, p^+_4, p^+_5\} = A \cap \partial\hat{A}$ and take a conic $Q(x, y)$ through them. Since all these points are in the boundary of a convex set, we have that the other four singular points are in the same connected component $\mathbb{R}^2 \setminus \{Q(x, y) = 0\}$. Assume now that there is a point p^+_6 in the 1st level of A. Then taking $L_{6,r}$ where p^-_r is a point in $A \cap \partial\hat{A}_1$ contiguous with p^+_6 and applying the Euler Jacobi formula to $Q(x, y)L_{6,r}$ we reach to a contradiction. So the configuration must be $(5+; 3-; +)$.

Consider the cubic system (1) with

$$
\begin{aligned}
P(x,y) &= 6y - 2x^2y - 5y^2 + y^3,\\
Q(x,y) &= -16x + 4x^3 + 15xy - 3xy^2.
\end{aligned} \tag{3}
$$

The cubic system (1) with P and Q given in (3) has the singular points

$$(-2,0),\ (0,0),\ (2,0),\ (1,4),\ (-1,4),\ (-1,1),\ (1,1),\ (0,3),\ (0,2),$$

in the configuration $(5+;3-;+)$.

Case $K = 4$. We distinguish between the configurations $(4+;5*)$, $(4+;4*;1*)$ and $(4+;3*;2*)$.

Configuration $(4+;5*)$. Take the points $\{p_1^+, p_2^+, p_3^+, p_4^+\} = A \cap \partial\hat{A}$. Denote the remaining points in the 1st level by p_5, p_6, p_7, p_8, p_9 (they are named so that they are contiguous). We claim that we can select the point p_5 in such a way that the unique configuration is $(4+;2+,-;2-)$. Without loss of generality we can assume that $p_5 = p_5^+$. Applying the Euler-Jacobi formula first to the cubic $C = L_{12}L_{34}L_{67}$ we get that p_5, p_8 and p_9 have different signs. So we have three possibilities: either $p_8 = p_8^-$, $p_9 = p_9^-$, or $p_8 = p_8^-$, $p_9 = p_9^+$ (and then $p_6 = p_6^-$ and $p_7 = p_7-$), or $p_8 = p_8^+$ and $p_9 = p_9^-$ (and then $p_7 = p_7^-$ and $p_6 = p_6^-$). Applying the Euler Jacobi formula to $C = L_{12}L_{34}L_{59}$, we get that the second case is not possible and applying the Euler Jacobi formula to $C = L_{12}L_{34}L_{56}$ we get that in the first case we must have $p_7 = p_7^+$ and $p_6 = p_6^-$. In short only the first and third cases are possible leading to the configuration (rearranging the indexes if necessary) $(4+;2+,-;2-)$.

Consider the cubic system (1) with

$$
\begin{aligned}
P(x,y) = &-\frac{2156x^3}{4625} + \frac{403x^2y}{148} + \frac{58551x^2}{18500} + \frac{2353xy}{1850} + \frac{11989x}{9250}\\
&+ y^3 - 13y - 12,\\
Q(x,y) = &-\frac{2329x^3}{24050} - \frac{69x^2y}{4810} + \frac{833x^2}{12025} - xy^2 - \frac{582xy}{2405} + \frac{9274x}{12025}.
\end{aligned} \tag{4}
$$

The cubic system (1) with P and Q given in (4) has the singular points

$$
\begin{aligned}
&(0,-3),\ (3,-\frac{1}{2}),\ (0,4),\ (-2,\frac{2}{5}),\ (-1,-\frac{9}{10}),\ (0,-1),\\
&(-\frac{14110835063}{95002171235}, -\frac{949301469327}{950021712350}),\ (1,-1),\ (2,\frac{3}{5}),
\end{aligned}
$$

in the configuration $(4+;2+,-;2-)$.

Configuration $(4+;4*;1*)$. Take the points $\{p_1^+, p_2^+, p_3^+, p_4^+\} = A \cap \partial\hat{A}$ oriented in counterclockwise sense. Denote the remaining points in the 1st level by p_5, p_6, p_7, p_8 (oriented in counterclockwise sense) and the point in the 2nd level by p_9. We consider two cases: either $p_9 = p_9^+$ or $p_9 = p_9^-$. In the first case, we have the configuration $(4+;+,3-;+)$. In the second case, we cannot have two consecutive points in the 1st level being positive (say p_6, p_7) because then the other two would be negative and thus contradicting the Euler-Jacobi formula for the curve $C = L_{12}L_{34}L_{67}$). So, the configuration must be $(4+;+,-,+,-;-)$. Now we consider the two straight lines L_{68} and L_9 where L_9 is parallel to L_{68}. Note that the three singular points with negative index are on $C = L_{68}L_9$, so all the all remaining six singular points have positive index. If these six singular points are not contained between the two parallel lines applying the Euler–Jacobi formula to C we have a contradiction. If one of these six singular points q lies between the two parallel lines (it must be one of the points in the zero level) then we consider the curve $C = L_{68}L_qL_q$ and we get again a contradiction applying the Euler–Jacobi formula to C. Note that no more than one point of these six singular points can be between the two parallel lines. In short configuration $(4+;+,-,+,-;-)$ is not possible.

Consider the cubic system (1) with

$$
\begin{aligned}
P(x,y) &= -\frac{11x^3}{20} - \frac{71x^2y}{60} + \frac{x^2}{20} + \frac{11x}{5} + y^3 - \frac{3y^2}{5} + \frac{26y}{15} + \frac{8}{5}, \\
Q(x,y) &= \frac{19x^3}{30} + \frac{11x^2y}{10} + \frac{x^2}{30} - xy^2 + \frac{7x}{15} - \frac{2y^2}{5} - \frac{22y}{5} + \frac{16}{15},
\end{aligned} \tag{5}
$$

The cubic system (1) with P and Q given in (5) has the singular points

$$
\begin{gathered}
(-4,-5),\ (4,-2),\ (4,5),\ (-4,2),\ (-2,-\sqrt{3}), \\
(2,-\sqrt{3}),\ (2,\sqrt{3}),\ (-2,\sqrt{3}),\ (-1,0),
\end{gathered}
$$

in the configuration $(4+;+,3-;+)$.

Configuration $(4+;3*;2*)$. Take the points $\{p_1^+, p_2^+, p_3^+, p_4^+\} = A \cap \partial\hat{A}$. Denote the remaining points in the 1st level by p_5, p_6, p_7 and the point in the 2nd level by p_8, p_9. We consider three cases: either $p_8 = p_8^+$ and $p_9 = p_9^+$, or $p_8 = p_8^-$ and $p_9 = p_9^-$, or $p_8 = p_8^+$ and $p_9 = p_9^-$. In the first case, the configuration must be $(4+;3-;2+)$. The second case is not possible. Indeed, in the second case we must have that there are two positive indices in the 1st level and one negative index. Without loss of generality we can assume that $p_5 = p_5^-$ and $p_6 = p_6^+$, $p_7 = p_7^+$. Applying the Euler-Jacobi formula to $C(x,y) = L_{12}L_{34}L_{67}$ we get to a contradiction. Finally, in the third case, the unique possible configuration is $(4+;+,2-;+,-)$.

Consider the cubic system (1) with

$$
\begin{aligned}
P(x,y) &= -\frac{1}{7}y\left(-6x^2 + 5x + 7y^2 + 13\right), \\
Q(x,y) &= \frac{1}{175}(-126x^3 - 5\sqrt{2}x^2y - 63x^2 + 175xy^2 + 10\sqrt{2}xy \\
&\quad + 189x + 224y^2 + 40\sqrt{2}y),
\end{aligned} \tag{6}
$$

The cubic system (1) with P and Q given in (6) has the singular points

$$(-3,-\sqrt{8}),\ (4,-3),\ (4,3),\ (-2,\sqrt{3}),\ (-2,-\sqrt{3}),\ (1,0),$$
$$(-1.74..,1.41..),\ (-3/2,0),\ (0,0),$$

in the configuration $(4+;3-;2+)$.

On the other hand, consider the cubic system (1) with

$$\begin{aligned} P(x,y) &= 364121-11155x-377710x^2-2433x^3+81471y \\ &\qquad -12060xy-101961x^2y+367013y^2+100000y^3, \\ Q(x,y) &= -334477-44491x+385485x^2+95499x^3-1298y \\ &\qquad +21654xy+7362x^2y-386234y^2-100000xy^2, \end{aligned} \tag{7}$$

The cubic system (1) with P and Q given in (7) has the singular points

$$(-3.99..,-3.99..),\ (3.99..,-3,74..),\ (4.00..,4.00..),\ (-2.99..,2.82..),$$
$$(-3.84..,-3.50..),\ (2.00,-1.73..),\ (-0.99,-0.00..),\ (-2.99..,-2.82..),$$
$$(-2.00..,-1.73..)$$

in the configuration $(4+;+,2-;+,-)$.

Case $K=3$. We distinguish between the configurations $(3+;6*)$, $(3+;5*;1*)$ and $(3+;4*;2*)$ and $(3+;3*;3*)$.

Configuration $(3+;6*)$. Take the points $\{p_1^+,p_2^+,p_3^+\} = A\cap\partial\hat{A}$. Denote the remaining points in the 1st level by p_4,p_5,p_6,p_7,p_8,p_9. We have the following possible configurations: $(3+;+,-,+,-,+,-)$, $(3+;3+,3-)$, $(3+;2+,-,+,2-)$ or $(3+;2+,2-,+,-)$. We claim that the unique possible configuration is the first one. In the remaining three configurations there are two consecutive points with positive index followed with two consecutive points with negative index and the remaining ones are a point with positive index and another point with negative index, that we denote by a and b respectively. We consider the straight line L_a that separates the two pairs of points having the same index. Since the triangle with vertices p_1,p_2,p_3 contains in its interior the hexagon of vertices p_4,p_5,p_6,p_7,p_8,p_9, there exists a straight line L_b and a point p_i with $i\in\{1,2,3\}$ so that L_b leaves all the vertices of the hexagon and p_i on the same side of the straight line L_b having p_i on the same side with respect to the straight line L_a than the pair of consecutive points with positive index. Now applying the Euler Jacobi formula to $L_aL_bL_{p_jp_k}$ with $j,k\in\{1,2,3\}$ and j and k different from i, we get to a contradiction. This completes the proof of the claim and the unique possible configuration is $(3+;+,-,+,-,+,-)$.

Consider the cubic system (1) with

$$
\begin{aligned}
P(x,y) &= 21279 - 12413x - 12413x^2 + 3546x^3 - 26868y \\
&\quad + 40210xy + 105277x^2y + 18619y^2 - 100000y^3, \\
Q(x,y) &= -148447 + 86594x + 86594x^2 - 24741x^3 + 211840y \\
&\quad + 61354xy - 3086x^2y + 170108y^2 + 100000xy^2.
\end{aligned} \tag{8}
$$

The cubic system (1) with P and Q given in (8) has the singular points

$$
\begin{aligned}
&(-2.99.., -2.82..),\ (4.00.., 0.00..),\ (-3.00.., 2.82..),\ (-2.00.., -1.73..),\\
&(0.99.., -1.00..),\ (0.99.., 0.00..),\ (0.00.., 0.49..),\ (-0.73.., 0.67..),\\
&(-1.5.., 0.00..),
\end{aligned}
$$

in the configuration $(3+; +, -, +, -, +, -)$.

Configuration $(3+; 5*; 1*)$. Take the points $\{p_1^+, p_2^+, p_3^+\} = A \cap \partial \hat{A}$ oriented in counterclockwise sense. Denote the remaining points in the 1st level by p_4, p_5, p_6, p_7, p_8 also oriented in counterclockwise sense and the point in the 2nd level by p_9. We consider the two possible cases: $p_9 = p_9^-$ and $p_9 = p_9^+$.

When $p_9 = p_9^-$ there are only two possible configurations $(3+; 3+, 2-; -)$ and $(3+; 2+, -, +, -; -)$. We claim that none of these two configurations are possible. For the first one, note that without loss of generality we can assume that $p_4 = p_4^-$, $p_5 = p_5^-$, $p_6 = p_6^+$, $p_7 = p_7^+$ and $p_8 = p_8^+$. We consider the straight line L_9 so that it separates the two pairs p_4, p_5 (with negative index) from the two pairs p_6, p_7 (with positive index). Note that there exists p_i with $i \in \{1,2,3\}$ so that L_8 leaves all the vertices of the pentagon and p_i on the same side of the straight line L_8 having p_i on the same side with respect to the straight line L_9 than the pair of points p_6 and p_7. Now applying the Euler Jacobi formula to $L_8 L_9 L_{p_j p_k}$ with $j, k \in \{1,2,3\}$ and j and k different from i, we get to a contradiction. So, configuration $(3+; 3+, 2-; -)$ is not possible.

Now we show that the configuration $(3+; 2+, -, +, -; -)$ is also not possible. Without loss of generality we can assume that $p_4 = p_4^+$, $p_5 = p_5^+$, $p_6 = p_6^-$, $p_7 = p_7^+$ and $p_8 = p_8^-$. Consider the straight line L_9 so that it separates p_4, p_5 from p_6. Note that there exists p_i with $i \in \{1,2,3\}$ so that L_9 leaves p_i and p_4, p_5 on the same side of the straight line. Now applying the Euler Jacobi formula to $L_9 L_{78} L_{p_j p_k}$ with $j, k \in \{1,2,3\}$ and j and k different from i, we get to a contradiction. So, no configuration with $p_9 = p_9^-$ is possible.

When $p_9 = p_9^+$ there are only the possible configurations $(3+; 2+, 3-; +)$ and $(3+; +, 2-, +, -; +)$. We will show that the unique possible configuration is the second one and that the configuration $(3+; 2+; 3-; +)$ is not possible. To do so, note that without loss of generality we can assume that $p_4 = p_4^-$, $p_5 = p_5^-$, $p_6 = p_6^-$, $p_7 = p_7^+$ and $p_8 = p_8^+$. Now, we consider the straight line L_9 so that it separates the two pairs p_5, p_6 (with negative index) from the two pairs p_7, p_8 (with positive index). Note that there exists p_i with $i \in \{1,2,3\}$ so that L_4 leaves all the vertices of the pentagon and p_i on the same side of the straight line L_4 having p_i on the same side with respect to the straight line L_9 than the pair of points p_7 and p_8. Now applying the Euler Jacobi formula to $L_4L_9L_{p_jp_k}$ with $j,k \in \{1,2,3\}$ and j and k different from i, we get to a contradiction. So, configuration $(3+;2+;3-;+)$ is not possible.

In short the unique possible configuration is $(3+;+,2-,+,-;+)$.

Consider the cubic system (1) with

(9)
$$\begin{aligned} P(x,y) &= 157383 - 11533x - 196418x^2 - 27501x^3 - 198571y - 70171xy \\ &\quad - 133817x^2y - 41046y^2 + 100000y^3, \\ Q(x,y) &= -7870 - 101786x - 39801x^2 + 54114x^3 - 420688y - 106999xy \\ &\quad - 12022x^2y - 214956y^2 - 100000xy^2. \end{aligned}$$

The cubic system (1) with P and Q given in (9) has the singular points it has the singular points

$$\begin{aligned} &(-3.99.., 4.99..),\ (-3.99.., -3.99..),\ (3.99.., -2.99..),\ (-2.00.., -1.73..), \\ &(2.00.., -1.73..),\ (0.99.., -0.19..),\ (-0.99.., 0.00..),\ (-1.50.., -0.50..), \\ &(0.57.., -1.62..), \end{aligned}$$

in the configuration $(3+;+,2-,+,-;+)$.

Configuration $(3+;4*;2*)$. Take the points $\{p_1^+, p_2^+, p_3^+\} = A \cap \partial \hat{A}$. Denote the remaining points in the 1st level by p_4, p_5, p_6, p_7 oriented in counter-clockwise sense and the points in the 2nd level by p_8, p_9. Clearly, in the 2nd level we can have either $p_8 = p_8^+$ and $p_9 = p_9^+$; or $p_8 = p_8^-$ and $p_9 = p_9^-$; or $p_8 = p_8^+$ and $p_9 = p_9^-$.

When $p_8 = p_8^-$, $p_9 = p_9^-$ the unique possible configuration is $(3+;3+,-;2-)$. We will show that such a configuration is not possible. Indeed, without loss of generality we can assume that $p_4 = p_4^+$, $p_5 = p_5^+$, $p_6 = p_6^+$ and $p_7 = p_7^-$. Consider the straight lines $L_{58}L_{79}$. Note that the points p_4^+ and p_6^+ are not contained between these two straight lines. Moreover, there exists p_i with $i \in \{1,2,3\}$ so that p_i is in the same side of the two straight lines of L_{58} and L_{79}. So, applying the Euler Jacobi formula to $C = L_{58}L_{79}L_{p_jp_k}$ with $j,k \in \{1,2,3\}$ and j and k different from i, we get to a contradiction. So, this case is not possible.

When $p_8 = p_8^+$, $p_9 = p_9^+$ the unique possible configuration is $(3+;+,3-;2+)$. We will show that such a configuration is not possible. Indeed, without loss of generality we can assume that $p_4 = p_4^-$, $p_5 = p_5^-$, $p_6 = p_6^-$ and $p_7 = p_7^+$. Note that there exists $a, b \in \{4,6\}$ with $a \neq b$ and p_i with $i \in \{1,2,3\}$ so that we can choose L_b in such a way that p_i and p_7, p_8, p_9 lie on the same sides of L_b and L_{a5}. Now applying the Euler-Jacobi formula to $L_b L_{5a} L_{p_j p_k}$ with $j, k \in \{1, 2, 3\}$ and j and k different from i, we get to a contradiction. So, this case is also not possible.

Finally in the case $p_8 = p_8^+$ and $p_9 = p_9^-$ there exist two possible configurations $(3+; +, -, +, -; +, -)$ and $(3+; 2+, 2-; +, -)$. We claim that the configuration $(3+; +, -, +, -; +, -)$ is not possible. Indeed, without loss of generality we can assume that $p_4 = p_4^+$, $p_5 = p_5^-$, $p_6 = p_6^+$ and $p_7 = p_7^-$. We take the straight lines $L_{59} L_{78}$. Note that the points p_4^+ and p_6^+ are not contained between these two straight lines. Moreover, there exists p_i with $i \in \{1, 2, 3\}$ so that p_i is in the same side of the two straight lines of L_{59} and L_{78}. So, applying the Euler Jacobi formula to $C = L_{59} L_{78} L_{p_j p_k}$ with $j, k \in \{1, 2, 3\}$ and j and k different from i, we get to a contradiction. This proves the claim.

In short the unique possible configuration is $(3+; 2+, 2-; +, -)$.

Consider the cubic system (1) with

$$
\begin{aligned}
P(x,y) &= 3.31278 + 2.27547x + 1.88693x^2 + 0.674273x^3 + 5.96836y \\
&\quad + 3.04208xy + 1.5789x^2y + 1.93759y^2 - y^3, \\
Q(x,y) &= 8.42839 + 5.10452x + 0.660667x^2 - 0.0927681x^3 + 11.9413y \\
&\quad + 4.28205xy + 0.338297x^2y + 3.77409y^2 + xy^2.
\end{aligned} \tag{10}
$$

The cubic system (1) with P and Q given in (10) has the singular points

$$
\begin{aligned}
&(45.73.., -24.47..),\ (-3.99.., 5.00..),\ (3.99.., -2.99..),\ (1.85.., -2.00..), \\
&(1.99.., -1.73..),\ (1.00.., -1.50..),\ (-3.99.., -4.00..),\ (-1.50.., -0.50..), \\
&(-2.52.., -2.17..)
\end{aligned}
$$

in the configuration $(3+; 2+, 2-; +, -)$.

Configuration $(3+; 3*; 3*)$. Take the points $\{p_1^+, p_2^+, p_3^+\} = A \cap \partial \hat{A}$. Denote the remaining points in the 1st level by p_4, p_5, p_6 oriented in counterclockwise sense and the point in the 2nd level by p_7, p_8, p_9 also oriented in counterclockwise sense. Clearly, in the 2nd level we can have either $p_7 = p_7^+$, $p_8 = p_8^+$, $p_9 = p_9^+$; or $p_7 = p_7^-$, $p_8 = p_8^-$, $p_9 = p_9^-$; or $p_7 = p_7^+$, $p_8 = p_8^+$, $p_9 = p_9^-$, or $p_7 = p_7^+$, $p_8 = p_8^-$, $p_9 = p_9^-$.

We will show that the cases $p_7 = p_7^-$, $p_8 = p_8^-$, $p_9 = p_9^-$, or $p_7 = p_7^+$, $p_8 = p_8^+$, $p_9 = p_9^+$ are not possible. Assume that the closest point of the set $\{p_4, p_5, p_6\}$ to p_1 is p_4. Then either the straight line L_{15} or L_{16} leaves the other two points of the set $\{p_4, p_5, p_6\}$ on the same side. Assume that it is the straight line L_{15}, otherwise the proof follows in a similar way. Then applying the Euler-Jacobi formula to $C = L_{15} L_{26} L_{34}$ we get to a contradiction because L_{26} and L_{34} also leave the points of the set $\{p_3, p_4, p_5\}$ on the same side.

Now we consider the case $p_7 = p_7^+$, $p_8 = p_8^+$, $p_9 = p_9^-$. In this case the unique possible configuration is $(3+; +, 2-; 2+, -)$. There exists $b \in \{4, 5, 6\}$ so that L_{b9} leaves p_7 and p_8 in the same side of L_{b9} (or they are in the same line). There exists $p_i \in \{1, 2, 3\}$ so that p_i is in the same side of the lines L_{b9} and L_{ac} where $a, c \in \{4, 5, 6\}$ with a and c different from b. Now applying the Euler-Jacobi formula with $C = L_b L_{ac} L_{p_j, p_k}$ with $j, k \in \{1, 2, 3\}$ and j and k different from i, we get to a contradiction. Hence, this case is not possible.

In short, the unique possible configuration is $(3+; 2+, -; +, 2-)$.

Consider the cubic system (1) with

(11)
$$\begin{aligned} P(x, y) &= -1.07194 - 0.921633x + 1.25039y + 1.62034x^2 + 2.38257xy \\ &\quad + 1.98048y^2 + 0.921139x^3 + 1.92938x^2y - y^3, \\ Q(x, y) &= 2.85826 + 3.76702x + 5.96687y + 1.87487x^2 + 4.94349xy \\ &\quad + 3.13978y^2 + 0.355045x^3 + 1.20183x^2y + xy^2. \end{aligned}$$

The cubic system (1) with P and Q given in (11) has the singular points

$$\begin{gathered} (-4.00.., 5.20..),\ (-2.52.., -2.17..),\ (4.92.., 3.88..),\ (-1.96.., 0.00..), \\ (-1.67.., -0.54..),\ (4.00.., -2.99..),\ (0.99.., -4.49..),\ (1.85.., -2.00..), \\ (1.99.., -2.08..) \end{gathered}$$

in the configuration $(3+; 2+, -; +, 2-)$. This concludes the proof of the theorem.

4. Proof of Theorem 3

In this case we have 5 points with positive index and 4 points with negative one. We will study each possible configuration separately.

Configuration (9): Assume that the subscripts of the points in A are in such a way that $p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8$ and p_9 are ordered in $\partial \hat{A}$ in counter-clockwise sense. We claim that we can select the point p_1 in such a way that the unique configuration is $(2+, -, +, -, +, -, +, -)$. Without loss of generality we can assume that $p_1 = p_1^+$. We apply the Euler-Jacobi formula to the cubic $L_{34}L_{56}L_{78}$, and we get three possibilities: either $p_2 = p_2^+$ and $p_9 = p_9^-$, or $p_2 = p_2^-$ and $p_9 = p_9^+$, or $p_2 = p_2^-$ and $p_9 = p_9^-$. Here we only study the first case because the other two can be analyzed in the same way. Applying the Euler-Jacobi formula first to the cubic $L_{34}L_{56}L_{89}$ we get that $p_7 = p_7^-$, second to the cubic $L_{34}L_{67}L_{89}$ we obtain $p_5 = p_5^-$, and third to the cubic $L_{45}L_{67}L_{89}$ we get that $p_3 = p_3^-$. Therefore $p_4 = p_4^+$, $p_6 = p_6^+$ and $p_8 = p_8^+$. So the claim is proved.

Consider the cubic system (1) with

$$
\begin{aligned}
&P(x,y) = -2 + y + 2x^2 + y^2 - x^2y - y^3,\\
(12)\quad &Q(x,y) = 2.3288 - 0.536384x + 0.84651y - 2.22807x^2 + 0.420694xy\\
&\qquad - 1.50219y^2 + 0.662308x^3 - 0.376157x^2y + xy^2.
\end{aligned}
$$

The cubic system (1) with P and Q given in (12) has the singular points

$$
\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \quad (2,2), \quad (1.5,2), \quad (1,2), \quad (-0.98499209.., 0.17259946..),
$$

$$
\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \quad \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \quad \left(-\sqrt{\frac{5}{8} - \frac{\sqrt{5}}{8}}, \frac{1}{4}\left(-1-\sqrt{5}\right)\right),
$$

$$
\left(\sqrt{\frac{5}{8} - \frac{\sqrt{5}}{8}}, \frac{1}{4}\left(-1-\sqrt{5}\right)\right).
$$

in the configuration $(2+,-,+,-,+,-,+,-)$.

Configuration $(8;1)$ Assume that the subscripts of the points in A are in such a way that $p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8$ are ordered in $\partial\hat{A}$ in counterclockwise sense and denote by p_9 the point in the 1st level. We claim that we can select the point p_1 in such a way that the unique configurations are $(+,-,+,-,+,-,+,-;+)$ and $(3+,-,+,-,+,-;-)$. We have two possibilities: either $p_9 = p_9^+$ or $p_9 = p_9^-$. Without loss of generality we can assume that $p_1 = p_1^+$. We apply the Euler-Jacobi formula to the cubic $L_{34}L_{56}L_{78}$, and we get three possibilities, either $p_2 = p_2^-$ and $p_9 = p_9^+$, or $p_2 = p_2^-$ and $p_9 = p_9^-$, or $p_2 = p_2^+$ and $p_9 = p_9^-$.

In the first case, applying the Euler-Jacobi formula first to the cubic $L_{23}L_{45}L_{67}$, second to the cubic $L_{23}L_{45}L_{78}$, and third to the cubic $L_{23}L_{56}L_{78}$ we get that $p_8 = p_8^-$, $p_6 = p_6^-$ and $p_4 = p_4^-$. Therefore $p_3 = p_3^+$, $p_5 = p_5^+$ and $p_7 = p_7^+$ and so the configuration is $(+,-,+,-,+,-,+,-;+)$.

Now we will show that the other two configurations yield the configuration $(3+,-,+,-,+,-;-)$. We will work only with the second one since the third one can be analyzed in the same way and we reach to the same conclusion. Hence, assume that $p_2 = p_2^-$ and $p_9 = p_9^-$. Applying the Euler-Jacobi formula first to the cubic $L_{18}L_{34}L_{56}$, second to the cubic $L_{18}L_{34}L_{67}$, and third to the cubic $L_{18}L_{45}L_{67}$ we obtain $p_7 = p_7^+$, $p_5 = p_5^+$ and $p_3 = p_3^+$. So, either p_4, or p_6, or p_8 are positive, but then we have the configuration $(3+,-,+,-,+,-;-)$. In short, the unique two possible configurations are $(+,-,+,-,+,-,+,-;+)$ and $(3+,-,+,-,+,-;-)$.

Consider the cubic system (1) with

$$
\begin{aligned}
P(x,y) &= -2 + y + 2x^2 + 2y^2 - x^2y - y^3,\\
Q(x,y) &= 2.12317 - 0.15626x + 0.572613y - 2.07381x^2 + 0.762756xy\\
&\quad - 1.58413y^2 + 0.383518x^3 - 0.305409x^2y + xy^2
\end{aligned}
\tag{13}
$$

The cubic system (1) with P and Q given in (13) has the singular points

$$
\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \quad (4,2), \quad (2,2), \quad (1,2), \quad 0.42432215.., -0.90551129..),
$$

$$
\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \quad \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \quad \left(-\sqrt{\frac{5}{8} - \frac{\sqrt{5}}{8}}, \frac{1}{4}\left(-1-\sqrt{5}\right)\right),
$$

$$
\left(\sqrt{\frac{5}{8} - \frac{\sqrt{5}}{8}}, \frac{1}{4}\left(-1-\sqrt{5}\right)\right).
$$

in the configuration $(+,-,+,-,+,-,+,-;+)$. On the other hand, consider the cubic system (1) with

(14)

$$
\begin{aligned}
P(x,y) &= -0.0434348 - 3.61689x + 3.60614y + 0.532168x^2 - 3.25473xy\\
&\quad + 2.77975y^2 + 2.6527x^3 - 1.67318x^2y - y^3,\\
Q(x,y) &= 1.98022 + 0.107989x + 0.382209y - 1.96657x^2 + 1.00055xy\\
&\quad - 1.6411y^2 + 0.189712x^3 - 0.256227x^2y + xy^2.
\end{aligned}
$$

The cubic system (1) with P and Q given in (14) has the singular points

$$
\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \quad (2,2), \quad (\tfrac{3}{2}, \tfrac{3}{2}), \quad (1,2), \quad 0.42432215.., -0.90551129..),
$$

$$
\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \quad \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \quad \left(-\sqrt{\frac{5}{8} - \frac{\sqrt{5}}{8}}, \frac{1}{4}\left(-1-\sqrt{5}\right)\right),
$$

$$
\left(\sqrt{\frac{5}{8} - \frac{\sqrt{5}}{8}}, \frac{1}{4}\left(-1-\sqrt{5}\right)\right).
$$

in the configuration $(3+,-,+,-,+,-;-)$.

Configuration $(7;2)$ Assume that the subscripts of the points in A are in such a way that $p_1, p_2, p_3, p_4, p_5, p_6, p_7$ are ordered in $\partial\hat{A}$ in counterclockwise sense and denote by p_8, p_9 the points in the 1st level. We claim that we can select the point p_1 in such a way that the unique configuration is $(2+,-,+,-,+,-,+;+,-)$.

We first note that if $p_8 = p_8^+$, $p_9 = p_9^+$ or $p_8 = p_8^-$ and $p_9 = p_9^-$ applying the Euler-Jacobi formula successively to $L_{12}L_{34}L_{56}$, $L_{17}L_{23}L_{45}$, $L_{67}L_{12}L_{34}$, $L_{56}L_{71}L_{23}$ and L_{456712} we reach to a contradiction. So we can assume that $p_8 = p_8^-$ and $p_9 = p_9^+$. Then the unique possible configurations are $(4+, 3-; +, -)$, $(3+, -, +, 2-; +, -)$, $(3+, 2-, +, -; +, -)$, $(2+, -, 2+, 2-; +, -)$, and $(2+, -, +, -, +, -, +; +, -)$.

We will show that the first four configurations are not possible which will prove the claim. Let p_{k_0} denote the point in the 0 level with negative index so that it is between two points with positive index for the configurations $(3+, -, +, 2-; +, -)$, $(3+, 2-, +, -; +, -)$, $(2+, -, 2+, 2-; +, -)$ and for the configuration $(4+, 3-; +, -)$ it is such that it has a point with positive index on one side and a point with negative index on the other side. Now let p_{k_1} be the closest point to L_{8k_0}. If p_{k_1} has negative index then there is only one more point in the 0-level with negative index that we denote by p_{k_2}. In this case applying the Euler-Jacobi formula to $L_{8k_0}L_{k_1}L_{k_2}$ with L_{k_1} being parallel to L_{8k_0} and L_{k_2} being such that all the points in the 0 and 1st level are on the same side of L_{k_2}, we reach to a contradiction. If p_{k_1} has positive index then there are two consecutive points in the 0-level with negative index that we denote by p_{k_2} and p_{k_3}. In this case applying the Euler-Jacobi formula to $L_{8k_0}L_{k_1}L_{k_2k_3}$ with L_{k_1} being parallel to L_{8k_0} we reach to a contradiction. In short, the claim is proved and the unique possible configuration is $(2+, -, +, -, +, -, +; +, -)$.

Consider the cubic system (1) with

$$\begin{aligned} P(x,y) &= 25.1098 + 25.5381x - 3.11093y - 2.73297x^2 + 0.110389xy \\ &\quad - 34.0843y^2 - 33.9773x^3 + 42.5851x^2y + y^3, \\ Q(x,y) &= 10.2122 + 11.2379x - 1.78176y - 1.61998x^2 + 2.01866xy \\ &\quad - 14.7742y^2 - 13.9715x^3 + 16.5187x^2y + xy^2. \end{aligned} \tag{15}$$

The cubic system (1) with P and Q given in (15) has the singular points

$$\begin{gathered} (1.99.., 1.99), \quad (1.49.., 1.99), \quad (1.00.., 1.50..), \\ (-0.86.., -0.49..), \quad (-0.70.., -0.49..), \quad (0.24.., -0.94..), \\ (0.42.., -0.90..), \quad (0.58.., -0.80..), \quad (0.86.., -0.50..) \end{gathered}$$

in the configuration $(2+, -, +, -, +, -, +; +, -)$.

Configuration $(6; 3)$ Assume that the subscripts of the points in A are in such a way that $p_1, p_2, p_3, p_4, p_5, p_6$ are ordered in $\partial\hat{A}$ in counterclockwise sense and denote by p_7, p_8, p_9 the points in the 1st level. First note that applying the Euler-Jacobi formula to $L_{12}L_{34}L_{56}$ we conclude that p_7, p_8, p_9 cannot have the same sign, so either $p_7 = p_7^+$, $p_8 = p_8^+$, $p_9 = p_9^-$, or $p_7 = p_7^+$, $p_8 = p_8^-$, $p_9 = p_9^-$. We will consider both cases separately.

Assume first that $p_7 = p_7^+$, $p_8 = p_8^-$ and $p_9 = p_9^-$. The unique possible configurations are $(4+, 2-; +, 2-)$, $(2+, -, 2+, -; +, 2-)$, $(3+, -, +, -; +, 2-)$. We will show that the configuration $(4+, 2-; +, 2-)$ is not possible. Consider the straight line L_{89}. We denote by k_0 the integer in $\{1, 2, 3, 4, 5, 6, 7\}$

such that p_{k_0} is the closest point to L_{89}. If p_{k_0} has negative index then there is only one more point in the 0-level with negative index that we denote by p_{k_1}. In this case applying the Euler-Jacobi formula to $L_{89}L_{k_0}L_{k_1}$ with L_{k_0} being parallel to L_{89} and L_{k_1} being such that all the points in the 0 and 1st level are on the same side of L_{k_1}, we reach to a contradiction. If p_{k_1} has positive index then there are two consecutive points in the 0-level with negative index that we denote by p_{k_1} and p_{k_2}. In this case applying the Euler-Jacobi formula to $L_{89}L_{k_0}L_{k_1k_2}$ with L_{k_0} being parallel to L_{89} we reach to a contradiction. In short, only the configurations $(2+,-,2+,-;+,2-)$ and $(3+,-,+,-;+,2-)$ are possible.

Finally, consider the case $p_7 = p_7^+$, $p_8 = p_8^+$, $p_9 = p_9^-$. The unique possible configurations are

$$(3+,3-;2+,-), \quad (2+,-,+,2-;2+,-), \quad (2+,2-,+,-;2+,-) \tag{16}$$

and $(+,-,+,-,+,-;2+,-)$. We will show that none of them are possible.

First we show that the configuration $(+,-,+,-,+,-;2+,-)$ is not possible. Consider the straight line L_{79} and without loss of generality we can assume that p_8 is on the right-hand side of the straight line L_{79} (if the three points are colinear then the argument is even simpler and also holds in the same lines). Note that there exists at least two points that we call p_1 and p_2 so that $p_1 = p_1^+$, $p_2 = p_2^-$ in such a way that p_1^+ is on the right-hand side of L_{79} and p_2^- is on the left-hand side of L_{79}. We denote the rest of the points on the 0-level by p_3, p_4, p_5 and p_6 being p_3, p_4 consecutive and p_5, p_6 also consecutive. Now applying the Euler-Jacobi formula to $L_{34}L_{56}L_{79}$ we reach to a contradiction. So, this configuration is not possible.

Now we consider the configurations in (16). Let p_{k_0} denote the point in the 0 level with negative index so that it is between two points with positive index for the configurations $(2+,-,+,2-;2+,-)$, $(2+,2-,+,-;2+,-)$, and for the configuration $(3+,3-;2+,-)$ it is such that it has a point with positive index on one side and a point with negative index on the other side. Now let p_{k_1} be the closest point to L_{9k_0}. If p_{k_1} has negative index then there is only one more point in the 0-level with negative index that we denote by p_{k_2}. In this case applying the Euler-Jacobi formula to $L_{9k_0}L_{k_1}L_{k_2}$ with L_{k_1} being parallel to L_{9k_0} and L_{k_2} being such that all the points in the 0 and 1st level are on the same side of L_{k_2}, we reach to a contradiction. If p_{k_1} has positive index then there are two consecutive points in the 0-level with negative index that we denote by p_{k_2} and p_{k_3}. In this case applying the Euler-Jacobi formula to $L_{9k_0}L_{k_1}L_{k_2k_3}$ with L_{k_1} being parallel to L_{9k_0} we reach to a contradiction. So, none of the configurations in (16) are possible.

In short the unique possible configurations are $(2+,-,2+,-;+,2-)$ and $(3+,-,+,-;+,2-)$.

Consider the cubic system (1) with

$$
\begin{aligned}
P(x,y) &= -\frac{1}{25} + x(y-x-1)(x+y+1),\\
Q(x,y) &= (x^2+y^2-3)(y-4x-\frac{1}{2}).
\end{aligned}
\tag{17}
$$

The cubic system (1) with P and Q given in (17) has the singular points

$$
\begin{aligned}
&(0.60..,1.62),\quad (0.02..,1.73),\quad (-1.62..,0.60..),\\
&(-1.62..,-0.60..),\quad (0.02..,-1.73..),\quad (0.60..,-1.62..),\\
&(0.19..,1.27..),\quad (-0.04..,0.30..),\quad (-0.27..,-0.61..)
\end{aligned}
$$

in the configuration $(2+,-,2+,-;+,2-)$.

Consider the cubic system (1) with

$$
\begin{aligned}
P(x,y) &= \frac{1}{50} - y^2 + x^2(x+1),\\
Q(x,y) &= (y+\frac{x}{2}+\frac{1}{4})(y-\frac{x}{2}-\frac{1}{4})(y-\frac{5}{8}x-\frac{139}{160}).
\end{aligned}
\tag{18}
$$

The cubic system (1) with P and Q given in (18) has the singular points

$$
\begin{aligned}
&(1.06..,1.53),\quad (-0.74..,0.40),\quad (-0.92..,0.29..),\\
&(-0.96..,0.23..),\quad (-0.96..,-0.23..),\quad (0.34..,-0.42..),\\
&(0.34..,0.42..),\quad (-0.12..,0.18..),\quad (-0.12..,-0.18..)
\end{aligned}
$$

in the configuration $(3+,-,+,-;+,2-)$.

Configuration $(5;4)$. We denote by $\{p_1,\ldots,p_5\} = A\cap\partial\hat{A}$ and take a conic $Q(x,y)$ through them. Since all these points are in the boundary of a convex set, we have that the other four singular points are in the same connected component $\mathbb{R}^2\setminus\{Q(x,y)=0\}$. Now we denote by p_6,p_7,p_8,p_9 the points in the 1st level ordered in counterclockwise sense. Now applying the Euler-Jacobi formula successively to QL_{67}, QL_{78}, QL_{89} and QL_{69} we get that p_8 and p_9 have different index, p_6 and p_9 have different index, p_6 and p_7 have different index and p_7 and p_8 have different index. In short we can assume without loss of generality that $p_6=p_6^+$, $p_7=p_7^-$, $p_8=p_8^+$ and $p_9=p_9^-$. Then the unique possible configurations are $(2+,-,+,-;+,-,+,-)$ or $(3+,2-;+,-,+,-)$.

We will show that the configuration $(3+,2-;+,-,+,-)$ is not possible. Indeed, without loss of generality we can assume that $p_1=p_1^+$, $p_2=p_2^+$, $p_3=p_3^+$, $p_4=p_4^-$ and $p_5=p_5^-$. We take the straight line L_{79} and the point p_k with $k\in\{1,2,3,4,5,6,8\}$ being the closest point to L_{79}. Applying the Euler-Jacobi formula to $L_{79}L_kL_{45}$ with L_k being parallel to L_{79}, we reach to a contradiction. In short the unique possible configuration is $(2+,-,+,-;+,-,+,-)$.

The cubic system (1) with

$$P(x,y) = 3x^3 + 5xy^2 - 4x^2 - 10y^2 - 45x + 90, \qquad (19)$$
$$Q(x,y) = y(111 - 40x - 10y)(-97 + 40x - 10y).$$

The cubic system (1) with P and Q given in (19) has the singular points

$$(1.82.., 3.80), \quad (-4.10.., 0), \quad (1.62.., -3.19..),$$
$$(2.89.., -0.47..), \quad (3, 0), \quad (2.60.., 0.66..),$$
$$(2.58.., 0.65..), \quad (2.43.., 0.04..), \quad (2.43.., 0)$$

in the configuration $(2+, -, +, -; +, -, +, -)$.

Configuration $(5; 3; 1)$. We denote by $\{p_1, \ldots, p_5\} = A \cap \partial \hat{A}$ ordered in counterclockwise sense and take a conic $Q(x,y)$ through them. Since all these points are in the boundary of a convex set, we have that the other four singular points are in the same connected component $\mathbb{R}^2 \setminus \{Q(x,y) = 0\}$. Now we denote by p_6, p_7, p_8 the points in the 1st level ordered in counterclockwise sense and by p_9 the point in the 2nd level. We consider the two cases $p_9 = p_9^+$ and $p_9 = p_9^-$.

Assume first that $p_9 = p_9^-$. Now applying the Euler-Jacobi formula successively to QL_{67}, QL_{78} and QL_{68} we get that $p_6 = p_6^+$, $p_7 = p_7^+$ and $p_8 = p_8^+$. So, the unique possible configurations are $(2+, 3-; 3+; -)$ and $(+, 2-, +, -; 3+; -)$. We will show that none of the configurations are possible.

Consider first the configuration $(2+, 3-; 3+; -)$. Without loss of generality we can assume that $p_1 = p_1^+$, $p_2 = p_2^+$, $p_3 = p_3^-$, $p_4 = p_4^-$ and $p_5 = p_5^-$. We take the straight line L_{59} and the point p_k with $k \in \{1, 2, 6, 7, 8\}$ being the closest point to L_{59}. Applying the Euler-Jacobi formula to $L_{59}L_kL_{34}$ with L_k being parallel to L_{59}, we reach to a contradiction.

Consider now the configuration $(+, 2-, +, -; 3+; -)$. Without loss of generality we can assume that $p_1 = p_1^+$, $p_2 = p_2^-$, $p_3 = p_3^-$, $p_4 = p_4^+$ and $p_5 = p_5^-$. We take the straight line L_{59} and the point p_k with $k \in \{1, 4, 6, 7, 8\}$ being the closest point to L_{59}. Applying the Euler-Jacobi formula to $L_{59}L_kL_{23}$ with L_k being parallel to L_{59}, we reach to a contradiction. In short, none of the configurations $(2+, 3-; 3+; -)$ and $(+, 2-, +, -; 3+; -)$ are possible.

Assume now that $p_9 = p_9^+$. Now applying the Euler-Jacobi formula successively to QL_{67}, QL_{78} and QL_{68} we get that $p_6 = p_6^-$, $p_7 = p_7^-$ and $p_8 = p_8^-$. So the unique possible configuration is $(4+, -; 3-; +)$.

The cubic system (1) with

$$P(x,y) = (8x^2 - y^2 - 1)(y - 2x + 1), \qquad (20)$$
$$Q(x,y) = (y^2 - x^2 - 1)(2 - y).$$

The cubic system (1) with P and Q given in (20) has the singular points

$$\begin{aligned}&(1, 5.20), \quad (-0.79.., 2), \quad (-0.53.., -1.13..),\\ &(0.53.., -1.13..), \quad (0.79.., 2), \quad (1.33.., 1.66..),\\ &(-0.53.., 1.13), \quad (0, -1), \quad (0.53.., 1.13..)\end{aligned}$$

in the configuration $(4+, -; 3-; +)$.

Configuration $(4; 5)$. Take the points $\{p_1, p_2, p_3, p_4\} = A \cap \partial\hat{A}$ in counterclockwise sense at the vertices of a quadrilateral. Denote the remaining points in the 1st level by p_5, p_6, p_7, p_8, p_9 in counterclockwise at the vertices of a pentagon inside the previous quadrilateral. We claim that the unique configurations are $(3+, -; +, 2-, +, -)$, $(+, -, +, -, 2+, -, +, -)$. Without loss of generality we can assume that $p_5 = p_5^+$. Applying the Euler-Jacobi formula first to the cubic $L_{12}L_{34}L_{67}$ we get that at least one of the indices of p_8 and p_9 has different sign from the index of p_5. So we have three possibilities: either $p_8 = p_8^-$, $p_9 = p_9^-$, or $p_8 = p_8^-$, $p_9 = p_9^+$, or $p_8 = p_8^+$ and $p_9 = p_9^-$. Applying the Euler-Jacobi formula to $L_{12}L_{34}L_{56}$ and $L_{12}L_{34}L_{89}$ we get that in the first case the unique possible configuration for the convex pentagon formed by the points $\{p_5, p_6, p_7, p_8, p_9\}$ is $(+, 2-, +, -)$. In the second case applying the Euler-Jacobi formula to $L_{12}L_{34}L_{78}$ and $L_{12}L_{34}L_{59}$ we get that the unique possible configurations for the convex pentagon is $(2+, -, +, -)$. In the third case applying the Euler-Jacobi formula to $L_{12}L_{34}L_{58}$ we obtain that at least one of the indices of p_6 and p_7 is negative and this provides that the unique possible configurations for the convex pentagon are $(+, 2-, +, -)$ and $(2+, -, +, -)$.

So, the possible configurations for $(4; 5)$ are

$$(3+, -; +, 2-, +, -), \quad (2+, 2-; 2+, -, +, -), \quad (+, -, +, -; 2+, -, +, -).$$

We will show that the configuration $(2+, 2-; 2+, -, +, -)$ is not possible. Consider the straight line L_{79}. Denote by p_{k_0} with $k_0 \in \{1, 2, 3, 4, 5, 6, 8\}$ the closest point to L_{78}. If p_{k_0} has negative index then there is only one more point in the 0-level with negative index that we denote by p_{k_1}. In this case applying the Euler-Jacobi formula to $L_{79}L_{k_0}L_{k_1}$ with L_{k_0} being parallel to L_{79} and L_{k_1} being such that all the points in the 0 and 1st level are on the same side of L_{k_1}, we reach to a contradiction. If p_{k_0} has positive index then there are two consecutive points in the 0-level with negative index that we denote by p_{k_1} and p_{k_2}. In this case applying the Euler-Jacobi formula to $L_{79}L_{k_0}L_{k_1k_2}$ with L_{k_0} being parallel to L_{79} we reach to a contradiction. So, the unique possible configurations are $(3+, -; +, 2-, +, -)$ and $(+, -, +, -; 2+, -, +, -)$.

The cubic system (1) with

$$
\begin{aligned}
P(x,y) &= 1.20193 + 3.10832x - 64.6443y - 0.618247x^2 + 41.203xy \\
&\quad - 1.43166y^2 - 0.317855x^3 - 6.10151x^2y + y^3 \\
Q(x,y) &= 19.1338 - 11.4492x - 18.5024y - 0.549995x^2 + 18.2762xy \\
&\quad - 1.99188y^2 + 0.822778x^3 - 4.37617x^2y + xy^2.
\end{aligned} \tag{21}
$$

The cubic system (1) with P and Q given in (21) has the singular points

$$
\begin{gathered}
(1.82..,3.80..), \quad (3,1), \quad (-4.10..,0), \quad (2.61..,0.66..), \\
(2.60..,0.66..), \quad (2.48..,0.55..), \quad (1.62..,-3.19..), \\
(2.49..,-0.27..), \quad (2.4,-0.8)
\end{gathered}
$$

in the configuration $(3+,-;+,2-,+,-)$.

The cubic system (1) with

$$
\begin{aligned}
P(x,y) &= 3x^3 + 5xy^2 - 4x^2 - 10y^2 - 45x + 90, \\
Q(x,y) &= y\Big(\frac{104}{5} - 8x + y\Big)\Big(y + 8x - \frac{112}{5}\Big).
\end{aligned} \tag{22}
$$

The cubic system (1) with P and Q given in (22) has the singular points

$$
\begin{gathered}
(1.96..,6.71), \quad (-4.10..,0), \quad (1.93..,-5.34..), \\
(3,0), \quad (2.86..,-0.52..), \quad (2.71..,0.67..), \\
(2.68..,0.68..), \quad (2.43..,0), \quad (2.52..,-0.57..)
\end{gathered}
$$

in the configuration $(+,-,+,-;2+,-,+,-)$.

Configuration $(4;4;1)$. Take the points $\{p_1,p_2,p_3,p_4\} = A \cap \partial\hat{A}$ oriented in counterclockwise sense. Denote the remaining points in the 1st level by p_5,p_6,p_7,p_8 (also oriented in counterclockwise sense) and the point in the 2nd level by p_9. We consider two cases: either $p_9 = p_9^+$ or $p_9 = p_9^-$.

Assume first that $p_9 = p_9^+$. Applying the Euler-Jacobi formula successively to $L_{12}L_{34}L_{56}$, $L_{12}L_{34}L_{67}$, $L_{12}L_{34}L_{78}$ and $L_{12}L_{34}L_{58}$ we get that there cannot be two consecutive points with positive index. So, the possible configurations are

$$
(4+;4-,+), \quad (3+,-;+,3-;+), \quad (+,-,+,-;+,-,+,-;+) \tag{23}
$$

and $(2+,2-;+,-,+,-;+)$. We will show that the last configuration is not possible. Indeed, without loss of generality we can assume that $p_1 = p_1^+$, $p_2 = p_2^+$, $p_3 = p_3^-$, $p_4 = p_4^-$, $p_5 = p_5^+$, $p_6 = p_6^-$, $p_7 = p_7^+$ and $p_8 = p_8^-$. Take the straight line L_{68}. Denote by p_{k_0} with $k_0 \in \{1,2,3,4,7,9\}$ the closest point to L_{68}. If p_{k_0} has negative index then there is only one more point in the 0-level with negative index that we denote by p_{k_1}. In this case applying the Euler-Jacobi formula to $L_{68}L_{k_0}L_{k_1}$ with L_{k_0} being parallel to L_{68} and L_{k_1} being such that all the points in the 0, 1st and 2nd level are on the same side of L_{k_1}, we reach to a contradiction. If p_{k_0} has positive index then there are two consecutive points in the 0-level with negative index that we denote by p_{k_1} and p_{k_2}. In this case applying the Euler-Jacobi formula to $L_{68}L_{k_0}L_{k_1k_2}$ with L_{k_0} being parallel to L_{68} we reach to a contradiction. So, only the configurations with $p_9 = p_9^+$ are the ones in (23).

Assume now that $p_9 = p_9^-$. Applying the Euler-Jacobi formula successively to $L_{1,2}L_{3,4}L_{5,6}$, $L_{1,2}L_{3,4}L_{6,7}$, $L_{1,2}L_{3,4}L_{7,8}$ and $L_{1,2}L_{3,4}L_{5,8}$ we get that there cannot be two consecutive points with negative index. Therefore, the possible configurations are

$$(+, 3-; 4+; -), \quad (3+, -; +, -, +, -; -), \quad (2+, 2-; 3+, -; -) \tag{24}$$

and $(+, -, +, -; 3+, -; -)$. We will show that none of the configurations in (24) are possible.

For the configuration $(+, 3-; 4+; -)$, without loss of generality we can assume that $p_1 = p_1^+$. Take the straight line L_{k9} with $k \in \{2, 3, 4\}$ so that it leaves three points with positive index on one side of the straight line (p_1 and two points in the 1st level that we denote by l_1, l_2). Now applying the Euler-Jacobi formula to $L_{k9}L_{i_1i_2}L_{j_1j_2}$ with $i_1, i_2 \in \{2, 3, 4\}$, $i_1, i_2 \neq k$ and $j_1, j_2 \in \{5, 6, 7, 8\}$ with $\{j_1, j_2\} \neq \{l_1, l_2\}$ we reach to a contradiction.

For the configuration $(3+, -; +, -, +, -; -)$ we can assume without loss of generality that $p_1 = p_1^+$, $p_2 = p_2^-$, $p_3 = p_3^+$, $p_4 = p_4^+$, $p_5 = p_5^-$, $p_6 = p_6^+$, $p_7 = p_7^-$, $p_8 = p_8^+$. Applying the Euler-Jacobi formula to $L_2L_{59}L_7$ or to $L_2L_5L_{79}$ depending on whether p_9 is closer to p_5 or to p_7 (here L_2 is the straight line passing through p_2 so that leaves all the points on the same side of the straight line, L_7 is the straight line passing through p_7 parallel to L_{59} and L_5 is the straight line passing through p_5 parallel to L_{79}), we reach to a contradiction.

For the configuration $(3+, -; +, -, +, -; -)$, we can assume without loss of generality that $p_5 = p_5^-$, $p_6 = p_6^+$, $p_7 = p_7^-$, $p_8 = p_8^+$. Take the straight line L_{59}. Note that it leaves three points with positive index on one side of the straight line (one of the points is either p_6 or p_8 and the other two points are in the 0-level that we denote by l_1, l_2. Without loss of generality we can assume that the point in the 1st level is p_6 since if it is p_8 the argument follows in the same manner). Now take L_7 so that it leaves the three points p_6, p_{l_1}, p_{l_2} on the same side of the straight line and the point p_8 on the other side of the straight line and $L_{59}(p_i)L_7(p_i) > 0$ with $i \in \{6, l_1, l_2\}$. Denote by l_3, l_4 the remaining two points in the 0-level different from l_1, l_2. Note that they are consecutive. Now applying the Euler-Jacobi formula to $L_{59}L_7L_{l_3l_4}$ we reach to a contradiction.

For the configuration $(2+, 2-; 3+, -; -)$ we can assume without loss of generality $p_8 = p_8^-$. Take the straight line L_{89} and denote by p_{k_0} with $k_0 \in \{1, 2, 3, 4, 5, 6, 7\}$ the closest point to L_{89}. If p_{k_0} has negative index then there is only one more point in the 0-level with negative index that we denote by p_{k_1}. In this case applying the Euler-Jacobi formula to $L_{89}L_{k_0}L_{k_1}$

with L_{k_0} being parallel to L_{89} and L_{k_1} being such that all the points in the 0, 1st and 2nd level are on the same side of L_{k_1}, we reach to a contradiction. If p_{k_0} has positive index then there are two consecutive points in the 0-level with negative index that we denote by p_{k_1} and p_{k_2}. In this case applying the Euler-Jacobi formula to $L_{89}L_{k_0}L_{k_1k_2}$ with L_{k_0} being parallel to L_{89} we reach to a contradiction. In short, only the configuration $(+,-,+,-;3+,-;-)$ is possible.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 1.7136 + 0.572791x + 3.00238x^2 - 2y^2 - 2.00477x^3 \\ &\quad - 1.2864y + 0.00238498x^2y + y^3, \\ Q(x,y) &= 1.09364 - 0.187289x - 0.406356y + 2.92225x^2 - 3.84449x^3 \\ &\quad + 1.42225x^2y - 1.5y^2 + xy^2. \end{aligned} \tag{25}$$

The cubic system (1) with P and Q given in (25) has the singular points

$$\begin{gathered} (1.5, 2), \quad (1.33.., 1.66..), \quad (0.79.., 1.5), \\ (0.53.., 1.13..), \quad (-0.53.., -1.13..), \quad (0, -1), \\ (0.53.., -1.13..), \quad (-0.53.., 1.13..), \quad (-0.22.., 0.78..) \end{gathered}$$

in the configuration $(4+;4-;+)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 11.2335 + 0.052227x - 51.794y - 1.32865x^2 + 29.0751xy \\ &\quad - 1.64635y^2 - 0.164314x^3 - 3.30897x^2y + y^3 \\ Q(x,y) &= 32.4222 - 17.1813x - 14.5172y - 1.2652x^2 + 14.6159xy \\ &\quad - 2.23433y^2 + 1.18119x^3 - 3.49778x^2y + xy^2. \end{aligned} \tag{26}$$

The cubic system (1) with P and Q given in (26) has the singular points

$$\begin{gathered} (5.16.., 3.14..), \quad (1.82.., 3.80..), \quad (3, 1), \quad (-4.10.., 0), \\ (2.60.., 0.66..), \quad (2.58.., 0.55..), \quad (1.62.., -3.19..), \\ (2.49.., -0.47..), \quad (2.43.., -0.8) \end{gathered}$$

in the configuration $(3+,-;+,3-;+)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= 1.19391 - 1.32373x - 0.901198y + 2.60813x^2 \\ &\quad + 0.876263xy - 1.0951y^2 + 1.15746x^3 + -2.98422x^2y + y^3, \\ Q(x,y) &= 6.831 + 11.1318x - 2.08151y + 1.39162x^2 \\ &\quad - 6.9797xy - 8.91252y^2 - 17.6276x^3 + 19.4634x^2y + xy^2. \end{aligned} \tag{27}$$

The cubic system (1) with P and Q given in (27) has the singular points

$$
\begin{aligned}
&(-0.79..,2), \quad (-0.53..,1.13..), \quad (-0.53..,-1.13..),\\
&(0,-1), \quad (1.5,2), \quad (1.31..,1.67..), \quad (0.79..,1.3..),\\
&(0.53..,1.10..), \quad (1.33..,1.66..).
\end{aligned}
$$

in the configuration $(+,-,+,-;+,-,+,-;+)$.

The cubic system (1) with

$$
\begin{aligned}
P(x,y) &= 3.74972 + 3.42597x - 0.418861y - 3.19156x^2\\
&\quad - 2.03937xy - 3.16858y^2 - 0.338597x^3 + 2.45895x^2y + y^3\\
Q(x,y) &= -17.0041 - 31.7873x - 4.1965y + 42.1811x^2\\
&\quad + 18.9981xy + 12.8076y^2 + 4.4236x^3 - 33.5519x^2y + xy^2.
\end{aligned} \tag{28}
$$

The cubic system (1) with P and Q given in (28) has the singular points

$$
\begin{aligned}
&(-13.18..,-0.39..), \quad (1.5,2), \quad (1.33..,1.46),\\
&(0.79..,1.44), \quad (0.53..,1.40..), \quad (-0.79..,2)\\
&(0,-1), \quad (-0.53..,1.13..), \quad (0.53..,-1.13..)
\end{aligned}
$$

in the configuration $(+,-,+,-;3+,-;-)$.

Configuration $(4;3;2)$. Take the points $\{p_1,p_2,p_3,p_4\} = A \cap \partial\hat{A}$ oriented in counterclockwise sense. Denote the remaining points in the 1st level by p_5,p_6,p_7 (also oriented in counterclockwise sense) and the point in the 2nd level by p_8,p_9. We consider three cases: either $p_8 = p_8^+$, $p_9 = p_9^+$; or $p_8 = p_8^-$, $p_9 = p_9^-$; or $p_8 = p_8^-$ and $p_9 = p_9^+$.

Assume first that $p_8 = p_8^+$, $p_9 = p_9^+$. In this case applying the Euler-Jacobi formula successively to $L_{12}L_{34}L_{56}$, $L_{1,2}L_{34}L_{67}$ and $L_{12}L_{34}L_{57}$ we get that $p_5 = p_5^-$, $p_6 = p_6^-$ and $p_7 = p_7^-$. In this case, the unique possible configuration is $(4+,-;3-;2+)$.

Assume now that $p_8 = p_8^-$, $p_9 = p_9^-$. In this case applying the Euler-Jacobi formula successively to $L_{12}L_{34}L_{56}$, $L_{1,2}L_{34}L_{67}$ and $L_{12}L_{34}L_{57}$ we get that $p_5 = p_5^+$, $p_6 = p_6^+$ and $p_7 = p_7^+$. In this case , the unique possible configurations are $(2+,2-;3+;2-)$ or $(+,-,+,-;3+;2-)$. We will show that the configuration $(2+,2-;3+;2-)$ is not possible. Take the straight line L_{89} and denote by p_{k_0} with $k_0 \in \{1,2,3,4,5,6,7\}$ the closest point to L_{89}. If p_{k_0} has negative index then there is only one more point in the 0-level with negative index that we denote by p_{k_1}. In this case applying the Euler-Jacobi formula to $L_{89}L_{k_0}L_{k_1}$ with L_{k_0} being parallel to L_{89} and L_{k_1} being such that all the points in the 0, 1st and 2nd level are on the same side of L_{k_1}, we reach to a contradiction. If p_{k_0} has positive index then there are two consecutive points in the 0-level with negative index that we denote by p_{k_1} and p_{k_2}. In this case applying the Euler-Jacobi formula to $L_{89}L_{k_0}L_{k_1k_2}$ with L_{k_0} being parallel to L_{89} we reach to a contradiction.

Finally, consider the case in which $p_8 = p_8^-$ and $p_9 = p_9^+$. We consider different cases: either $p_5 = p_5^+$, $p_6 = p_6^+$ and $p_7 = p_7^+$, or $p_5 = p_5^-$, $p_6 = p_6^-$ and $p_7 = p_7^-$, or $p_5 = p_5^+$, $p_6 = p_6^-$ and $p_7 = p_7^-$, or $p_5 = p_5^+$, $p_6 = p_6^+$ and $p_7 = p_7^-$. In the first case the configuration must be $(+, 3-; 3+; +, -)$. In the second case the configurations must be $(4+; 3-; +, -)$, in the third case the configuration must be $(3+, -; +, 2-; +, -)$ and in the fourth case the configurations can be $(2+, 2-; 2+, -; +, -)$ and $(+, -, +, -; 2+, -; +, -)$. We will show that configurations $(+, 3-; 3+; +, -)$ and $(2+, 2-; 2+, -; +, -)$ are not possible.

For the configuration $(+, 3-; 3+; +, -)$ without loss of generality we can assume that $p_1 = p_1^+$ and $p_2 = p_2^-$. Take the straight line L_{28} and denote by p_{k_0} with $k_0 \in \{1, 3, 4, 5, 6, 7, 9\}$ the closest point to L_{78}. If p_{k_0} has negative index then there is only one more point in the 0-level with negative index that we denote by p_{k_1}. In this case applying the Euler-Jacobi formula to $L_{28}L_{k_0}L_{k_1}$ with L_{k_0} being parallel to L_{28} and L_{k_1} being such that all the points in the 0, 1st and 2nd level are on the same side of L_{k_1}, we reach to a contradiction. If p_{k_0} has positive index then there are two consecutive points in the 0-level with negative index that we denote by p_{k_1} and p_{k_2}. In this case applying the Euler-Jacobi formula to $L_{28}L_{k_0}L_{k_1k_2}$ with L_{k_0} being parallel to L_{28} we reach to a contradiction. So, this case is not possible.

Now we show that the configuration $(2+, 2-; 2+, -; +, -)$ is also not possible. Take the straight line L_{78} and denote by p_{k_0} with $k_0 \in \{1, 2, 3, 4, 5, 6, 9\}$ the closest point to L_{78}. If p_{k_0} has negative index then there is only one more point in the 0-level with negative index that we denote by p_{k_1}. In this case applying the Euler-Jacobi formula to $L_{78}L_{k_0}L_{k_1}$ with L_{k_0} being parallel to L_{78} and L_{k_1} being such that all the points in the 0, 1st and 2nd level are on the same side of L_{k_1}, we reach to a contradiction. If p_{k_0} has positive index then there are two consecutive points in the 0-level with negative index that we denote by p_{k_1} and p_{k_2}. In this case applying the Euler-Jacobi formula to $L_{78}L_{k_0}L_{k_1k_2}$ with L_{k_0} being parallel to L_{78} we reach to a contradiction. In short the unique possible configurations are $(4+; 3-; +, -)$, $(3+, -; +, 2-; +, -)$ and $(+, -, +, -; 2+, -; +, -)$.

The cubic system (1) with

$$
\begin{aligned}
P(x,y) &= 1.9467 + 0.106601x - 1.0533y + 2.18655x^2 - 2y^2 \\
&\quad - 0.373105x^3 - 0.813448x^2y + y^3, \\
Q(x,y) &= 1.44598 - 0.891962x - 0.0540188y + 1.68907x^2 - 1.5y^2 \\
&\quad - 1.37813x^3 + 0.189066x^2y + xy^2.
\end{aligned} \tag{29}
$$

The cubic system (1) with P and Q given in (29) has the singular points

$$\begin{gathered}(3.91..,4.38..),\quad (1.5,2),\quad (1.33..,1.66..),\\ (0.79..,1.3),\quad (-0.53..,-1.13..),\quad (0.53..,1.13..),\\ (0,-1),\quad (-0.53..,1.13..),\quad (0,53..,-1.13..)\end{gathered}$$

in the configuration $(4+;3-;+,-)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) &= 4.44579-3.87209x-93.8975y+0.0772746x^2+70.385xy\\ &\quad -1.27865y^2+0.313112x^3-13.2176x^2y+y^3,\\ Q(x,y) &= 31.5935-17.6604x-19.6577y-1.09355x^2+19.6595xy\\ &\quad -2.18944y^2+1.23948x^3-4.70753x^2y+xy^2.\end{aligned}\tag{30}$$

The cubic system (1) with P and Q given in (30) has the singular points

$$\begin{gathered}(1.82..,3.80..),\quad (3,1),\quad (-4.10..,0),\quad (2.60..,0.66..),\\ (2.58..,0.55..),\quad (1.62..,-3.19..),\quad (2.52..,0.26..),\\ (2.43,0),\quad (2.49..,-0.47..)\end{gathered}$$

in the configuration $(3+,-;+,2-;+,-)$.

The cubic system (1) with

$$\begin{aligned}P(x,y) &= 3.15594+2.30642x-0.515948y-2.07794x^2-1.44045xy\\ &\quad -2.67189y^2+0.200156x^3+1.20557x^2y+y^3,\\ Q(x,y) &= -12.0555-22.457x-3.38738y+32.9002x^2+14.0066xy\\ &\quad +8.66815y^2-0.0663528x^3-23.1063x^2y+xy^2.\end{aligned}\tag{31}$$

The cubic system (1) with P and Q given in (31) has the singular points

$$\begin{gathered}(1.5,2),\quad (1.33..,1.46),\quad (0.79..,1.41)\\ (0.53..,1.40..),\quad (0.53..,1.40..),\quad (-0.79..,2),\\ (0,1),\quad (-0.53..,1.13..),\quad (0.53..,-1.13..)\end{gathered}$$

in the configuration $(+,-,+,-;2+,-;+,-)$.

Configuration (3; 6) Take the points $\{p_1,p_2,p_3\}=A\cap\partial\hat{A}$ oriented in counter-clockwise sense. Denote the points in the 1st level by $C=\{p_4,p_5,p_6,p_7,p_8,p_9\}$. Applying the Euler-Jacobi formula to $L_{12}L_3L_{45}$ we get that there cannot be four consecutive points being all positive or all negative. So at most, there can be three consecutive points with the same index. Moreover, we cannot have the configuration $(3+,3-)$ in the 1st level because then, if we denote by p_7, p_8, p_9 the three consecutive points with negative index, applying the Euler-Jacobi formula to $L_{12}L_3L_{79}$ we reach to a contradiction. Hence, the unique possible configuration for the convex hexagon formed by the points in C is $(3+,-,+,-)$, $(2+,-,2+,-)$, $(2+,-,+,2-)$, $(2+,2-,+,-)$, $(+,2-,+,2-)$,

$$(+,-,+,-,+,-) \quad \text{and} \quad (+,3-,+,-). \tag{32}$$

We claim that the unique possible configurations for the convex hexagon formed by the points $\{p_4, p_5, p_6, p_7, p_8, p_9\}$ are the ones in (32). Now we prove the claim. We will consider the other five cases separately.

If the configuration for the convex hexagon formed by the points in C is $(3+,-,+,-)$ then the total configuration must be $(+,2-;3+,-,+,-)$. Without loss of generality we can assume that $p_4 = p_4^-$ and $p_6 = p_6^-$. Consider the straight line L_{46} and denote by p_{k_0} with $k_0 \in \{1,2,3,5,7,8,9\}$ the closest point to L_{46}. If p_{k_0} has negative index then there is only one more point in the 0-level with negative index that we denote by p_{k_1}. In this case applying the Euler-Jacobi formula to $L_{46}L_{k_0}L_{k_1}$ with L_{k_0} being parallel to L_{46} and L_{k_1} being such that all the points in the 0, 1st and 2nd level are on the same side of L_{k_1}, we reach to a contradiction. If p_{k_0} has positive index then there are two consecutive points in the 0-level with negative index that we denote by p_{k_1} and p_{k_2}. In this case applying the Euler-Jacobi formula to $L_{46}L_{k_0}L_{k_1k_2}$ with L_{k_0} being parallel to L_{46} we reach to a contradiction.

If the configuration for the convex hexagon formed by the points in C is $(2+,-,2+,-)$ then the total configurations must be $(+,2-;2+,-,2+,-)$. Let p_{k_0}, p_{k_1} denote the points in the 1st level with negative index that are between two points with positive index. Denote by p_{k_2} with $k_2 \in \{1,2,3,4,5,6,7,8,9\}$ with $k_2 \neq k_0$, $k_2 \neq k_1$ closest to $L_{k_0k_1}$. If p_{k_2} has negative index then there is only one more point in the 0-level with negative index that we denote by p_{k_3}. In this case applying the Euler-Jacobi formula to $L_{k_0k_1}L_{k_2}L_{k_3}$ with L_{k_2} being parallel to $L_{k_0k_1}$ and L_{k_3} being such that all the points in the 0 and 1st level are on the same side of L_{k_3}, we reach to a contradiction. If p_{k_2} has positive index then there are two consecutive points in the 0-level with negative index that we denote by p_{k_3} and p_{k_4}. In this case applying the Euler-Jacobi formula to $L_{k_0k_1}L_{k_2}L_{k_3k_4}$ with L_{k_2} being parallel to $L_{k_0k_1}$ we reach to a contradiction.

If the configuration for the convex hexagon formed by the points in C is $(2+,-,+,2-)$ or $(2+,2-,+,-)$ then the total configurations must be $(2+,-;2+,-,+,2-)$ or $(2+,-;2+,2-,+,-)$. Without loss of generality we can denote by $p_4 = p_4^+$, $p_5 = p_5^+$, $p_6 = p_6^-$, $p_7 = p_7^+$, $p_8 = p_8^-$, $p_9 = p_9^-$. Take the straight lines L_{59}, L_{68} for the first configuration and L_{48}, L_{69} for the second one. Note that there exists a point, p_{k_0} with positive index in the 0-level that is on the same sides of both straight lines (taking the corresponding straight lines depending on the configuration). Denote by p_{k_1}, p_{k_2} the remaining points in the 0-level. Now applying the Euler-Jacobi formula to $L_{59}L_{68}L_{k_1k_2}$ for the first configuration and to $L_{48}L_{69}L_{k_1k_2}$ for the second one, we reach to a contradiction.

Finally, if the configuration for the convex hexagon formed by the points in C is $(+,2-,+,2-)$ then the total configuration must be $(3+;+,2-,+,2-)$. Without loss of generality we can denote by $p_4 = p_4^+$, $p_5 = p_5^-$, $p_6 = p_6^-$, $p_7 = p_7^+$, $p_8 = p_8^-$ and $p_9 = p_9^-$ (oriented in counterclockwise sense). Take the straight lines L_{59} and L_{68}. Note that there exists a point p_{k_0} in the 0-level so that it is in both sides of L_{59} and L_{68}. Denote by p_{k_1}, p_{k_2} the remaining points in the 0-level. Then applying the Euler-Jacobi formula to $L_{59}L_{68}L_{k_1k_2}$ we reach to a contradiction. In short we have proved the claim and the unique possible configurations are $(2+,-;+,-,+,-,+,-)$ and $(3+;+,3-,+,-)$.

The cubic system (1) with

$$
\begin{aligned}
P(x,y) &= 0.0700114x - 0.820177y + 0.618091x^2 - 2.34582xy \\
&\quad + 0.0637765y^2 + 1.07808x^3 - 0.866707x^2y + y^3, \\
Q(x,y) &= 0.428369x - 3.72284y + 2.00156x^2 - 1.86912xy \\
&\quad - 3.67881y^2 + 0.0266153x^3 + 1.50949x^2y + xy^2
\end{aligned}
\tag{33}
$$

The cubic system (1) with P and Q given in (33) has the singular points

$$
\begin{array}{lllll}
(1.7, 1.3), & (1.2, 0.64), & (0.2, -1.2), & (0.6, 0.2), & (-0.5, 0.1) \\
(-1.63.., 1.29..), & (-0.75.., 0.28..), & (-1.1, 0.65), & (0, 0). &
\end{array}
$$

in the configuration $(2+,-;+,-,+,-,+,-)$.

The cubic system (1) with

$$
\begin{aligned}
P(x,y) &= 11.0069 - 3.2419x - 30.9182y - 2.42678x^2 + 22.2105xy \\
&\quad + 0.826471y^2 + 0.761473x^3 - 4.03378x^2y + y^3, \\
Q(x,y) &= 6.50717 - 1.60833x - 15.4643y - 1.3909x^2 + 11.6013xy \\
&\quad - 2.61894y^2 + 0.401328x^3 - 2.16772x^2y + xy^2
\end{aligned}
\tag{34}
$$

The cubic system (1) with P and Q given in (34) has the singular points

$$
\begin{array}{lll}
(4.96.., 2.71..), & (2.86.., 0.61..), & (2.71.., 0.67..), \\
(2.68.., 0.68..), & (3, 0), & (-2.10.., 0), \\
(2.86.., -0.52..), & (2.52.., -0.57..), & (1.93.., -2.34..)
\end{array}
$$

in the configuration $(3+;+,3-,+,-)$.

Configuration $(3;5;1)$. Take the points $\{p_1, p_2, p_3\} = A \cap \partial\hat{A}$ oriented in counterclockwise sense. Denote the remaining points in the 1st level by p_4, p_5, p_6, p_7, p_8 also oriented in counterclockwise sense and the point in the 2nd level by p_9. We consider two possible cases $p_9 = p_9^+$ and $p_9 = p_9^-$.

When $p_9 = p_9^+$, applying the Euler-Jacobi formula to $L_{12}L_3L_{45}$, $L_{12}L_3L_{56}$, $L_{12}L_3L_{67}$, $L_{12}L_3L_{78}$ and $L_{12}L_3L_{48}$ we obtain that there cannot be 3 consecutive points in the 1st level with positive index. So the possible configurations are $(2+,-;2+,3-;+)$, $(+,2-;2+,-,+,-;+)$, $(2+,-;+,2-,+,-;+)$ and $(3+;+,4-;+)$. We will show that the configurations $(+,2-;2+,-,+,-;+)$ and $(2+,-;+,2-,+,-;+)$ are not possible.

For the configuration $(+,2-;2+,-,+,-;+)$, without loss of generality we can assume that $p_1 = p_1^+$, $p_2 = p_2^-$, $p_3 = p_3^-$, $p_4 = p_4^+$, $p_5 = p_5^+$, $p_6 = p_6^-$, $p_7 = p_7^+$, $p_8 = p_8^-$. Take the straight line L_{68} and denote by p_{k_0} with $k_0 \in \{1,2,3,4,5,7,9\}$ the closest point to L_{68}. If p_{k_0} has negative index then there is only one more point in the 0-level with negative index that we denote by p_{k_1}. In this case applying the Euler-Jacobi formula to $L_{68}L_{k_0}L_{k_1}$ with L_{k_0} being parallel to L_{68} and L_{k_1} being such that all the points in the 0 and 1st level are on the same side of L_{k_1}, we reach to a contradiction. If p_{k_0} has positive index then there are two consecutive points in the 0-level with negative index that we denote by p_{k_1} and p_{k_2}. In this case applying the Euler-Jacobi formula to $L_{68}L_{k_0}L_{k_1k_2}$ with L_{k_0} being parallel to L_{68} we reach to a contradiction.

For the configuration $(2+,-;+,2-,+,-;+)$ we note that without loss of generality we can assume that $p_1 = p_1^-$ and $p_2 = p_2^+$, $p_3 = p_3^+$. Moreover, $p_4 = p_4^-$, $p_5 = p_5^+$, $p_6 = p_6^-$, $p_7 = p_7^-$ and $p_8 = p_8^+$ and there exists $k \in \{\emptyset, 5, 8\}$ (if $k = \emptyset$ then $L_{k4} = L_4$ chosen in a convenient way) so that $C = L_{67}L_{k4}$ satisfies $C(p_1)C(p_j) < 0$ for $j \in \{5,8,9\}$ with $j \neq k$. Then, applying the Euler-Jacobi formula to $L_{67}L_{k4}L_{23}$ we reach to a contradiction.

Take now $p_9 = p_9^-$. Proceeding as above, we obtain that there cannot be 3 points in the 1st level with negative index. So, the possible configurations are $(2+,-;3+,2-;-)$, $(3+;+,2-,+,-;-)$, $(+,2-;4+,-;-)$ and $(2+,-;2+,-,+,-;-)$. We will show that configurations $(2+,-;3+,2-;-)$ and $(+,2-;4+,-;-)$ are not possible.

For the configuration $(2+,-;3+,2-;-)$ we can denote $p_1 = p_1^-$ and $p_7 = p_7^-$, $p_8 = p_8^-$ (note that the rest of the points in the 0-level and in the 1st level are positive). Now we consider the straight lines L_{89}, L_{k7} where $k \in \{2,3,4,5,6\}$ so that there are no points in $\{2,3,4,5,6\}$ between these two straight lines. Now applying the Euler-Jacobi formula to $L_1L_{89}L_{k7}$ we reach to a contradiction.

For the configuration $(+,2-;4+,-;-)$, without loss of generality denote by p_5 the point with negative index in the 1-st level. Take the straight line L_{59} and denote by p_{k_0} with $k_0 \in \{1,2,3,4,6,7,9,8\}$ the closest point to L_{59}. If p_{k_0} has negative index then there is only one more point in the 0-level with negative index that we denote by p_{k_1}. In this case applying the Euler-Jacobi formula to $L_{59}L_{k_0}L_{k_1}$ with L_{k_0} being parallel to L_{59} and L_{k_1} being such that all the points in the 0 and 1st level are on the same side of L_{k_1}, we reach to a contradiction. If p_{k_0} has positive index then there are two consecutive points in the 0-level with negative index that we denote by p_{k_1} and p_{k_2}. In this case applying the Euler-Jacobi formula to $L_{59}L_{k_0}L_{k_1k_2}$ with L_{k_0} being parallel to L_{59} we reach to a contradiction.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= -25.835 + 15.2203x + 25.4367y + 0.399967x^2 - 20.647xy \\ &\quad - 2.07302y^2 - 0.915852x^3 + 4.31994x^2y + y^3, \\ Q(x,y) &= 3.15824 - 1.75253x - 3.00217y - 0.0468047x^2 + 3.13052xy \\ &\quad - 2.40622y^2 + 0.0954355x^3 - 0.782147x^2y + xy^2. \end{aligned} \tag{35}$$

The cubic system (1) with P and Q given in (35) has the singular points

$$\begin{aligned} &(7.95, 2.50..), \quad (3,1), \quad (2.4, 1.2), \\ &(2.39, 1.05), \quad (2.38.., 1.04), \quad (-4.10.., 0.1), \\ &(2.67.., -0.14..), \quad (2.49.., -0.27..), \quad (2.4, -0.25) \end{aligned}$$

in the configuration $(2+, -; 2+, 3-; +)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= -0.0231623x + 0.433626y - 0.0938062x^2 - 0.789841xy \\ &\quad + 1.38804y^2 + 0.415785x^3 - 1.35566x^2y + y^3, \\ Q(x,y) &= 0.213746x - 0.83474y + 0.361726x^2 + 1.71502xy \\ &\quad - 0.628414y^2 - 1.49897x^3 + 0.383194x^2y + xy^2. \end{aligned} \tag{36}$$

The cubic system (1) with P and Q given in (36) has the singular points

$$\begin{aligned} &(1.7, 1.3), \quad (1.2, 0.81), \quad (0.2, -1.2), \quad (0.6, 0.2), \quad (0,0), \\ &(0.55.., 0.12..), \quad (-0.5, 0.1), \quad (-1.63.., 1.29..), \quad (-1.1, 0.65) \end{aligned}$$

in the configuration $(3+; +, 4-; +)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= -0.101538 - 0.011492x - 0.36926y + 0.642709x^2 \\ &\quad - 0.0529155xy + 0.892157y^2 + 0.0525865x^3 - 1.35813x^2y \\ &\quad + y^3, \\ Q(x,y) &= -0.382507 + 0.121336x - 2.02418y + 2.0943x^2 \\ &\quad + 6.76853xy - 0.5582y^2 - 3.83656x^3 - 0.341746x^2y + xy^2. \end{aligned} \tag{37}$$

The cubic system (1) with P and Q given in (37) has the singular points

$$\begin{aligned} &(1.7, 1.3), \quad (1.2, 0.64), \quad (0.2, -1.2), \quad (0.39.., 0.45..), \quad (0, -0.2) \\ &(0.6, 0.2), \quad (-0.5, 0.1), \quad (-1.63.., 1.29..), \quad (-1.1, 0.65) \end{aligned}$$

in the configuration $(3+; +, 2-, +, -; -)$.

The cubic system (1) with

$$\begin{aligned} P(x,y) &= -4.0025 + 12.0174x - 4.7607y - 6.24871x^2 - 0.249178xy \\ &\quad - 1.15051y^2 + 0.860309x^3 + 0.819933x^2y + y^3, \\ Q(x,y) &= 13.008 - 14.4565x + 0.0233284y + 5.11221x^2 + 1.06544xy \\ &\quad - 2.13972y^2 - 0.573151x^3 - 0.447514x^2y + xy^2. \end{aligned} \tag{38}$$

The cubic system (1) with P and Q given in (38) has the singular points

$$\begin{array}{l} (1.82.., 1.80..), \quad (4.20.., 0), \quad (3.34.., 0.46..), \\ (2.8, 0.7), \quad (2.60.., 0.76..), \quad (2.48.., 0.79) \\ (2.59.., -0.27), \quad (0.62.., -1.79.), \quad (2.4, -0.8) \end{array}$$

in the configuration $(2+, -; 2+, -, +, -; -)$.

Configuration $(3; 4; 2)$. Take the points $\{p_1, p_2, p_3\} = A \cap \partial\hat{A}$. Denote the remaining points in the 1st level by p_4, p_5, p_6, p_7 and the points in the 2nd level by p_8, p_9. Clearly we can have either $p_8 = p_8^+$, $p_9 = p_9^+$, or $p_8 = p_8^-$, $p_9 = p_9^-$, or $p_8 = p_8^+$ and $p_9 = p_9^-$.

When $p_8 = p_8^+$, $p_9 = p_9^+$. Assuming that the points p_4, p_5, p_6, p_7 are oriented in counterclockwise sense, applying the Euler-Jacobi formula to $L_{12}L_3L_{45}$, $L_{12}L_3L_{56}$, $L_{12}L_3L_{67}$ and to $L_{12}L_3L_{47}$ we get that only the following configurations are possible

$$(2+, -; +, 3-; 2+), \quad (3+; 4-; 2+), \quad (+, 2-; +, -, +, -; 2+).$$

We will show that the last two configurations are not possible.

For the configuration $(3+; 4-; 2+)$ we note that there exist $k \in \{1, 2, 3\}$ (that without loss of generality we name p_1) and $k_0, k_1 \in \{4, 5, 6, 7\}$ being consecutive (that without loss of generality we name p_4, p_5) so that the straight lines L_{2p_4} and L_{3p_5} satisfy: the points p_6, p_7, p_8, p_9 are between them and the point p_1 is in the opposite side of these points for both straight lines. Now, applying the Euler-Jacobi formula to $L_{2p_4}L_{3p_5}L_{67}$ we reach to a contradiction.

For the configuration $(+, 2-; +, -, +, -; 2+)$ we note that without loss of generality we can assume that $p_1 = p_1^+$ and $p_4 = p_4^+$, $p_5 = p_5^-$, $p_6 = p_6^+$ and $p_7 = p_7^-$. We observe that there exists $k \in \{5, 7\}$ so that L_k separates either p_2 or p_3 from the rest of the points. Without loss of generality we can assume that the point that separates L_k is p_2. Then applying the Euler-Jacobi formula to $L_kL_{13}L_j$ with $j \in \{5, 7\}$, $j \neq k$, we reach to a contradiction. So this case is not possible.

When $p_8 = p_8^-$, $p_9 = p_9^-$. Assuming that the points p_4, p_5, p_6, p_7 are oriented in counterclockwise sense, applying the Euler-Jacobi formula to

$L_{12}L_3L_{45}$, $L_{12}L_3L_{56}$, $L_{12}L_3L_{67}$ and to $L_{12}L_3L_{47}$ we get that only the following configurations are possible

$$(+, 2-; 4+; 2-), \quad (2+, -; 3+, -; 2-), \quad (3+; +, -, +, -; 2-).$$

We will show that none of these configurations are possible.

For the configuration $(+, 2-; 4+; 2-)$ we take the straight line L_{89} and denote by p_{k_0} with $k_0 \in \{1, 2, 3, 4, 5, 6, 7\}$ the closest point to L_{89}. If p_{k_0} has negative index then there is only one more point in the 0-level with negative index that we denote by p_{k_1}. In this case applying the Euler-Jacobi formula to $L_{89}L_{k_0}L_{k_1}$ with L_{k_0} being parallel to L_{89} and L_{k_1} being such that all the points in the 0 and 1-st level are on the same side of L_{k_1}, we reach to a contradiction. If p_{k_0} has positive index then there are two consecutive points in the 0-level with negative index that we denote by p_{k_1} and p_{k_2}. In this case applying the Euler-Jacobi formula to $L_{89}L_{k_0}L_{k_1k_2}$ with L_{k_0} being parallel to L_{89} we reach to a contradiction.

For the configuration $(2+, -; 3+, -; 2-)$ we can denote $p_1 = p_1^-$ and $p_4 = p_4^-$ (note that the rest of the points in the 0-level and in the 1st level are positive). Now we consider the straight lines L_{48} and L_{k9} where $k \in \{2, 3, 5, 6, 7\}$ so that there are no more points in $\{2, 3, 5, 6, 7\}$ between these two straight lines. Now applying the Euler-Jacobi formula to $L_1L_{48}L_{k9}$ we reach to a contradiction.

Finally, for the configuration $(3+; +, -, +, -; 2-)$. Without loss of generality we can assume that $p_4 = p_4^+$, $p_5 = p_5^-$, $p_6 = p_6^+$ and $p_7 = p_7^-$. Consider the straight lines L_{58} and L_{79} (we can rename p_8 and p_9 so that the above straight lines do not intersect). Note that there exists $k \in \{1, 2, 3\}$ so that p_k is on the same side of both straight lines L_{58} and L_{79}. Then applying the Euler-Jacobi formula to $L_{58}L_{79}L_{ij}$ with $i, j \in \{1, 2, 3\}$ and $i, j \neq k$ we reach to a contradiction. In short, no configuration of the form $(3*; 4*, 2-)$ is possible.

Assume now that $p_8 = p_8^-$ and $p_9 = p_9^+$. The possible configurations are

$$(3+; +, 3-; +, -), \quad (2+, -; 2+, 2-; +, -),$$

and

$$(2+, -; +, -, +, -; +, -), \quad (3-; 4+; +, -), \quad (+, 2-; 3+, -; +, -)$$

We will show that the last three configurations are not possible.

For the configuration $(3-; 4+; +, -)$, note that there exists $k \in \{4, 5, 6, 7\}$ so that L_{k8} leaves two points of the 1st level and p_9 in one side and one point (that we denote by p_p) on the other side. Moreover, there exists $\ell \in \{1, 2, 3\}$ so that $L_{\ell p}$ leaves all the points in the 1st and 2nd levels of the configuration on one side of the straight line. Now applying the Euler-Jacobi formula to $L_{\ell p}L_{k8}L_{ij}$ with $i, j \in \{1, 2, 3\}$ with $i, j \neq \ell$ we reach to a contradiction.

For the configuration $(2+,-;+,-,+,-;+,-)$, denote by $p_4 = p_4^+$, $p_5 = p_5^-$, $p_6 = p_6^+$ and $p_7 = p_7^-$ (assume that the points p_4, p_5, p_6, p_7 are oriented in counterclockwise sense). Take the straight lines $L_{58}L_{79}$. Note that the point p_4 is on the same side of the straight lines L_{58} and L_{79}, and the same happens for p_6. Now note that there exists $k \in \{1,2,3\}$ with $p_k = p_k^+$ so that it is also on the same side of the straight lines L_{58} and L_{79}. Then applying the Euler-Jacobi formula to $L_{58}L_{79}L_{ij}$ with $i,j \in \{1,2,3\}$ being $i,j \neq k$ we reach to a contradiction.

For the configuration $(+,2-;3+,-;+,-)$ we denote by $p_7 = p_7^-$ the point in the 1-st level with negative index. Take the straight line L_{78} and denote by p_{k_0} with $k_0 \in \{1,2,3,4,5,6,9\}$ the closest point to L_{78}. If p_{k_0} has negative index then there is only one more point in the 0-level with negative index that we denote by p_{k_1}. In this case applying the Euler-Jacobi formula to $L_{78}L_{k_0}L_{k_1}$ with L_{k_0} being parallel to L_{78} and L_{k_1} being such that all the points in the 0 and 1-st level are on the same side of L_{k_1}, we reach to a contradiction. If p_{k_0} has positive index then there are two consecutive points in the 0-level with negative index that we denote by p_{k_1} and p_{k_2}. In this case applying the Euler-Jacobi formula to $L_{78}L_{k_0}L_{k_1k_2}$ with L_{k_0} being parallel to L_{78} we reach to a contradiction.

The cubic system (1) with

$$\begin{aligned}
P(x,y) &= -19.06 + 14.078x - 27.9208y + 0.133169x^2 + 12.6251xy \\
&\quad - 1.14841y^2 - 1.0794x^3 - 0.559453x^2y + y^3 \\
Q(x,y) &= -1.07288 + 0.925648x - 7.18843y - 0.0137971x^2 + 5.40235xy \\
&\quad - 2.5207y^2 - 0.073858x^3 - 1.00929x^2y + xy^2.
\end{aligned} \tag{39}$$

The cubic system (1) with P and Q given in (39) has the singular points

$$\begin{aligned}
&(4.82..,2.80..), \quad (3,1), \quad (-4.10..,0), \\
&(2.60..,0.66..), \quad (2.51..,0.69), \quad (2.55..,0.63..), \\
&(2.48..,0.55..), \quad (2.49..,-0.27..), \quad (2.4,-0.8)
\end{aligned}$$

in the configuration $(2+,-;+,3-;2+)$.

The cubic system (1) with

$$\begin{aligned}
P(x,y) &= 0.0790021x - 0.108409y + 0.322009x^2 - 1.08066xy \\
&\quad + 0.865459y^2 + 0.502493x^3 - 1.24144x^2y + y^3, \\
Q(x,y) &= 0.457237x - 1.43742y + 1.05087x^2 + 2.19319xy \\
&\quad - 1.10469y^2 - 1.82155x^3 + 0.306245x^2y + xy^2.
\end{aligned} \tag{40}$$

The cubic system (1) with P and Q given in (40) has the singular points

$$(1.7, 1.3),\quad (1.2, 0.64),\quad (0.2, -1.2),\quad (0, 0),\quad (0.3, 0.2),$$
$$(0.40.., 0.31..),\quad (-0.5, 0.1),\quad (-1.63.., 1.29..),\quad (-1.1, 0.65)$$

in the configuration $(3+; +, 3-; +, -)$.

The cubic system (1) with

(41)
$$\begin{aligned} P(x,y) &= -41.9913 + 30.0846x - 49.7037y + 0.450059x^2 + 19.3914xy \\ &\quad - 2.85838y^2 - 2.28458x^3 + 0.306496x^2y + y^3 \\ Q(x,y) &= -4.41011 + 3.25512x - 10.3585y + 0.0323204x^2 + 6.38705xy \\ &\quad - 2.76956y^2 - 0.24925x^3 - 0.883271x^2y + xy^2. \end{aligned}$$

The cubic system (1) with P and Q given in (41) has the singular points

$$(4.82.., 2.80..),\quad (3, 1),\quad (-4.10.., 0),$$
$$(2.58.., 0.75..),\quad (2.61.., 0.66..),\quad (2.60.., 0.66..),$$
$$(2.48.., 0.55..),\quad (2.49.., -0.27..),\quad (2.4, -0.8)$$

in the configuration $(2+, -; 2+, 2-; +, -)$.

Configuration $(3; 3; 3)$ Take the points $\{p_1, p_2, p_3\} = A \cap \partial\hat{A}$. Denote the remaining points in the 1st level by p_4, p_5, p_6 oriented in counterclockwise sense and the points in the 2nd level by p_7, p_8, p_9 also oriented in counterclockwise sense. Clearly, in the 2nd level we can have either $p_7 = p_7^+$, $p_8 = p_8^+$, $p_9 = p_9^+$; or $p_7 = p_7^-$, $p_8 = p_8^-$, $p_9 = p_9^-$; or $p_7 = p_7^+$, $p_8 = p_8^+$, $p_9 = p_9^-$, or $p_7 = p_7^+$, $p_8 = p_8^-$, $p_9 = p_9^-$.

We will show that the cases $p_7 = p_7^-$, $p_8 = p_8^-$, $p_9 = p_9^-$, or $p_7 = p_7^+$, $p_8 = p_8^+$, $p_9 = p_9^+$ are not possible. Assume that the closest point of the set $\{p_4, p_5, p_6\}$ to p_1 is p_4. Then either the straight line L_{15} or L_{16} leaves the other two points of the set $\{p_4, p_5, p_6\}$ on the same side. Assume that it is the straight line L_{15}, otherwise the proof follows in a similar way. Then applying the Euler-Jacobi formula to $C = L_{15}L_{26}L_{34}$ we get to a contradiction because L_{26} and L_{34} also leave the points of the set $\{p_3, p_4, p_5\}$ on the same side.

Assume now that $p_7 = p_7^-$, $p_8 = p_8^+$ and $p_9 = p_9^+$. The unique possible configurations in this case are $(3-; 3+; 2+, -)$, $(+, 2-; 2+, -; 2+, -)$, $(3+; 3-; 2+, -)$, and $(2+, -; +, 2-; 2+, -)$. We will show that the configurations $(3-; 3+; 2+, -)$ and $(+, 2-; 2+, -; 2+, -)$ are not possible.

For the configuration $(3-; 3+; 2+, -)$, we take the straight line L_7 so that it leaves four points with positive index on one side of the straight line and two points with positive index on the other side of the straight line (if the three points $p_7 = p_7^-$, $p_8 = p_8^+$ and $p_9 = p_9^+$ are on a straight line then the straight line L_{789} leaves at least two points of the 1st level on one side of the straight line and we denote the third point by p_p. Now there exists $k \in \{1,2,3\}$ so that L_{kp} leaves all the points of the 1st level on one side of L_{kp}. Then applying the Euler-Jacobi formula to $L_{789}L_{kp}L_{ij}$ with $i,j \in \{1,2,3\}$ and $i,j \neq k$ we reach to a contradiction). Denote by $p \in \{4,5,6\}$ the unique in the 1st level that is on one side of the straight line L_7. Then there exists $k \in \{1,2,3\}$ so that L_{kp} leaves all the points of the 1st level on one side of L_{kp}. Applying the Euler-Jacobi formula to $L_7L_{kp}L_{ij}$ with $i,j \in \{1,2,3\}$ and $i,j \neq k$ we reach to a contradiction.

We will show that the configuration $(+,2-;2+,-;2+,-)$ is not possible. Indeed, denote by $p_1 = p_1^+$ in the 0 level and $p_6 = p_6^-$ in the 1st level. Take L_{69}. Then there exists $k \in \{1,4,5,7,8\}$ so that taking L_k passing through the point p_k and being parallel to L_{69}, there is no other point of $\{1,4,5,7,8\}$ between the straight lines L_k and L_{69}. Then applying the Euler-Jacobi formula to $L_kL_{69}L_{23}$ we reach to a contradiction.

Finally, assume that $p_7 = p_7^-$, $p_8 = p_8^-$ and $p_9 = p_9^+$. The unique possible configurations in this case are $(+,2-;3+;+,2-)$, $(2+,-;2+,-;+,2-)$ and $(3+;+,2-;+,2-)$. We will show that the configuration $(+,2-;3+;+,2-)$ is not possible. Indeed, denote by $p_1 = p_1^+$ and take the straight line L_{78}. Then there exists $k \in \{1,4,5,6,9\}$ so that taking L_k passing through the point p_k and being parallel to L_{78}, there is no other point of $\{1,4,5,6,9\}$ between the straight lines L_k and L_{78}. Then applying the Euler-Jacobi formula to $L_kL_{78}L_{23}$ we reach to a contradiction.

The cubic system (1) with

$$
\begin{aligned}
P(x,y) &= 0.507522 + 0.477393x - 3.07401y + 0.495041x^2 - 13.8065xy \\
&\quad - 4.07675y^2 + 6.20386x^3 + 1.58957x^2y + y^3, \\
Q(x,y) &= 2.71688 + 2.60917x - 15.7881y + 1.34284x^2 - 63.2208xy \\
&\quad - 25.844y^2 + 27.466x^3 + 14.6585x^2y + xy^2.
\end{aligned} \tag{42}
$$

The cubic system (1) with P and Q given in (42) has the singular points

$$
\begin{gathered}
(1.7,1.3), \quad (1.2,0.64), \quad (0.2,-1.2), \quad (0.6,0.2), \quad (-0.5,0.1), \\
(-0.15..,-0.41..), \quad (-1.63..,1.29..), \quad (0,0.14), \quad (-1.1,0.65)
\end{gathered}
$$

in the configuration $(3+;3-;2+,-)$.

The cubic system (1) with

$$
\begin{aligned}
P(x,y) &= 2.75306 + 1.56901x - 30.609y - 0.575369x^2 + 17.882xy \\
&\quad - 0.84234y^2 - 0.193547x^3 - 2.25008x^2y + y^3, \\
Q(x,y) &= 2.13916 - 0.916338x - 7.58429y - 0.118131x^2 + 6.17644xy \\
&\quad - 2.47563y^2 + 0.0565861x^3 - 1.25824x^2y + xy^2.
\end{aligned} \tag{43}
$$

The cubic system (1) with P and Q given in (43) has the singular points

$$
\begin{gathered}
(4.82.., 2.80..), \quad (3,1), \quad (-4.10.., 0), \\
(2.60.., 0.76), \quad (2.51.., 0.69), \quad (2.35.., 0.66..) \\
(2.48.., 0.55..), \quad (2.49.., -0.27..), \quad (2.4, -0.8)
\end{gathered}
$$

in the configuration $(2+, -; +, 2-; 2+, -)$.

The cubic system (1) with

$$
\begin{aligned}
P(x,y) &= (x - 201/100)(x^2+y^2) + x, \\
Q(x,y) &= y(\frac{16}{5} - 4x + 2y)(-\frac{16}{5} + 4x + 2y).
\end{aligned} \tag{44}
$$

The cubic system (1) with P and Q given in (44) has the singular points

$$
\begin{gathered}
(1.74.., -1.89..), \quad (1.10.., 0), \quad (0.85.., -0.10..) \\
(0.90.., 0), \quad (0.69.., -0.21..), \quad (0.85.., 0.10..) \\
(0.69.., 0.21..), \quad (1.74.., 1.89..), \quad (0,0)
\end{gathered}
$$

in the configuration $(2+, -; 2+, -; +, 2-)$.

The cubic system (1) with

$$
\begin{aligned}
P(x,y) = {} & 0.846752 - 2.84767x - 2.03307y + 3.1496x^2 \\
& + 5.47874xy + 0.173954y^2 - 1.1457x^3 - 3.64529x^2y \\
& + y^3, \\
Q(x,y) = {} & 0.392774 - 0.320917x + 0.244422y - 0.549028x^2 \\
& - 0.427336xy - 1.92931y^2 + 0.468558x^3 + 0.164535x^2y \\
& + xy^2.
\end{aligned} \tag{45}
$$

The cubic system (1) with P and Q given in (45) has the singular points

$$
\begin{gathered}
(1.74.., 1.89..), \quad (1.74.., -1.89..), \quad (1.10.., 0) \\
(0.85.., 0.10..), \quad (0.90.., 0), \quad (0.1, 0.5) \\
(0.85.., -0.10..), \quad (-0.19.., 0.53..), \quad (0.69.., -0.21..)
\end{gathered}
$$

in the configuration $(3+; +, 2-; +, 2-)$.

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