ON THE LIMIT CYCLES OF A CLASS OF DISCONTINUOUS PIECEWISE LINEAR DIFFERENTIAL SYSTEMS

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Abstract. In this paper we consider discontinuous piecewise linear differential systems whose discontinuity set is a straight line L which does not pass through the origin and it is formed by the two linear differential systems $\dot{x} = Ax \pm b$. We study the limit cycles of this class of discontinuous piecewise linear differential systems. We do this study analyzing the fixed points of the return map of the system defined on the straight line L.

1. Introduction and statement of the main results

The theory of discontinuous piecewise differential systems is in constant development due to its applicability in different areas of the knowledge such as ecology, mechanic and electrical engineering, see for instance [6]. However even in the planar case there are important questions unsolved for this class of differential system as to know the number of their limit cycles.

In 2010 Han and Zhang [4] conjecture that piecewise linear systems with only two regions have at most two limit cycles. In 2012 Huan and Yang [5] investigate the number of limit cycles of planar piecewise linear systems with two regions sharing the same equilibrium. Moreover they provide a numerical example to illustrate the existence of three limit cycles, thus have a negative answer to the conjecture by Han and Zhang. In 2012 Llibre and Ponce [10] provide a rigorous proof of the existence of such three limit cycles. This was the first example that a discontinuous differential piecewise linear systems with two regions can have three limit cycles.

Many others researchers have analyzed the existence of limit cycles for a piecewise linear systems with two regions separated by a straight line. In [3] is proved that a such piecewise linear system has at most two limit cycles when the singularities are both virtual focus or center. In [1] the authors consider a piecewise linear system separated by a straight line with singularities of type real saddle and prove that this system has at most two limit cycles.

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Consider the 2×2 real matrix A^+ and A^-, and b^+, $b^- \in \mathbb{R}^2$. We define the planar piecewise discontinuous linear systems

$$\dot{x} = \begin{cases} X(x) = A^+ x + b^+ & \text{if} \quad x_1 > 0, \\ Y(x) = A^- x + b^- & \text{if} \quad x_1 < 0, \end{cases} \tag{1}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$. Note that $L = h^{-1}(0) = \{x \in \mathbb{R}^2; x_1 = 0\}$ split the plane in two open regions $S^+ = \{x \in \mathbb{R}^2; x_1 > 0\}$ and $S^- = \{x \in \mathbb{R}^2; x_1 < 0\}$. We say that a limit cycle of system (1) is a *crossing limit cycle* if it share no points with the sliding set of the system.

In [2] Cespedes studies systems (1) satisfying $div(X) div(Y) \geq 0$, i.e. the product of the divergences of the subsystems X and Y is non–negative, and show that such systems have at most two limit cycles. Moreover this author exhibited an example with exactly two crossing limit cycles. In this paper we consider $S_+ = \{(x_1, x_2) \in \mathbb{R}^2 x_: 1 > 1\}$ and $S_- = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 1\}$ and piecewise linear systems with two zones given by

$$\dot{x} = \begin{cases} X(x) = Ax + b & \text{if} \quad x \in S_+, \\ Y(x) = Ax - b & \text{if} \quad x \in S_-, \end{cases} \tag{2}$$

where $b \in \mathbb{R}^2 \setminus \{0\}$. In this case the discontinuity of system (2) is the straight line $L = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 1\}$. Assuming that there are no singularities of system (2) in L we shall study the existence of crossing limit cycles.

Set $\Phi_\pm(t, x)$ the flow of the system X and Y, respectively. The flow Φ_+ is said *transversal* to L at point p if $X(p)$ is not contained in L. If $X(p) \in L$ then the point p is called a *contact point* of the flow with L. Analogous definitions hold for system Y. We say that $p \in \mathbb{R}^2$ is a *real singularity* of system (2) if $p = (x_1, x_2)$ is such that either $x_1 > 1$ and $X(p) = 0$, or $x_1 < 1$ and $Y(p) = 0$. On the other hand p is a *virtual singularity* if $p = (x_1, x_2)$ is such that either $x_1 > 1$ and $Y(p) = 0$, or $x_1 < 1$ and $X(p) = 0$. It follows that the discontinuous system (2) can be of type *virtual–virtual*, *virtual–real* and *real–real* depending if the singularities of the systems X and Y are virtual or real.

In this work d and t denote the determinant and trace of the matrix A, respectively. Our main results are the following.

Theorem A. *Consider the discontinuous piecewise linear differential system* (2)*. If the contact points of system* (2) *with the straight line L coincides then system* (2) *has no limit cycles.*

Theorem B. *Assume that the discontinuous piecewise linear differential system* (2) *is of type virtual–virtual. If the contact points of the systems X and Y*

are distinct, then system (2) *has at most one limit cycle. There exists necessary and sufficiently conditions for the existence of exactly one limit cycle.*

Observe that system (2) satisfy $div(X)div(Y) \geq 0$, but this work provides new results with respect the ones obtained in chapter 2 of [2].

One of the most important tools in the study of the periodic orbits are the Poincaré maps. These maps characterize the behavior of the flows in the neighborhood of periodic orbits. Moreover, there exists a correspondence between the limit cycles of a system and the fixed points of same Poincaré map. In [8] Llibre and Teruel studied Poincaré maps for piecewise linear differential systems in $\mathbb{R}^n$. In [9] these authors determine the Poincaré maps and analyze the existence of crossing limit cycles of planar piecewise linear differential systems defined in three regions separated by two straight lines L_+ and L_- which are symmetric with respect to the origin. In order to prove Theorems A and B we determine the Poincaré maps of the subsystems X and Y of system (2) with respect to the straight line L, and we use these maps to determine the return map of system (2) in L. After that we analyze the existence of fixed points of this return map.

The paper is organized as follows. In section 2 we provide the basic results that we shall need to prove the mains results. Section 3 is split in three subsections, in subsection 3.1 and 3.2 we study existence of the Poincaré maps for the non–homogeneous systems X and Y, respectively. In subsection 3.3 we define the return map for the discontinuous piecewise linear differential system (2). In section 4 we study the existence of limit cycle for the discontinuous system (2) and prove our main results.

2. Preliminary results

Consider a linear differential system

$$\dot{x} = Ax, \tag{3}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$. Given $p, v \in \mathbb{R}^2$ let $L = \{p + \lambda v : \lambda \in \mathbb{R}\}$ be a straight line that does not pass through the origin, and let n be a unit vector orthogonal to L such that $n^T p > 0$, we say that n is oriented in the opposite sense to the origin. Furthermore if $n\dot{q} \neq 0$ the flow of system (3) is transversal to the line L at a point $q \in L$, where $\dot{q}$ denotes the vector field of the differential system (3) evaluated at q. Otherwise q is a contact point of the flow with the straight line L.

The following result is proved in Proposition 4.2.7 of [9].

Proposition 1. *Consider the differential system* (3) *with* $A \in GL(\mathbb{R}^2)$. *Let* L *be a straight line in the phase plane not passing through the origin,* p *a contact* *point of the flow in L, and $x(t)$ the solution of the system such that $x(0) = p$. Then*

(a) *if* $\det(A) > 0$ *there exists* $\varepsilon > 0$ *such that* $\{x(t); t \in (-\varepsilon, \varepsilon)\} \subset S_0 \cup L$,
(b) *if* $\det(A) < 0$ *there exists* $\varepsilon > 0$ *such that* $\{x(t); t \in (-\varepsilon, \varepsilon)\} \subset S \cup L$,

where S_0 and S are the half–planes bounded by the straight line L, being S_0 the half–plane containing the origin.

A transversal flow to L at a point $q \in L$ is said to have *outside orientation* if $n^T\dot{q} > 0$, and it is said to have *inside orientation* if $n^T\dot{q} < 0$. Thus we define the following subsets in L

$$L^I = \{q \in L : n^T\dot{q} \leq 0\} \text{ and } L^O = \{q \in L : n^T\dot{q} \geq 0\}.$$

If $\det A \neq 0$ and there exists a contact point p of the flow of system (3) in L, then L^I and L^O are two half–lines such that the flow over L^I and L^O has opposite sense and $L^I \cap L^O = p$. This is part of the next result that is proved in Proposition 4.2.5 of [9].

Proposition 2. *Consider the differential system* (3) *with $A \in GL(\mathbb{R}^2)$. Let L be a straight line in the phase plane not passing through the origin.*

(a) *There exists at most one contact point of the flow with L.*
(b) *Let p be a contact point of the flow with L. If* $\det A > 0$, *then* $L^I = \{p + \lambda\dot{p} : \lambda \geq 0\}$ *and* $L^O = \{p + \lambda\dot{p} : \lambda \leq 0\}$. *If* $\det A < 0$, *then* $L^I = \{p + \lambda\dot{p} : \lambda \leq 0\}$ *and* $L^O = \{p + \lambda\dot{p} : \lambda \geq 0\}$.
(c) *If $L^I \neq \emptyset$ and $L^O \neq \emptyset$, then there exists exactly one contact point of the flow with L.*
(d) *If the flow has no contact points with L, then either $L^I = L$ and $L^O = \emptyset$, or $L^I = \emptyset$ and $L^O = L$.*

Poincaré maps of a homogeneous linear system

Consider the system (3) and two parallel straight lines in the plane L_+ and L_- which are symmetric with respect to origin. Notice that the lines L_+ and L_- split the plane in three regions S_0, S_+, and S_-, where S_0 is the open strip containing the origin and S_+ and S_- the half–planes bounded by L_+ and L_-, respectively. Moreover, we can define the following subsets,

$$
\begin{aligned}
Dom_{++} = \ & \big\{q \in L_+ : \exists t_q > 0 \text{ such that } e^{At_q}q \in L_+ \text{ and either} \\
& \ e^{At}q \subset S_+ \text{ or } e^{At}q \subset S_0 \ \forall t \in (0, t_q) \ \} \cup CP_+, \\
Dom_{+-} = \ & \big\{q \in L_+ : \exists t_q > 0 \text{ such that } e^{At_q}q \in L_- \text{ and} \\
& e^{At}q \subset S_0 \ \forall t \in (0, t_q) \ \}, \\
Dom_{--} = \ & \big\{q \in L_- : \exists t_q > 0 \ \text{ such that } e^{At_q}q \in L_- \text{ and either} \\
& e^{At}q \subset S_- \text{ or } e^{At}q \subset S_0 \ \forall t \in (0, t_q)\big\} \cup CP_-, \\
Dom_{-+} = \ & \big\{q \in L_- : \exists t_q > 0 \text{ such that } e^{At_q}q \in L_+, \text{ and} \\
& e^{At}q \subset S_0 \ \forall t \in (0, t_q)\big\},
\end{aligned}
\tag{4}
$$

where CP_+ and CP_- are either empty, sets or consist of contact points of the flow with the lines L_+ and L_-, respectively. We have the following result that is proved in Lemma 4.3.2 of [9].

Proposition 3. *Let $\dot{x} = Ax$ be a planar linear differential system with A non identically zero. Consider L_+ and L_- two parallel straight lines symmetric with respect to the origin and let Dom_{jk} be the sets defined in (4). Assume that for some $j, k \in \{+, -\}$ the set $Dom_{jk} \neq \emptyset$. Then there exists a unique contact point p^+ of the flow in L_+, and $p^- = -p^+$ is the unique contact point of the flow in L_-.*

Provided that $Dom_{jk} \neq \emptyset$, for $j, k \in \{+, -\}$, we define the Poincaré map $\Pi_{jk} : Dom_{jk} \subset L_j \longrightarrow L_k$ of the linear differential system (3) associated to the lines L_j and L_k as $\Pi_{jk}(q) = e^{At_q}q$. In what follows we present some results on the domains of definition of these Poincaré maps and necessary and sufficient conditions for the existence of these maps.

Notice that when $Dom_{++} \neq \emptyset$ and $Dom_{--} \neq \emptyset$, the contact points p_+ and p_- split L_+ and L_- into respective half–lines L_+^I, L_+^O, L_-^I, and L_-^O. We have the following results.

Proposition 4. *Consider a planar linear differential system $\dot{x} = Ax$ with A non identically zero. Set L_+ and L_- two parallel straight lines symmetric with respect to the origin and let Dom_{jk} be the sets defined in (4). Suppose that $Dom_{jk} \neq \emptyset$ for every $j, k \in \{+, -\}$.*

(i) *If* $\det A > 0$*, then*

$$\begin{aligned}
&\Pi_{++} : Dom_{++} \subset L_+^O \longrightarrow L_+^I,\\
&\Pi_{+-} : Dom_{+-} \subset L_+^I \longrightarrow L_-^O,\\
&\Pi_{--} : Dom_{--} \subset L_-^O \longrightarrow L_-^I,\\
&\Pi_{-+} : Dom_{-+} \subset L_-^I \longrightarrow L_+^O.
\end{aligned} \tag{5}$$

(ii) *If* $\det A < 0$*, then*

$$\begin{aligned}
&\Pi_{++} : Dom_{++} \subset L_+^I \longrightarrow L_+^O,\\
&\Pi_{+-} : Dom_{+-} \subset L_+^I \longrightarrow L_-^O,\\
&\Pi_{--} : Dom_{--} \subset L_-^I \longrightarrow L_-^O,\\
&\Pi_{-+} : Dom_{-+} \subset L_-^I \longrightarrow L_+^O.
\end{aligned} \tag{6}$$

(iii) *If* $\det A = 0$ *then* $Dom_{++} = \{p^+\}$, $Dom_{--} = \{p^-\}$, $\Pi_{--}(p^-) = \{p^-\}$, *and* $\Pi_{++}(p^+) = \{p^+\}$

$$\begin{aligned}
&\Pi_{+-} : Dom_{+-} \subset L_+^I \longrightarrow L_-^O,\\
&\Pi_{-+} : Dom_{-+} \subset L_-^I \longrightarrow L_+^O.
\end{aligned} \tag{7}$$

Proposition 5. *Let $\dot{x} = Ax$ be a planar linear differential system with A non identically zero. Consider L_+ and L_- two parallel straight lines symmetric with respect to the origin. The Poincaré maps which appear in the statement of Proposition 4 are defined if and only if the flow of the system has a unique contact point with L_+.*

The proofs of the two previous propositions can be found in Propositions 4.3.3 and 4.3.4 of [9], respectively.

2.1. **Qualitative behavior of the Poincaré maps.** In this subsection we analyze the qualitative behavior of the Poincaré maps defined by the flow of system (3) and associated to straight lines L_+ and L_-.

Assume that the Poincaré maps Π_{jk} are defined, for $j, k \in \{+, -\}$. By Proposition 5 there exists a unique contact point p of the flow of system (3) with the straight line L_+ and, therefore $-p$ is the unique contact point of the flow of system (3) with the straight line L_-. Thus $L_+ = \{p + \lambda\dot{p} : \lambda \in \mathbb{R}\}$ and $L_- = \{-p + \lambda\dot{p} : \lambda \in \mathbb{R}\}$.

Observe that, by Proposition 2(b), if $\det A > 0$ then

$$L_+^I = \{p + a\dot{p} : a \geq 0\}, \qquad L_+^O = \{p - a\dot{p} : a \geq 0\},$$
$$L_-^I = \{-p - a\dot{p} : a \geq 0\}, \quad L_-^O = \{-p + a\dot{p} : a \geq 0\}, \tag{8}$$

and if $\det A < 0$ then

$$L_+^I = \{p - a\dot{p} : a \geq 0\}, \qquad L_+^O = \{p + a\dot{p} : a \geq 0\},$$
$$L_-^I = \{-p + a\dot{p} : a \geq 0\}, \quad L_-^O = \{-p - a\dot{p} : a \geq 0\}. \tag{9}$$

It follows that given any point q on L_+ or on L_- we can associated a unique $a \geq 0$, called the *coordinate of* q.

Consider $q_1 \in L_j$ and $q_2 \in L_k$ such that $\Pi_{jk}(q_1) = q_2$, where $j, k \in \{+, -\}$. Let a_1 and a_2 be the coordinates of q_1 and q_2, respectively. Then we define the Poincaré maps π_{jk} by $\pi_{jk}(a_1) = a_2$. Thus to know the qualitative behavior of the map π_{jk} is equivalent to know the qualitative behavior of the Poincaré map Π_{jk}.

The next results, proved in Lemma 4.3.5 and Proposition 4.3.7 of [9], respectively, provide some properties of the Poincaré maps π_{jk}.

Proposition 6. *Consider the linear differential system* (3) *and let* L_+ *and* L_- *two parallel straight lines which are symmetric with the respect to the origin. Suppose that the Poincaré maps* π_{jk} *with* $j, k \in \{+, -\}$ *are defined. Then*

(a) *the maps* π_{++} *and* π_{--} *coincides.*
(b) *the maps* π_{+-} *and* π_{-+} *coincides.*
(c) *the Poincaré maps* π^*_{jk} *associated to the flow of the system* $\dot{x} = -Ax$ *and to the lines* L_+ *and* L_- *are defined, and they satisfy* $\pi^*_{jk} = \pi^{-1}_{jk}$.
(d) π_{jk} *and* π^{-1}_{jk} *are analytic functions.*

Proposition 7. *Consider the linear differential system* (3) *and let* L_+ *and* L_- *two parallel straight lines which are symmetric with the respect to the origin. Assume that the Poincaré maps* π_{jk} *with* $j, k \in \{+, -\}$ *are defined. If* $M \in GL(\mathbb{R}^2)$*, then the maps* π_{jk} *are invariant under the change of coordinates* $y = Mx$.

Assume that the maps π_{++} and π_{+-} are defined. Since these maps are invariant under linear changes of coordinates, see Proposition 7, in what follows we consider A given in its real Jordan normal form. Moreover we denote the eigenvalues of A by λ_1 and λ_2. In what follows we characterize the behavior of the Poincaré map π_{++} depending on t and d. Moreover we characterize the behavior of composition $\widetilde{\pi}_{++} = \pi_{-+} \circ \pi_{--} \circ \pi_{+-}$ when $d > 0$ and $t^2 - 4d < 0$.

Saddle: $d < 0$.

The next two results are proved in Proposition 4.4.15 and Corollary 4.4.16 of [9], respectively.

Proposition 8. *Assume that $d < 0$, $t \geq 0$. Then the eigenvalues of the matrix A satisfy $\lambda_1 > 0 > \lambda_2$. Let π_{++} be the Poincaré map defined by the flow of the linear system $\dot{x} = Ax$ and associated to the parallel straight lines L_+ and L_- symmetric with respect to the origin. If $t = 0$, then π_{++} is the identity map on the interval $\left[0, \lambda_1^{-1}\right)$, and if $t > 0$, then*

(a) $\pi_{++} : \left[0, \lambda_1^{-1}\right) \to [0, |\lambda_2|^{-1})$, $\pi_{++}(0) = 0$, $\lim\limits_{a \nearrow \lambda_1^{-1}} \pi_{++}(a) = |\lambda_2|^{-1}$, *and* $\pi_{++}(a) > a$ *in* $(0, \lambda_1^{-1})$.
(b) *if* $a \in (0, \lambda_1^{-1})$, *then* $\pi'_{++}(a) > 1$. *Furthermore,* $\lim\limits_{a \searrow 0} \pi'_{++}(a) = 1$ *and* $\lim\limits_{a \nearrow \lambda_1^{-1}} \pi'_{++}(a) = +\infty$.
(c) *if* $a \in (0, \lambda_1^{-1})$, *then* $\pi''_{++}(a) > 0$.
(d) *the graph of* π_{++} *has a vertical asymptote at* $a = \lambda_1^{-1}$.
(e) π_{++} *is implicitly defined by the equation*

(10)
$$\left(\frac{2 + \pi_{++}(a)(t - \sqrt{t^2 - 4d})}{2 - a(t - \sqrt{t^2 - 4d})}\right)^{\dfrac{t + \sqrt{t^2 - 4d}}{t - \sqrt{t^2 - 4d}}} = \frac{2 + \pi_{++}(a)(t + \sqrt{t^2 - 4d})}{2 - a(t + \sqrt{t^2 - 4d})}.$$

(f) *The qualitative behavior of the graph of* π_{++} *is represented in Figure 1-(a).*

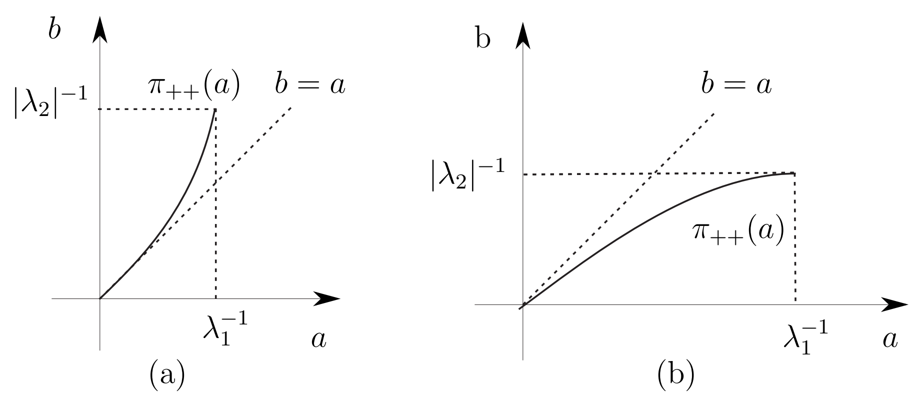

FIGURE 1. Qualitative behavior of the Poincaré map π_{++}; (a) $t > 0$ and (b) $t < 0$.

Proposition 9. *Assume that $d < 0$, $t < 0$. Then the eigenvalues of the matrix A satisfy $\lambda_1 > 0 > \lambda_2$. Let π_{++} be the Poincaré map defined by the flow of the linear system $\dot{x} = Ax$ and associated to the parallel straight lines L_+ and L_-*

symmetric with respect to the origin. If $t = 0$, then π_{++} is the identity map on the interval $\left[0, \lambda_1^{-1}\right)$, and if $t > 0$, then

(a) $\pi_{++} : \left[0, \lambda_1^{-1}\right) \to [0, |\lambda_2|^{-1})$, $\pi_{++}(0) = 0$, $\lim_{a \nearrow \lambda_1^{-1}} \pi_{++}(a) = |\lambda_2|^{-1}$, *and* $\pi_{++}(a) < a$ *in* $(0, \lambda_1^{-1})$.
(b) *if* $a \in (0, \lambda_1^{-1})$, *then* $0 < \pi'_{++}(a) < 1$ *and* $\lim_{a \searrow 0} \pi'_{++}(a) = 1$.
(c) *if* $a \in (0, \lambda_1^{-1})$, *then* $\pi''_{++}(a) < 0$.
(d) *the graph of* π_{++} *has a horizontal asymptote at* $a = \lambda_1^{-1}$ *when* a *tends to* $+\infty$.
(e) π_{++} *is implicitly defined by equation* (10).
(f) *The qualitative behavior of the graph of* π_{++} *is represented in Figure 1-(b).*

Diagonal node: $d > 0$ and $t^2 - 4d > 0$.

The following results are proved in Proposition 4.4.1 and Corollary 4.4.2 of [9], respectively.

Proposition 10. *Assume that $d > 0$, $t > 0$, and $t^2 - 4d > 0$. Then the eigenvalues of the matrix A satisfy $\lambda_1 > \lambda_2 > 0$. Let π_{++} be the Poincaré map defined by the flow of the linear system $\dot{x} = Ax$ and associated to the parallel straight lines L_+ and L_- symmetric with respect to the origin. Then*

(a) $\pi_{++} : \left[0, \lambda_1^{-1}\right) \to [0, +\infty)$, $\pi_{++}(0) = 0$, $\lim_{a \nearrow \lambda_1^{-1}} \pi_{++}(a) = +\infty$, *and* $\pi_{++}(a) > a$ *in* $(0, \lambda_1^{-1})$.
(b) *if* $a \in (0, \lambda_1^{-1})$, *then* $\pi'_{++}(a) > 1$ *and* $\lim_{a \searrow 0} \pi'_{++}(a) = 1$.
(c) *if* $a \in (0, \lambda_1^{-1})$, *then* $\pi''_{++}(a) > 0$.
(d) *the graph of* π_{++} *has a vertical asymptote at* $a = \lambda_1^{-1}$.
(e) π_{++} *is implicitly defined by the equation*

(11)
$$\left(\frac{2 + \pi_{++}(a)(t - \sqrt{t^2 - 4d})}{2 - a(t - \sqrt{t^2 - 4d})}\right)^{\frac{t + \sqrt{t^2 - 4d}}{t - \sqrt{t^2 - 4d}}} = \frac{2 + \pi_{++}(a)(t + \sqrt{t^2 - 4d})}{2 - a(t + \sqrt{t^2 - 4d})}.$$

(f) *The qualitative behavior of the graph of* π_{++} *is represented in Figure 2-(a).*

Proposition 11. *Assume that $d > 0$, $t < 0$, and $t^2 - 4d > 0$. Then the eigenvalues of the matrix A satisfy $\lambda_2 < \lambda_1 < 0$. Let π_{++} be the Poincaré map defined by the flow of the linear system $\dot{x} = Ax$ and associated to the parallel straight lines L_+ and L_- symmetric with respect to the origin. Then*

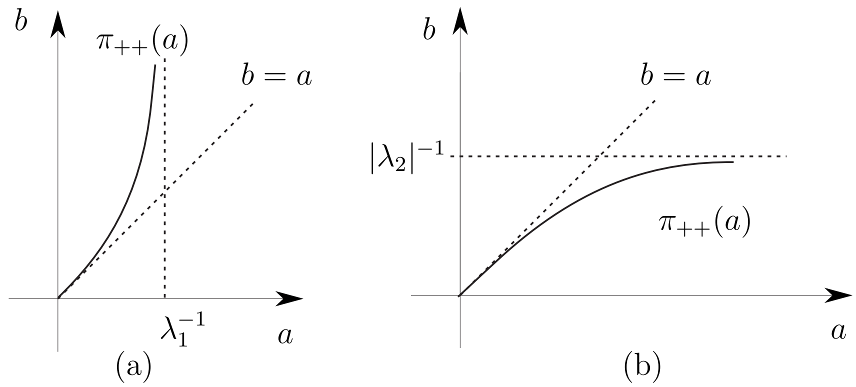

FIGURE 2. Qualitative behavior of the Poincaré map π_{++}; (a) $t > 0$ and (b) $t < 0$.

(a) $\pi_{++} : [0, +\infty) \to [0, |\lambda_2|^{-1})$, $\pi_{++}(0) = 0$, $\lim_{a \nearrow +\infty} \pi_{++}(a) = |\lambda_2|^{-1}$, *and* $\pi_{++}(a) < a$ *in* $(0, |\lambda_2|^{-1})$.
(b) *if* $a \in (0, +\infty)$, *then* $0 < \pi'_{++}(a) < 1$ *and* $\lim_{a \searrow 0} \pi'_{++}(a) = 1$.
(c) *if* $a \in (0, +\infty)$, *then* $\pi''_{++}(a) < 0$.
(d) *the graph of* π_{++} *has a horizontal asymptote at* $b = |\lambda_2|^{-1}$.
(e) π_{++} *is implicitly defined by equation* (11).
(f) *The qualitative behavior of the graph of* π_{++} *is represented in Figure 2-(b).*

Non–diagonal node: $d > 0$ and $t^2 - 4d = 0$.

The next two results are proved in Proposition 4.4.6 and Corollary 4.4.7 of [9], respectively.

Proposition 12. *Assume that* $d > 0$, $t > 0$, *and* $t^2 - 4d = 0$. *Then the eigenvalues of the matrix* A *satisfy* $\lambda_1 = \lambda_2 = \lambda$. *Suppose that* A *does not diagonalizable and let* π_{++} *be the Poincaré map defined by the flow of the linear system* $\dot{x} = Ax$ *and associated to the parallel straight lines* L_+ *and* L_- *symmetric with respect to the origin. Then*

(a) $\pi_{++} : [0, \lambda^{-1}) \to [0, +\infty)$, $\pi_{++}(0) = 0$, $\lim_{a \nearrow \lambda^{-1}} \pi_{++}(a) = +\infty$, *and* $\pi_{++}(a) > a$ *in the interval* $(0, \lambda^{-1})$.
(b) *if* $a \in (0, \lambda^{-1})$, *then* $\pi'_{++}(a) > 1$ *and* $\lim_{a \searrow 0} \pi'_{++}(a) = 1$.
(c) *if* $a \in (0, \lambda_1^{-1})$, *then* $\pi''_{++}(a) > 0$.
(d) *the graph of* π_{++} *has a vertical asymptote at* $a = \lambda^{-1}$.

(e) *π_{++} is implicitly defined by the equation*

$$\frac{t\pi_{++}(a)+2}{2-at} = e^{\frac{2t(\pi_{++}(a)+a)}{(t\pi_{++}(a)+2)(2-at)}}. \tag{12}$$

(f) *The qualitative behavior of the graph of π_{++} is represented in Figure 2-(a).*

Proposition 13. *Suppose that $d>0$, $t<0$, and $t^2-4d=0$. Then the eigenvalues of the matrix A satisfy $\lambda_2=\lambda_1=\lambda<0$. Assume that A does not diagonalize and let π_{++} be the Poincaré map defined by the flow of the linear system $\dot{x}=Ax$ and associated to the parallel straight lines L_+ and L_- symmetric with respect to the origin. Then*

(a) *$\pi_{++}:[0,+\infty)\to[0,|\lambda|^{-1})$, $\pi_{++}(0)=0$, $\lim_{a\nearrow+\infty}\pi_{++}(a)=|\lambda|^{-1}$, and $\pi_{++}(a)<a$ on the interval $(0,+\infty)$.*
(b) *if $a\in(0,+\infty)$, then $0<\pi'_{++}(a)<1$ and $\lim_{a\searrow 0}\pi'_{++}(a)=1$.*
(c) *if $a\in(0,+\infty)$, then $\pi''_{++}(a)<0$ in $(0,+\infty)$.*
(d) *when a tends to $+\infty$, the graph of π_{++} has a horizontal asymptote at $b=|\lambda|^{-1}$.*
(e) *π_{++} is implicitly defined by equation (12).*
(f) *The qualitative behavior of the graph of π_{++} is represented in Figure 2-(b).*

Degenerate node: $d=0$.

In this case A has one null eigenvalue and other equal to t. Hence the matrix A has two different real Jordan normal forms. One being for $t\neq 0$ and the other one for $t=0$. In any case the behavior of the Poincaré map is defined only in a contact point, see Proposition 4-(iii). Thus the map π_{++} is defined only at zero and $\pi_{++}(0)=0$.

Center and focus: $d>0$ and $t^2-4d<0$.

The results for the centers and foci are proved on sections 4.4 and 4.5 of [9].

The following results are Proposition 4.4.11 and Corollary 4.4.12 of [9], respectively.

Proposition 14. *Suppose that $d>0$, $t\geq 0$, and $t^2-4d<0$. Then the matrix A has a pair of complex eigenvalues. Let π_{++} be the Poincaré map defined by the flow of the linear system $\dot{x}=Ax$ and associated to the parallel straight lines L_+ and L_- symmetric with respect to the origin. If $t=0$ then π_{++} is the identity in $[0,+\infty)$, and if $t>0$ then*

(a) $\pi_{++} : [0,+\infty) \to [0,+\infty)$, $\pi_{++}(0) = 0$, $\lim_{a\nearrow+\infty} \pi_{++}(a) = +\infty$, *and* $\pi_{++}(a) > a$ *on the interval* $(0,+\infty)$.
(b) *if* $a \in (0,+\infty)$, *then* $\pi'_{++}(a) > 1$ *and* $\lim_{a\searrow 0} \pi'_{++}(a) = 1$.
(c) *if* $a \in (0,+\infty)$, *then* $\pi''_{++}(a) > 0$.
(d) *when* a *tends to* $+\infty$, *the graph of* π_{++} *has an asymptote at* $b = ae^{\gamma\pi} - t(1+e^{\gamma\pi})/d$, *where* $\gamma = t/\sqrt{4d-t^2}$.
(e) π_{++} *is implicitly defined by the equation*

$$\frac{1+t\pi_{++}(a)+d\pi_{++}(a)^2}{1-ta+da^2} = e^{2\gamma\arctan\left(\frac{(a+\pi_{++}(a))\beta}{(\pi_{++}(a)-a)\alpha+1-ad\pi_{++}(a)}\right)}. \tag{13}$$

(f) *The qualitative behavior of the graph of* π_{++} *is represented in Figure 3-(a).*

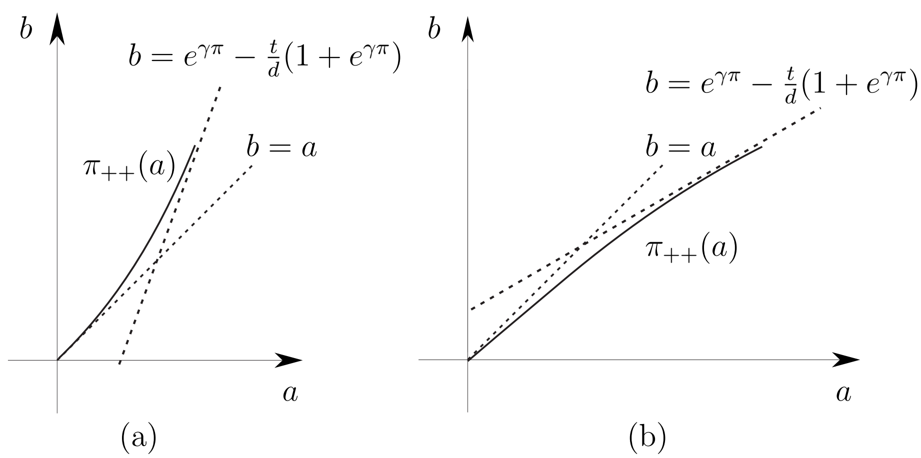

FIGURE 3. Qualitative behavior of the Poincaré map π_{++}; (*a*) $t > 0$ and (*b*) $t < 0$.

Proposition 15. *Suppose that* $d > 0$, $t < 0$, *and* $t^2 - 4d < 0$. *Then the matrix* A *has a pair of complex eigenvalues. Let* π_{++} *be the Poincaré map defined by the flow of the linear system* $\dot{x} = Ax$ *and associated to the parallel straight lines* L_+ *and* L_- *symmetric with respect to the origin. Then*

(a) $\pi_{++} : [0,+\infty) \to [0,+\infty)$, $\pi_{++}(0) = 0$, $\lim_{a\nearrow+\infty} \pi_{++}(a) = +\infty$, *and* $\pi_{++}(a) < a$ *on the interval* $(0,+\infty)$.
(b) *if* $a \in (0,+\infty)$, *then* $0 < \pi'_{++}(a) < 1$ *and* $\lim_{a\nearrow 0} \pi'_{++}(a) = 1$.
(c) *if* $a \in (0,+\infty)$, *then* $\pi''_{++}(a) < 0$.
(d) *when* a *tends to* $+\infty$, *the straight line* $b = ae^{\gamma\pi} - t(1+e^{\gamma\pi})/d$ *is an asymptote of the graph of* π_{++}, *where* $\gamma = t/\sqrt{4d-t^2}$.

(e) π_{++} *is implicitly defined by equation* (13).
(f) *The qualitative behavior of the graph of* π_{++} *is represented in Figure 3-(b).*

Next results characterizes the behavior of $\widetilde{\pi}_{++} = \pi_{+-} \circ \pi_{--} \circ \pi_{-+}$. The proof can be found in Proposition 4.5.7 and Corollary 4.5.8 of [9], respectively.

Proposition 16. *Consider a matrix* $A \in GL(\mathbb{R}^2)$ *such that* $d > 0$, $t \geq 0$, *and* $t^2 - 4d < 0$, *and a vector* $b \in \mathbb{R}^2 \setminus \{0\}$. *Let* $\widetilde{\pi}_{++}$ *be the Poincaré map defined by the flow of the linear system* $\dot{x} = Ax + b$ *and associated to the parallel straight lines* L_+ *and* L_- *symmetric with respect to the origin. If* $t = 0$ *then* $\widetilde{\pi}_{++}$ *is the identity in* $[0, +\infty)$. *On the other hand if* $t > 0$ *then*

(a) *there exist a value* $b^* > 0$ *such that* $\widetilde{\pi}_{++} : [0, +\infty) \to [b^*, +\infty)$ *and* $\widetilde{\pi}_{++}(0) = b^*$. *Furthermore,* $\lim\limits_{a \nearrow +\infty} \widetilde{\pi}_{++}(a) = +\infty$ *and* $\widetilde{\pi}_{++}(a) > a$ *in* $(0, +\infty)$.
(b) *if* $a \in (0, +\infty)$, *then* $\widetilde{\pi}'_{++}(a) > 0$ *and* $\lim\limits_{a \searrow 0} \widetilde{\pi}'_{++}(a) = 0$.
(c) *if* $a \in (0, +\infty)$, *then* $\widetilde{\pi}''_{++}(a) > 0$.
(d) *the graph of* $\widetilde{\pi}_{++}$ *has an asymptote at* $b = ae^{\gamma\pi} + t(1 + e^{\gamma\pi})/d$ *when* a *tends to* $+\infty$, *where* $\gamma = t/\sqrt{4d - t^2}$.
(e) $\widetilde{\pi}_{++}$ *is implicitly defined by the equation*

$$\frac{1 - t\widetilde{\pi}_{++}(a) + d\widetilde{\pi}_{++}(a)^2}{1 + ta + da^2} = e^{2\gamma \arctan\left(\frac{(a + \widetilde{\pi}_{++}(a))\beta}{(\widetilde{\pi}_{++}(a) - a)\alpha - 1 + ad\widetilde{\pi}_{++}(a)}\right)}. \tag{14}$$

(f) *The qualitative behavior of the graph of map* $\widetilde{\pi}_{++}$ *is represented in Figure 4–(a).*

Proposition 17. *Consider a matrix* $A \in GL(\mathbb{R}^2)$ *such that* $d > 0$, $t < 0$, *and* $t^2 - 4d < 0$, *and a vector* $b \in \mathbb{R}^2 \setminus \{0\}$. *Let* $\widetilde{\pi}_{++}$ *be the Poincaré map defined by the flow of the linear system* $\dot{x} = Ax + b$ *and associated to the parallel straight lines* L_+ *and* L_- *symmetric with respect to the origin.*

(a) *there exist a value* $a^* > 0$ *such that* $\widetilde{\pi}_{++} : [a^*, +\infty) \to [0, +\infty)$ *and* $\widetilde{\pi}_{++}(a^*) = 0$. *Furthermore,* $\lim\limits_{a \nearrow +\infty} \widetilde{\pi}_{++}(a) = +\infty$ *and* $\widetilde{\pi}_{++}(a) < a$ *in* $(a^*, +\infty)$.
(b) *if* $a \in (a^*, +\infty)$, *then* $\widetilde{\pi}'_{++}(a) > 0$ *and* $\lim\limits_{a \searrow a^*} \widetilde{\pi}'_{++}(a) = +\infty$.
(c) *if* $a \in (a^*, +\infty)$, *then* $\widetilde{\pi}''_{++}(a) < 0$.
(d) *the graph of* $\widetilde{\pi}_{++}$ *has an asymptote at* $b = ae^{\gamma\pi} + t(1 + e^{\gamma\pi})/d$ *when* a *tends to* $+\infty$, *where* $\gamma = t/\sqrt{4d - t^2}$.
(e) $\widetilde{\pi}_{++}$ *is implicitly defined by equation* (14).

(f) *The qualitative behavior of the graph of map $\widetilde{\pi}_{++}$ is represented in Figure 4–(b).*

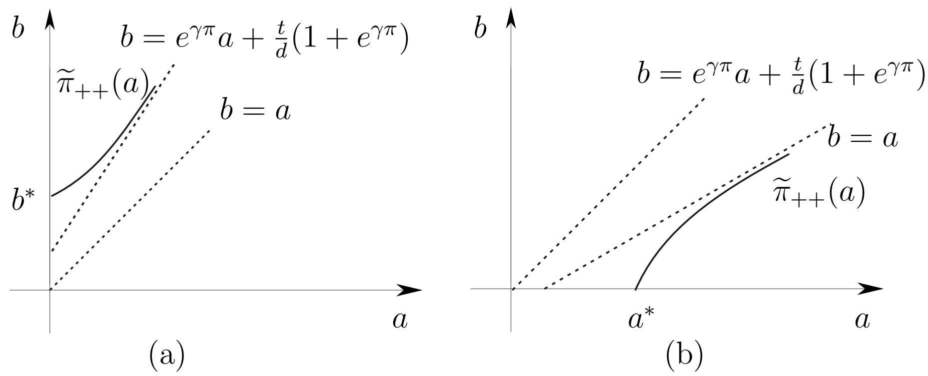

FIGURE 4. Qualitative behavior of the Poincaré map $\widetilde{\pi}_{++}$; (a) $t > 0$ and (b) $t < 0$.

3. Return map of discontinuous differential system

Consider the following discontinuous piecewise linear differential system

$$\dot{x} = \begin{cases} Ax + b, & \text{if} \quad x_1 > 1, \\ Ax - b, & \text{if} \quad x_1 < 1, \end{cases} \tag{15}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ and $b \in \mathbb{R}^2 \setminus \{0\}$. The discontinuity of this system is the straight line $L = \{(x_1, x_2) \in \mathbb{R}^2, x_1 = 1\}$. Assume that the singularities of system (15) not belong to L. If $d = 0$. In this section we analyze the Poincaré maps defined by the flow of system (15) and associated to the straight line L and we define the return map for this system.

In what follows we denote by n the unit orthogonal vector to the line L which is oriented in the direction opposite to the origin, and $S_+ = \{(x_1, x_2) \in \mathbb{R}^2, x_1 > 1\}$ and $S_- = \{(x_1, x_2) \in \mathbb{R}^2, x_1 < 1\}$ denote the half–planes bounded by L.

In the study of the Poincaré maps associated to the flow of the linear differential systems $\dot{x} = Ax \pm b$ with respect to the straight line L it by we denote $L_\pm$ to specify what system we are considering.

3.1. Poincaré maps of linear differential system $\dot{x} = Ax + b$.

In this subsection we study the Poincaré maps defined by the flow of the linear differential system

$$\dot{x} = Ax + b \tag{16}$$

and associated to the straight line L_+. Since L_+ does not pass through the origin it can be split into the subsets $L_+^I = \{q \in L : n^T \dot{q} \leq 0\}$ and $L_+^O = \{q \in L : n^T \dot{q} \geq 0\}$, where $\dot{q} = Aq + b$. Then the set $CP_+ = L_+^I \cap L_+^O$ consists of contact points of the flow of system (16) with L_+.

Set $\Phi_+(t, q)$ the flow of system (16) such that $\Phi_+(0, q) = q$. Define in L_+ the subset

$$
\begin{aligned}
Dom_{++} = \ & \left\{q \in L_+^O : \exists t_q \geq 0, \Phi_+(t_q, q) \in L_+^I \text{ and } \Phi_+(t, q) \subset S_+ \forall t \in (0, t_q)\right\} \\
& \cup CP_+.
\end{aligned}
$$

Assuming that $Dom_{++} \neq \emptyset$ we define the Poincaré map of the linear differential system (16) associated to the straight line L_+ by

$$
\begin{aligned}
\Pi_+ : \quad Dom_{++} \subset L_+^O & \longrightarrow L_+^I \\
q & \longmapsto \Phi_+(t_q, q).
\end{aligned} \tag{17}
$$

Suppose that the flow of system (16) has a unique contact point p with L_+. We have that $\Pi_+(p) = p$ because $p \in L_+^I$ and $p \in L_+^O$.

Let $e_+ = -A^{-1}b$ be the singularity of system (16). Applying the translation $y = x - e_+$ we rewrite system (16) as

$$
\dot{y} = Ay, \tag{18}
$$

and the straight line L_+ is transformed into the straight line L_+^*. If the Poincaré map Π_+, given in (17), is defined then it induces a Poincaré map Π_{++}^* associated to the flow of system (18) and to the straight line L_+^*. Clearly the converse statement is also true and, therefore the behavior of the map Π_+ can be obtained from the behavior of the map Π_{++}^*. We have the following result that are proved in Propositions 4.5.1 of [9].

Proposition 18. *Consider L_+ a straight line in the plane not passing through the origin and such that $e_+ = -A^{-1}b \notin L_+$. The Poincaré map Π_+ associated to the flow of system $\dot{x} = Ax + b$ and to the straight line L_+ is defined if and only if there exists a unique contact point p^+ of the flow with L_+, and L_+^I and L_+^O are non–empty half–lines such that $L_+ = L_+^I \cup L_+^O$.*

Since $e_+ \notin L_+$ we have that L_+^* does not pass through the origin, and we can define the symmetric line $-L_+^* = L_-^* = \{-q; q \in L_+^*\}$. Therefore the results of section 2 can be applied to study the Poincaré maps of system (18) associated to the straight lines L_+^* and L_-^* symmetric with respect to the origin. Set Π_{++}^* the Poincaré map induced by translation $y = x - e_+$, and Π_{jk} with $j, k \in \{+, -\}$ the Poincaré maps defined for homogeneous case, see Proposition 4.

Proposition 19. *Consider L_+ a straight line in the plane not passing through the origin and such that $e_+ = -A^{-1}b \notin L_+$. Assume that the Poincaré map Π_+ associated to the flow of system $\dot{x} = Ax + b$ and to the straight line L_+ is defined. Let Π^*_{++} be the Poincaré map of system* (18)*.*

(i) *Suppose that $e_+ \in S_-$.*
   - *If* $\det A > 0$*, then Π^*_{++} is the Poincaré map Π_{++}.*
   - *If* $\det A < 0$*, then Π^*_{++} is trivial, i.e. the map Π^*_{++} is only defined in a contact point of the L^*_+.*

(ii) *Suppose that $e_+ \in S_+$.*
   - *If* $\det A > 0$ *and $t^2 - 4d \geq 0$, then Π^*_{++} is trivial.*
   - *If* $\det A > 0$ *and $t^2 - 4d < 0$, then Π^*_{++} coincides with the composition $\Pi_{-+} \circ \Pi_{--} \circ \Pi_{+-}$.*
   - *Assume that* $\det A < 0$*. Then Π^*_{++} is the Poincaré map Π_{++}.*

Proof. (i) We have that the translation $y = x - e_+$ transforms L^I_+ into L^{*I}_+ and L^O_+ into L^{*O}_+ because $e_+ \in S_-$, see Figure 5-(a). Furthermore the domain Dom_{++} of Π^*_{++} is contained in L^{*O}_+.

Assuming that $\det A > 0$, we have that the Poincaré map defined by a linear flow having the domain contained in L^{*O}_+ is the map Π_{++} defined in subsection 2, see Proposition 4(i). On the other hand if $\det A < 0$ then, by Proposition 4(ii), the domain of the map Π^*_{++} is contained in L^{*I}_+ and, therefore the domain of definition of the map Π^*_{++} is contained in the intersection $L^{*I}_+ \cap L^{*O}_+$. That is the map Π^*_{++} is defined only in the contact point. Consequently the behavior of Π^*_{++} is trivial.

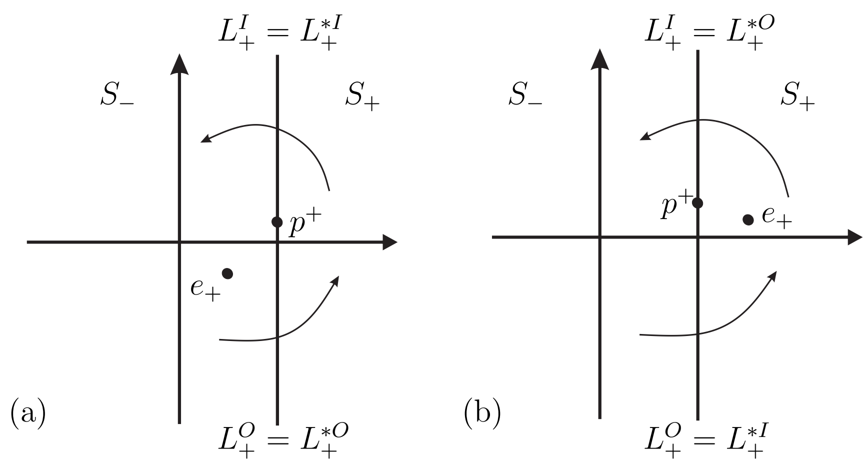

FIGURE 5. Relation between the half–lines L^O_+, L^I_+, L^{*O}_+, and L^{*I}_+ depending on (a) $e_+ \in S_-$, (b) $e_+ \in S_+$.

(ii) By hypothesis we have that $e_+ \in S_+$, thus the translation $y = x - e_+$ transforms L_+^I into L_+^{*O} and L_+^O into L_+^{*I}, see figure 5-(b).

Assuming that $\det A > 0$ Proposition 4(i) implies that there is no Poincaré map Π_{++}, associated to the flow of a homogeneous linear system and to two parallel straight lines L_+ and L_- symmetric with respect to the origin, defined on L_+^I with the image contained on L_+^O. This implies that either the behavior of Π_{++}^* is trivial or $\Pi_{++}^* = \Pi_{-+} \circ \Pi_{--} \circ \Pi_{+-}$. Notice that in the last case the orbits have to surround the origin. It follows that Π_{++}^* is trivial for $t^2 - 4d \geq 0$ and $\Pi_{++}^* = \Pi_{-+} \circ \Pi_{--} \circ \Pi_{+-}$ for $t^2 - 4d < 0$.

Assuming that $\det A < 0$ we have that Π_{++}^* coincides either with the Poincaré map Π_{++} or with the composition $\Pi_{-+} \circ \Pi_{--} \circ \Pi_{+-}$, see Proposition 4(ii). But, in the last case we need that the orbits surround the origin and this is a contradiction with $\det A < 0$. Therefore, Π_{++}^* is the map Π_{++}. □

By Proposition 19 the behavior of the map Π_{++}^* depends on whether $e_+ \in S_-$, or $e_+ \in S_+$ and $t^2 - 4d < 0$. In order to distinguish between these situations we will denote $\widetilde{\Pi}_{++}$ the map $\Pi_{-+} \circ \Pi_{--} \circ \Pi_{+-}$. Consequently we reduce the study of the Poincaré map Π_+ associated to the flow of system (16) and to the straight line L_+ to study the Poincaré maps Π_{++} and $\widetilde{\Pi}_{++}$ defined by the flow of the homogeneous linear system (18) and associated to the lines L_+^* and L_-^*. Therefore, to know the qualitative behavior of the Poincaré map Π_+ defined by system (16) is equivalent to know the behavior of Poincaré maps π_{++} and $\widetilde{\pi}_{++} = \pi_{-+} \circ \pi_{--} \circ \pi_{+-}$.

3.2. **The Poincaré maps of the linear differential system $\dot{x} = Ax - b$.** In this subsection we consider the linear differential equation

$$\dot{x} = Ax - b, \tag{19}$$

which is defined in the region S_- and we study the Poincaré map defined by the flow of the system and associated to the straight line $L = L_-$.

Notice that L_- does not pass through the origin, thus we split it into the the subsets $L_-^I = \{q \in L : n^T \dot{q} \leq 0\}$ and $L_-^O = \{q \in L : n^T \dot{q} \geq 0\}$, where $\dot{q} = Aq - b$. The set $CP_- = L_-^I \cap L_-^O$ consists of contact points of the flow of system (19) with L_-.

Set $\Phi_-(t, q)$ the flow of system (19) such that $\Phi_-(0, q) = q$. Then we define in L_- the following subset

$$\begin{aligned} Dom_{--} = \ & \big\{q \in L_-^I; \exists t_q \geq 0, \Phi_-(t_q, q) \in L_-^O \text{ and } \Phi_-(t, q) \subset S_- \forall t \in (0, t_q)\big\} \\ & \cup CP_-. \end{aligned}$$

Suppose that $Dom_{--} \neq \emptyset$ then we can define the Poincaré map of the linear differential system (19) associated to the straight line L_- by

$$\begin{array}{rrcl} \Pi_- : & Dom_{--} \subset L_-^I & \longrightarrow & L_-^O \\ & q & \longmapsto & \Phi_-(t_q, q). \end{array} \tag{20}$$

Suppose that the flow of system (19) has a unique contact point p with L_-. We have that $\Pi_-(p) = p$ because $p \in L_-^I$ and $p \in L_-^O$.

Let $e_- = A^{-1}b$ be the singularity of system (19). Notice that the translation $y = x - e_-$ transform system (19) in

$$\dot{y} = Ay, \tag{21}$$

and the line L_- is transformed into the straight line L_+^*. Therefore if the Poincaré map Π_-, given in (20), is defined then it induces a Poincaré map Π_{++}^* associated to the flow of system (21) and to the straight line L_+^*. Of course the converse statement is also true. Moreover, the behavior of the map Π_- can be obtained from the behavior of the map Π_{++}^*. We have the following result that can be proved in a similar way to proof of Proposition 18.

Proposition 20. *Consider L_- a straight line in the plane not passing through the origin and such that $e_- = A^{-1}b \notin L_-$. The Poincaré map Π_- associated to the flow of the system $\dot{x} = Ax - b$ and to the straight line L_- is defined if and only if there exists a unique contact point p^- of the flow in L_-, and L_-^I and L_-^O are non–empty half–lines such that $L_- = L_-^I \cup L_-^O$.*

Since $e_- \notin L_-$ then L_+^* does not pass through the origin and we can define the symmetric line $L_-^* = \{-q; q \in L_+^*\}$. Therefore the results of section 2 can be applied to study the Poincaré maps Π_{jk}, with $j, k \in \{+, -\}$, associated to the system (21) and to the straight lines L_+^* and L_-^* symmetric with respect to the origin.

Set Π_{++}^* the Poincaré map induced by translation $y = x - e_-$, we have the following proposition.

Proposition 21. *Consider L_- a straight line in the plane not passing through the origin and such that $e_- = A^{-1}b \notin L_-$. Assume that the Poincaré map Π_- associated to the flow of the system $\dot{x} = Ax - b$ and to the straight line L_- is defined. Let Π_{++}^* be the Poincaré map induced by the translation $y = x - e_-$.*

(i) *Suppose that $e_- \in S_-$.*
   - *If* $\det A > 0$ *and $t^2 - 4d < 0$, then Π_{++}^* coincides with the composition $\Pi_{-+} \circ \Pi_{--} \circ \Pi_{+-}$.*
   - *If* $\det A > 0$ *and $t^2 - 4d \geq 0$, then Π_{++}^* is trivial.*
   - *Suppose that* $\det A < 0$. *Then Π_{--}^* is the Poincaré map Π_{++}.*

(ii) *Suppose that* $e_- \in S_+$.
- *If* $\det A > 0$ *then* Π^*_{++} *is the Poincaré map* Π_{++}.
- *If* $\det A < 0$, *then* Π^*_{++} *is trivial.*

Proof. In case (i) the translation $y = x - e_-$ transforms L^I_- into L^{*I}_+ and L^O_- into L^{*O}_+, and in case (ii) the translation $y = x - e_-$ transforms L^I_- into L^{*O}_+ and L^O_- into L^{*I}_+. The rest of proof is similar to proof of Proposition 19. □

By Proposition 21 the behavior of the map Π^*_{++} depends on whether $e_- \in S_-$, or $e_- \in S_+$. In order to distinguish between these situation we will denote the map $\Pi_{-+} \circ \Pi_{--} \circ \Pi_{+-}$ by $\widetilde{\Pi}_{++}$.

We reduce the study of the Poincaré map Π_- associated to the flow of system (19) and to the straight line L_- to study the Poincaré maps Π_{++} and $\widetilde{\Pi}_{++}$ defined by the flow of the homogeneous linear system (21) and associated to two parallel lines L^*_- and L^*_+. Then to know the qualitative behavior of the Poincaré map Π_-, defined by system (19), is equivalent to know the behavior of of Poincaré maps π_{++} and $\widetilde{\pi}_{++} = \pi_{-+} \circ \pi_{--} \circ \pi_{+-}$.

Remark 22. *By Proposition 6, we have that* $\pi_{++} = \pi_{--}$ *and* $\widetilde{\pi}_{++} = \widetilde{\pi}_{--}$. *In order to not confuse the Poincaré maps* π_{jk}, *where* $j, k \in \{+, -\}$, *associated to system* (19), *with the maps studied on the previous section, we will use the notation* π_{--} *and* $\widetilde{\pi}_{--}$.

3.3. **Return Map of the discontinuous system** (15)**.** In this subsection we describe the return map Π defined by the flow of the discontinuous piecewise linear differential system (15) and associated to the straight line L as the composition of the Poincaré maps Π_+ and Π_- defined in subsections 3.1 and 3.2.

Notice that the half–lines L^I_+, L^O_+, L^I_-, and L^O_- are contained in the straight line L. Furthermore, when the Poincaré map Π_+ is defined we have that its domain is $Dom_{++} \subset L^O_+$ and its image is contained in L^I_+. Moreover, when the Poincaré map Π_- is defined we have that its domain is $Dom_{--} \subset L^I_-$ and its image is contained in L^O_-. Then, consider the subset $L^I_+ \cap L^I_-$ and $L^O_+ \cap L^O_-$ of the straight line L and define

$$\begin{aligned} Dom = \; & \big\{ q \in L^O_+ \cap L^O_-; \exists t_q > 0, \Phi(t_q, q) \in L^O_+ \cap L^O_- \text{ and } \forall t \in (0, t_q), \\ & \Phi(t, q) \notin L^O_+ \cap L^O_- \big\}, \end{aligned}$$

where $\Phi(t, q)$ is the solution of the discontinuous system (15). Observe that if $d = 0$ then $Dom = \emptyset$ and therefore the return map is not defined. In what follows we assume that $A \in GL(\mathbb{R}^2)$.

When $Dom \neq \emptyset$ we define the *return map* associated to the flow of the discontinuous system(15) and to the straight line L by

$$
\begin{aligned}
\Pi: \quad Dom \subset L_+^O \cap L_-^O &\longrightarrow L_+^O \cap L_-^O \\
q &\longmapsto \Phi(t_q, q).
\end{aligned} \tag{22}
$$

In what follows we refer Π_+ and Π_- the restriction of the Poincaré maps define in subsections 3.1 and 3.2 to subsets $L_+^O \cap L_-^O$ and $L_+^I \cap L_-^I$, respectively.

Theorem 23. *Consider the discontinuous piecewise linear differential system* (15). *Assume that the return map* Π *associated to the flow of system* (15) *and to the straight line* L *is defined.*

(i) *Suppose that* $e_+ \in S_-$ *and* $e_- \in S_+$.
  – *If* $\det A > 0$, *then* $\Pi = \Pi_- \circ \Pi_+$.
  – *If* $\det A < 0$, *then the return map* Π *is not defined.*

(ii) *Suppose that* $e_+ \in S_-$ *and* $e_- \in S_-$.
  – *If* $\det A > 0$ *and* $t^2 - 4d < 0$, *then the map* $\Pi = \Pi_- \circ \Pi_+$.
  – *If* $\det A < 0$ *or* $\det A > 0$ *and* $t^2 - 4d \geq 0$ *then return map* Π *is not defined.*

(iii) *Suppose that* $e_+ \in S_+$ *and* $e_- \in S_-$.
  – *If* $\det A < 0$ *or* $\det A > 0$ *and* $t^2 - 4d < 0$, *then* $\Pi = \Pi_- \circ \Pi_+$.
  – *If* $\det A > 0$ *and* $t^2 - 4d \geq 0$, *then the return map* Π *is not defined.*

Proof. Notice that $Dom \neq \emptyset$ implies that $Dom_{++} \neq \emptyset$ and $Dom_{--} \neq \emptyset$. Thus there exists a contact point p^+ of the flow of system (16) with the line L and a contact point p^- of the flow of system (19) with the line L. Therefore the Poincaré maps Π_+ and Π_- are defined, see Propositions 18 and 20. The result follows from Propositions 19 and 21. $\square$

In what follows we assume that the return map of the discontinuous system (15) is defined. That is $Dom \neq \emptyset$.

Let p^+ and p^- be the contact point of the system $\dot{x} = Ax \pm b$, respectively. By definition of L_+^{*O} and L_-^{*I} and from Figure 5 we have the following:

- If $\det A > 0$, $e_+ \in S_-$ and $e_- \in S_+$, then

$$
\begin{aligned}
L_+^I &= \{p^+ + a\dot{p}^+ : a \geq 0\}, \quad L_+^O = \{p^+ - a\dot{p}^+ : a \geq 0\}, \\
L_-^I &= \{p^- - a\dot{p}^- : a \geq 0\}, \quad L_-^O = \{p^- + a\dot{p}^- : a \geq 0\}.
\end{aligned} \tag{23}
$$

- If $\det A > 0$, $e_+ \in S_-$ and $e_- \in S_-$, then

$$\begin{aligned} L_+^I &= \{p^+ + a\dot{p^+} : a \geq 0\}, \quad L_+^O = \{p^+ - a\dot{p^+} : a \geq 0\}, \\ L_-^I &= \{p^- + a\dot{p^-} : a \geq 0\}, \quad L_-^O = \{p^- - a\dot{p^-} : a \geq 0\}. \end{aligned} \tag{24}$$

- If $\det A > 0$, $e_+ \in S_+$ and $e_- \in S_-$, then

$$\begin{aligned} L_+^I &= \{p^+ - a\dot{p^+} : a \geq 0\}, \quad L_+^O = \{p^+ + a\dot{p^+} : a \geq 0\}, \\ L_-^I &= \{p^- + a\dot{p^-} : a \geq 0\}, \quad L_-^O = \{p^- - a\dot{p^-} : a \geq 0\}. \end{aligned} \tag{25}$$

- If $\det A < 0$, $e_+ \in S_+$ and $e_- \in S_-$, then

$$\begin{aligned} L_+^I &= \{p^+ + a\dot{p^+} : a \geq 0\}, \quad L_+^O = \{p^+ - a\dot{p^+} : a \geq 0\}, \\ L_-^I &= \{p^- - a\dot{p^-} : a \geq 0\}, \quad L_-^O = \{p^- + a\dot{p^-} : a \geq 0\}. \end{aligned} \tag{26}$$

Since the behavior of maps Π_+ and Π_- are determined by the Poincaré maps π_{++}, π_{--}, $\widetilde{\pi}_{++}$ and $\widetilde{\pi}_{--}$, we will study the behavior of the map Π via so–defined Poincaré maps π_{jk}, where $j, k \in \{+, -\}$. Note that the behavior of those maps were studied in subsection 2.1.

We will use $\pi_\pm$ to identify which of the linear system $\dot{x} = Ax \pm b$ define the maps π_{jk} and we denote $p^\pm$ the respective contact points with respect to the straight line L. Thus either $\pi_+ = \pi_{++}$ or $\pi_+ = \widetilde{\pi}_{++}$, and either $\pi_- = \pi_{--}$ or $\pi_- = \widetilde{\pi}_{--}$.

Given $q \in Dom$, let r and r_0 be the coordinates of q and $\Pi(q)$ on the half–line L_+^O, respectively. We define the *return map* π as $\pi(r) = r_0$, that is the return map transforms the coordinate of q into the coordinate of $\Pi(q)$.

Consider $p \in L_+^I \cap L_-^I$ and $q \in L_+^O \cap L_-^O$, set r and s the coordinates of p on the half–lines L_+^I and L_-^I, respectively, and m and n the coordinates of q on the half–lines L_+^O and L_-^O, respectively. By previous definition of $L_\pm^I$ we have that

- if $\det A > 0$, $e_+ \in S_-$ and $e_- \in S_+$ then

$$p = p^+ + r\dot{p^+} = p^- - s\dot{p^-} \quad \text{and} \quad q = p^+ - m\dot{p^+} = p^- + n\dot{p^-}.$$

  Thus we can define the maps

$$f_1(r) = \frac{\left\| p^- - (p^+ + r\dot{p^+}) \right\|}{\left\| \dot{p^-} \right\|}, \quad h_1(n) = \frac{\left\| p^+ - (p^- + n\dot{p^-}) \right\|}{\left\| \dot{p^+} \right\|}. \tag{27}$$

- if $\det A > 0$, $e_+ \in S_-$ and $e_- \in S_-$, then

$$p = p^+ + r\dot{p^+} = p^- + s\dot{p^-} \quad \text{and} \quad q = p^+ - m\dot{p^+} = p^- - n\dot{p^-}.$$

Thus we can define the maps

$$f_2(r) = \frac{\left\|(p^+ + r\dot{p^+}) - p^-\right\|}{\left\|\dot{p^-}\right\|}, \quad h_2(n) = \frac{\left\|p^+ - (p^- - n\dot{p^-})\right\|}{\left\|\dot{p^+}\right\|}. \tag{28}$$

- if $\det A > 0$, $e_+ \in S_+$ and $e_- \in S_-$, then

$$p = p^+ - r\dot{p^+} = p^- + s\dot{p^-} \quad \text{and} \quad q = p^+ + m\dot{p^+} = p^- - n\dot{p^-}.$$

Thus we can define the maps

$$f_3(r) = \frac{\left\|(p^+ - r\dot{p^+}) - p^-\right\|}{\left\|\dot{p^-}\right\|}, \quad h_3(n) = \frac{\left\|(p^- - n\dot{p^-}) - p^+\right\|}{\left\|\dot{p^+}\right\|}. \tag{29}$$

- if $\det A < 0$, $e_+ \in S_+$ and $e_- \in S_-$, then

$$p = p^+ + r\dot{p^+} = p^- - s\dot{p^-} \quad \text{and} \quad q = p^+ - m\dot{p^+} = p^- + n\dot{p^-}.$$

Thus we can define the maps

$$f_4(r) = \frac{\left\|p^- - (p^+ + r\dot{p^+})\right\|}{\left\|\dot{p^-}\right\|}, \quad h_4(n) = \frac{\left\|p^+ - (p^- + n\dot{p^-})\right\|}{\left\|\dot{p^+}\right\|}. \tag{30}$$

Remark 24. *The map f_i transform the coordinate of p in L_+^I into the coordinate of p in L_-^I and h_i transform the coordinate of q in L_+^O into the coordinate of q in L_-^O. Clearly the inverse maps $f_i^{-1}(s)$ and $h_i^{-1}(m)$, $i = 1, 2, 3, 4$, are well defined.*

Suppose that the contact points satisfy $p^- \neq p^+$. If p^- belongs to L_+^I, then the maps f_i and h_i satisfy

$$f_i : [r^*, \infty) \longrightarrow [0, \infty), \quad h_i : [s^*, \infty) \longrightarrow [0, \infty), \tag{31}$$

where

$$r^* = \frac{\|p^- - p^+\|}{\left\|\dot{p^+}\right\|}, \quad s^* = \frac{\|p^+ - p^-\|}{\left\|\dot{p^-}\right\|}, \tag{32}$$

are the coordinates of the contact points p^- and p^+ with respect to the straight lines L_+ and L_-, respectively. On the other hand, if p^- belongs to L_+^O then the maps f_i and h_i satisfy

$$f_i : [0, \infty) \longrightarrow [s^*, \infty), \quad h_i : [0, \infty) \longrightarrow [r^*, \infty), \tag{33}$$

with r^* and s^* given in (32).

Finally, assuming that $p^- = p^+$ we get

$$f_i : [0,\infty) \longrightarrow [0,\infty), \quad h_i : [0,\infty) \longrightarrow [0,\infty), \tag{34}$$

Observe that f_i and h_i, $i = 1,2,3,4$, are increasing linear maps that tends to infinity when r and s tends to infinity, respectively. Moreover, we have the following results.

Lemma 25. *Consider the inverse map of h_i for $i = 1,2,3,4$. If $p^- = p^+$ then $h_i^{-1}(r) = f_i(r)$. If $p^- \neq p^+$ and*

- *$p^- \in L_+^O$ then $f_i(r) > h_i^{-1}(r)$ for every $r \in [\,r^*,\infty)$.*
- *$p^- \in L_+^I$ then $f_i(r) < h_i^{-1}(r)$ for every $r \in [\,r^*,\infty)$.*

Proof. Consider $e_+ \in S_-$ and $e_- \in S_-$. In this case we have

$$h_2^{-1}(r) = \frac{\left\| p^- - (p^+ - r\dot{p^+}) \right\|}{\left\| \dot{p^-} \right\|}.$$

Therefore $p^- = p^+$ implies that $h_2^{-1}(r) = f_2(r)$, see (28). On the other hand, if $p^- \neq p^+$, taking $r \in [\,r^*,\infty)$ we have that $p_1 = p^+ + r\dot{p^+} \in L_+^I$ and $p_2 = p^+ - r\dot{p^+} \in L_+^O$ are symmetric points with respect to contact point p^+. Furthermore

$$f_2(r) = \frac{\|p_1 - p^-\|}{\left\| \dot{p^-} \right\|}, \quad h_2^{-1}(r) = \frac{\|p^- - p_2\|}{\left\| \dot{p^-} \right\|}.$$

It follows that, if $p^- \in L_+^O$ we get that $\|p^- - p_2\| < \|p_1 - p^-\|$ and therefore $h_2^{-1}(r) < f_2(r)$. Otherwise, if $p^- \in L_+^I$ we get $\|p_1 - p^-\| < \|p^- - p_2\|$. Therefore $f_2(r) < h_2^{-1}(r)$.

The others cases can be proved in a similar way. $\square$

The following result provides the return map π as compositions of the Poincaré maps π_+, π_- and the maps f_i and h_i, $i = 1,2$.

Theorem 26. *Consider the discontinuous piecewise linear differential system* (15).

1. *Assume that $e_+ \in S_-$ and $e_- \in S_+$. If $d > 0$, then* $\pi(r) = h_1(\pi_{--}(f_1(\pi_{++}(r))))$.
2. *Assume that $e_+ \in S_-$ and $e_- \in S_-$. If $d > 0$ and $t^2 - 4d < 0$, then* $\pi(r) = h_2(\widetilde{\pi}_{--}(f_2(\pi_{++}(r))))$.
3. *Assume that $e_+ \in S_+$ and $e_- \in S_-$.*
   - *If $d > 0$ and $t^2 - 4d < 0$, then $\pi(r) = h_3(\widetilde{\pi}_{--}(f_3(\widetilde{\pi}_{++}(r))))$.*
   - *If $d < 0$ then $\pi(r) = h_4(\pi_{--}(f_4(\pi_{++}(r))))$.*

In the others cases we have that the return map π does not defined.

Proof. Given $q \in Dom$, we have that $q \in L_+^O \cap L_-^O$, $\Pi_+(q) \in L_+^I \cap L_-^I$, and $\Pi(q) = \Pi_-(\Pi_+(q)) \in L_+^O \cap L_-^O$. Assume that r and r_0 are the coordinate of q and $\Pi(q)$ in L_+^O, respectively. By definitions of π_+ and remark 24, it follows that $\pi_+(r)$ and $f_i(\pi_+(r))$ are the coordinates of $\Pi_+(q)$ in L_+^I and L_-^I, respectively. Furthermore, by definitions of π_- and remark 24, we have that $\pi_-(f_i(\pi_+(r)))$ and $h_i(\pi_-(f_i(\pi_+(r)))) = r_0$ are the coordinates of $\Pi(q))$ in L_-^O and L_+^O, respectively. Consequently, the return map π is given by $\pi(r) = r_0 = h_i(\pi_-(f_i(\pi_+(r))))$.

Assume that $e_+ \in S_-$ and $e_- \in S_+$. By Theorem 23-(i) the return map of the discontinuous system (15) is given by $\Pi = \Pi_- \circ \Pi_+$ for $d > 0$ and it is not defined when $d < 0$. Furthermore Proposition 19 implies that the behavior of the Poincaré map Π_+ is equivalent to behavior of Π_{++} for $d > 0$ and Π_+ is trivial for $d < 0$. The Proposition 21 and remark 22 implies that the behavior of the Poincaré map Π_- is equivalent to behavior of Π_{--} for $d > 0$ and Π_- is trivial for $d < 0$. Passing to Poincaré maps π_{jk}, $j, k \in \{+, -\}$, we have that $\pi_+ = \pi_{++}$ and $\pi_- = \pi_{--}$ for $d > 0$ and π_-, π_+ are defined only at zero for $d < 0$. Since $e_+ \in S_-$, $e_- \in S_+$ and $d > 0$, we have that the maps f_1, h_1 are defined, see equation (27). Therefore the return map is given by $\pi(r) = h_1(\pi_{--}(f_1(\pi_{++}(r))))$ for $d > 0$ and it is not defined for $d < 0$. We have proved the statement (1).

The others cases can be proved in a similar way and the result follows. □

4. Existence of limit cycle for discontinuous system (15)

From the definition of the return map π we have that the existence of periodic orbits for system (15) is equivalent to the existence of a fixed point of π. Moreover any limit cycle is associated to an isolated fixed point of π.

Theorem 27. *Consider the discontinuous piecewise linear differential system* (15). *If one of the statements below is satisfied, then system* (15) *has no limit cycles.*

(1) $e_+ \in S_-$, $e_- \in S_+$, *and* $d < 0$.
(2) $e_+ \in S_-$, $e_- \in S_-$, *and either* $d < 0$ *or* $d > 0$ *and* $t^2 - 4d \geq 0$.
(3) $e_+ \in S_+$, $e_- \in S_-$, $d > 0$ *and* $t^2 - 4d \geq 0$.

Proof. On these assumptions the return map π is not defined, see Theorem 26. Then system (15) has no limit cycles. □

In the rest of this paper we assume that the return map π for the discontinuous piecewise linear differential system (2) is defined.

Consider the following maps

$$g(r) = \pi_-(f_i(\pi_+(r))) - h_i^{-1}(r), \quad g_1(r) = f_i(\pi_+(r)) - \pi_-^{-1}(h_i^{-1}(r)), \tag{35}$$

$i = 1, 2, 3, 4$.

We have the following result.

Lemma 28. *The maps g and g_1 have a zero at r_0 if and only if r_0 is a fixed point of the return map π.*

Proof. Suppose that r_0 is a zero of the map $g(r)$. Then $\pi_-(f_i(\pi_+(r_0))) = h_i^{-1}(r_0)$ and this implies that $\pi(r_0) = h_i(\pi_-(f_i(\pi_+(r_0)))) = h_i(h_i^{-1}(r_0)) = r_0$.

Reciprocally, if r_0 is a fixed point of π then $h_i(\pi_-(f_i(\pi_+(r_0)))) = r_0$ from which follows that $\pi_-(f_i(\pi_+(r_0))) = h_i^{-1}(r_0)$.

The proof to g_1 is similar. □

Suppose that $p^- = p^+$, that is the contact points of system (16) and (19) coincide. We have the following results.

Theorem 29. *Consider the discontinuous piecewise linear differential system* (15)*, assume that $p^- = p^+$ and the return map π is defined.*

- *If $t \neq 0$ then π has not fixed points.*
- *If $t = 0$ then π is the identity map.*

Proof. According to Theorem 26 we should consider the following cases: (i) $e_+ \in S_-$ and $e_- \in S_+$, (ii) $e_+ \in S_-$ and $e_- \in S_-$, and (iii) $e_+ \in S_+$ and $e_- \in S_-$.

We shall prove the case (i), the others cases can be proved in a similar way, observing that the behavior of the maps $\widetilde{\pi}_{++}$ is characterized in Propositions 16 and 17.

(i) In this case the return map is given by $\pi(r) = h_1(\pi_{--}(f_1(\pi_{++}(r))))$, see Proposition 26-(1) if $d > 0$, and it is not defined if $d < 0$. Furthermore by Lemma 25 we have that $f_1(r) = h_1^{-1}(r)$. Then we can define $g(r) = \pi_{--}(f_1(\pi_{++}(r))) - h_1^{-1}(r)$.

Observe that if $d > 0$ the singularities of the piecewise linear differential system (15) can be diagonal nodes, non–diagonal nodes, centers or foci.

Assume that the singularities of system (15) are diagonal nodes, i.e. $d > 0$, $t^2 - 4d > 0$, and $t \neq 0$. If $t > 0$ then Proposition 10 implies that $\pi_{++}(r) > r$

and $\pi_{--}(r) > r$. Thus

$$\pi_{--}(f_1(\pi_{++}(r))) > f_1(\pi_{++}(r)) > f_1(r), \tag{36}$$

in the second inequality we have used that f_1 is increasing. It follows that $g(r) > 0$. On the other hand if $t < 0$ Proposition 11 implies that $\pi_{++}(r) < r$ and $\pi_{--}(r) < r$. Thus

$$\pi_{--}(f_1(\pi_{++}(r))) < f_1(\pi_{++}(r)) < f_1(r), \tag{37}$$

and it follows that $g(r) < 0$.

Assume that the singularities of system (15) are non–diagonal nodes, i.e. $d > 0$, $t^2 - 4d = 0$, and $t \neq 0$. If $t > 0$ then Proposition 12 implies that $\pi_{++}(r) > r$ and $\pi_{--}(r) > r$. Thus

$$\pi_{--}(f_1(\pi_{++}(r))) > f_1(\pi_{++}(r)) > f_1(r), \tag{38}$$

in the second inequality we have used that f_1 is increasing. It follows that $g(r) > 0$. On the other hand if $t < 0$ Proposition 13 implies that $\pi_{++}(r) < r$ and $\pi_{--}(r) < r$. Thus

$$\pi_{--}(f_1(\pi_{++}(r))) < f_1(\pi_{++}(r)) < f_1(r), \tag{39}$$

and it follows that $g(r) < 0$.

Finally assume that the singularities of system (15) are centers or foci, i.e. $d > 0$ and $t^2 - 4d < 0$. If $t > 0$ then Proposition 14 implies that $\pi_{++}(r) > r$ and $\pi_{--}(r) > r$. Then

$$\pi_{--}(f_1(\pi_{++}(r))) > f_1(\pi_{++}(r)) > f_1(r), \tag{40}$$

in the second inequality we have used that f_1 is increasing. It follows that $g(r) > 0$. On the other hand if $t < 0$ Proposition 15 implies that $\pi_{++}(r) < r$ and $\pi_{--}(r) < r$. Thus

$$\pi_{--}(f_1(\pi_{++}(r))) < f_1(\pi_{++}(r)) < f_1(r), \tag{41}$$

and it follows that $g(r) < 0$.

If $t = 0$, by Proposition 14 we have that the Poincaré maps π_{--} and π_{++} are the identity map. Moreover by Lemma 25 we have that $f_1(r) = h_1^{-1}(r)$. Therefore $g(r) = f_1(r) - h_1^{-1}(r) = 0$ and $\pi(r) = r$, i.e. the return map is the identity, see Lemma 28.

Consequently if $t \neq 0$ then either $g(r) > 0$, or $g(r) < 0$. Therefore the return map has not fixed points, see Lemma 28. $\square$

Proof of Theorem A. By Theorem 29 we conclude that either system (2) has no closed orbits, or it contains a continuum of closed orbits. Therefore system (2) has no limit cycle. $\square$

In what follows we assume that $p^- \neq p^+$.

Observe that if f_i and h_i^{-1} are defined at r then $f_1(r) = f_2(r) = f_4(r) = h_3^{-1}(r)$ and $f_3(r) = h_2^{-1}(r) = h_1^{-1}(r) = h_4^{-1}(r)$.

Suppose that $A = (a_{ij})$ and $b = (b_1, b_2)^T$, we can rewrite the maps f_2 and h_2^{-1} as

$$f_2(r) = \frac{|(a_{11}a_{22} - a_{12}a_{21} - a_{12}b_2 + a_{22}b_1)r + 2b_1|}{|a_{11}a_{22} - a_{12}a_{21} + a_{12}b_2 - a_{22}b_1|}, \tag{42}$$

$$h_2^{-1}(r) = \frac{|(a_{11}a_{22} - a_{12}a_{21} - a_{12}b_2 + a_{22}b_1)r - 2b_1|}{|a_{11}a_{22} - a_{12}a_{21} + a_{12}b_2 - a_{22}b_1|}. \tag{43}$$

Then if the singularities of system (15) are such that $e_+ \in S_-$ and $e_- \in S_+$, we get that

$$\frac{a_{12}b_2 - a_{22}b_1}{a_{11}a_{22} - a_{12}a_{21}} < -1.$$

Therefore

$$a_{11}a_{22} - a_{12}a_{21} + a_{12}b_2 - a_{22}b_1 < 0,$$
$$a_{11}a_{22} - a_{12}a_{21} - a_{12}b_2 + a_{22}b_1 > 0.$$

Since f_1 and h_1^{-1} are increasing maps, we conclude that

$$f_1'(r) = (h_1^{-1})'(r) = \frac{a_{11}a_{22} - a_{12}a_{21} - a_{12}b_2 + a_{22}b_1}{-a_{11}a_{22} + a_{12}a_{21} - a_{12}b_2 + a_{22}b_1}. \tag{44}$$

If $e_+ \in S_-$ and $e_- \in S_-$ we have that

$$-1 < \frac{a_{12}b_2 - a_{22}b_1}{a_{11}a_{22} - a_{12}a_{21}} < 1,$$

from which it follows that

$$a_{11}a_{22} - a_{12}a_{21} + a_{12}b_2 - a_{22}b_1 > 0,$$
$$a_{11}a_{22} - a_{12}a_{21} - a_{12}b_2 + a_{22}b_1 > 0.$$

Therefore

$$f_2'(r) = (h_2^{-1})'(r) = \frac{a_{11}a_{22} - a_{12}a_{21} - a_{12}b_2 + a_{22}b_1}{a_{11}a_{22} - a_{12}a_{21} + a_{12}b_2 - a_{22}b_1}. \tag{45}$$

Finally assume that $e_+ \in S_+$ and $e_- \in S_-$, thus

$$\frac{a_{12}b_2 - a_{22}b_1}{a_{11}a_{22} - a_{12}a_{21}} > 1,$$

and this implies that

$$a_{11}a_{22} - a_{12}a_{21} - a_{12}b_2 + a_{22}b_1 < 0.$$

Furthermore if $d > 0$ we obtain

$$a_{11}a_{22} - a_{12}a_{21} + a_{12}b_2 - a_{22}b_1 > 0$$

and if $d < 0$ then

$$a_{11}a_{22} - a_{12}a_{21} + a_{12}b_2 - a_{22}b_1 < 0.$$

Since f_i and h_i^{-1}, $i = 3, 4$, are increasing maps, we conclude that

$$f_3'(r) = (h_3^{-1})'(r) = \frac{-a_{11}a_{22} + a_{12}a_{21} + a_{12}b_2 - a_{22}b_1}{a_{11}a_{22} - a_{12}a_{21} + a_{12}b_2 - a_{22}b_1}, \tag{46}$$

$$f_4'(r) = (h_4^{-1})'(r) = \frac{-a_{11}a_{22} + a_{12}a_{21} + a_{12}b_2 - a_{22}b_1}{-a_{11}a_{22} + a_{12}a_{21} - a_{12}b_2 + a_{22}b_1}. \tag{47}$$

4.1. **Virtual–virtual case.** In this subsection we assume that the singularities of discontinuous piecewise linear differential system (15) are such that $e_+ \in S_-$ and $e_- \in S_+$ and we analyze the existence of limit cycles.

On the next results we study the existence of limit cycle for the discontinuous system (15).

Theorem 30. *Suppose that $d > 0$ and $p^- \in L_+^O$. If $t \geq 0$, then π has not fixed points. If $t < 0$, then π has one fixed point.*

Proof. By Theorem 26 we have that the return map π is given by $\pi(r) = h_1(\pi_{--}(f_1(\pi_{++}(r))))$. Then we get that

$$g(r) = \pi_{--}(f_1(\pi_{++}(r))) - h_1^{-1}(r). \tag{48}$$

Observe that, by equation (33), the map f_1 is define on the interval $[0, \infty)$ and its image is $[s^*, \infty)$. Moreover, the map h_1^{-1} is defined in $[r^*, \infty)$ whose image is the interval $[0, \infty)$.

In order to prove the result we consider two cases: (i) $t^2 - 4d \geq 0$; (ii) $t^2 - 4d < 0$.

(i) Assume that $t > 0$, by Propositions 10 and 12 there exist $\alpha_1 > 0$ and $\alpha_2 > 0$ such that

$$\pi_{++} : [0, \alpha_1) \longrightarrow [0, \infty)\,, \quad \pi_{--}\ [0, \alpha_2) \longrightarrow [0, \infty)\,.$$

Observe that the return map is only defined if $r^* < \alpha_1$ and $s^* < \alpha_2$ because r^* and s^* is the coordinate of p^- and p^+ with respect to L_+ and L_-, respectively. Thus g is defined for $r \in (r^*, a)$, where $a = \pi_{++}^{-1}(f_1^{-1}(\alpha_2))$. Furthermore the maps satisfy $\pi_{++}(r) > r$ and $\pi_{--}(r) > r$ and, therefore

$$\pi_{--}(f_1(\pi_{++}(r))) > f_1(\pi_{++}(r)) > f_1(r) > h_1^{-1}(r),$$

where we use that f_1 is an increasing map and Lemma 25. This implies that $g(r) > 0$, and by Lemma 28 there are not fixed points of π.

Now, if $t<0$ then Propositions 11 and 13 imply that there exist $\alpha_1>0$ and $\alpha_2>0$ such that

$$\pi_{++}:[0,\infty)\longrightarrow[0,\alpha_1)\ ,\quad \pi_{--}\,[0,\infty)\longrightarrow[0,\alpha_2)\ .$$

Furthermore, π_{++} and π_{--} are increasing maps and satisfy $\pi_{++}(r)<r$, $\pi_{--}(r)<r$, $\lim\limits_{r\nearrow+\infty}\pi_{++}(r)=\alpha_1$ and $\lim\limits_{r\nearrow+\infty}\pi_{--}(r)=\alpha_2$. It follows that the map $g(r)$ is defined for $r\in(r^*,\infty)$ and satisfies

$$\begin{aligned}&\lim_{r\searrow r^*}g(r)=\pi_{--}(f_1(\pi_{++}(r^*)))>0,\\&\lim_{r\nearrow+\infty}g(r)=\lim_{r\nearrow+\infty}\pi_{--}(f_1(\pi_{++}(r)))-\lim_{r\nearrow+\infty}h_1^{-1}(r)=-\infty.\end{aligned}$$

Therefore there exists a $r_0\in(r^*,+\infty)$ such that $g(r_0)=0$. Since h_1^{-1} is linear, Propositions 11-(c) and 13-(c) imply that $g''(r)<0$.

(ii) If $t=0$, by Propositions 14 and 15, we have that π_{++} and π_{--} are the identity in $[0,+\infty)$. Then $g(r)=f_1(r)-h_1^{-1}(r)$ is defined for $r\in(r^*,+\infty)$ and, by Lemma 25, $g(r)>0$. Therefore the return map π has no fixed points, see Lemma 28.

In what follows we assume that $t\neq 0$. In this case we have that

$$\pi_{++}:[0,\infty)\longrightarrow[0,\infty)\ ,\quad \pi_{--}\,[0,\infty)\longrightarrow[0,\infty)\ ,$$

are increasing maps such that $\lim\limits_{r\nearrow+\infty}\pi_{++}(r)=+\infty$ and $\lim\limits_{r\nearrow+\infty}\pi_{--}(r)=+\infty$ see Propositions 14 and 15. Then it follows that the map $g(r)$ is defined for $r\in(r^*,\infty)$.

Assume that $t>0$, by Proposition 14 we get that $\pi_{++}(r)>r$ and $\pi_{--}(r)>r$. Therefore

$$\pi_{--}(f_1(\pi_{++}(r)))>f_1(\pi_{++}(r))>f_1(r)>h_1^{-1}(r),$$

where we use that f_1 is an increasing map and Lemma 25. This implies that $g(r)>0$ and, by Lemma 28 there are not fixed points of π.

Now, for $t<0$ Proposition 15 implies that $\pi_{++}(r)<r$, $\pi_{--}(r)<r$, $\pi'_{++}(r)<1$, and $\pi'_{--}(r)<1$. It follows that

$$(\pi_{--}\circ f_1\circ\pi_{++})'(r)=\pi'_{--}(f_1(\pi_{++}(r)))f_1'(\pi_{++}(r))\pi'_{++}(r)<f_1'(\pi_{++}(r)),$$

where $f_1'(\pi_{++}(r))=f_1'(r)=(h_1^{-1})'(r)$ is a constant, see equation (44). Consequently,

$$\lim_{r\nearrow+\infty}g(r)=\lim_{r\nearrow+\infty}[\pi_{--}(f_1(\pi_{++}(r)))-h_1^{-1}(r)]=-\infty.$$

Furthermore, we have that $\lim_{r\searrow r^*} g(r) = \pi_{--}(f_1(\pi_{++}(r^*))) > 0$. Therefore there exist $r_0 \in (r^*, \infty)$ such that $g(r_0) = 0$. By Proposition 15-(c) we have that $g''(r) < 0$, and therefore r_0 is unique. By Lemma 28, we conclude that the return map π has a unique fixed point. $\square$

Theorem 31. *Suppose that $d > 0$ and $p^- \in L_+^I$. If $t > 0$ then the return map π has one fixed point, and if $t \le 0$ then π has not fixed points.*

Proof. By Theorem 26-(1) we have that the return map π is given by $\pi(r) = h_1(\pi_{--}(f_1(\pi_{++}(r))))$. Then we get that

$$g(r) = \pi_{--}(f_1(\pi_{++}(r))) - h_1^{-1}(r). \tag{49}$$

By equation (31) the map f_1 is defined on the interval $[r^*, \infty)$ and its image is $[0, \infty)$. Moreover the map h_1^{-1} is defined in $[0, \infty)$ whose image is the interval $[s^*, \infty)$.

In order to prove the result we consider two cases: (i) $t^2 - 4d \ge 0$; (ii) $t^2 - 4d < 0$.

(i) In this case we have that $t \neq 0$. Assume that $t > 0$, Propositions 10 and 12 imply that there exist $\alpha_1 > 0$ and $\alpha_2 > 0$ such that

$$\pi_{++} : [0, \alpha_1) \longrightarrow [0, \infty)\ , \quad \pi_{--}\ [0, \alpha_2) \longrightarrow [0, \infty)\ .$$

Moreover π_{++} and π_{--} are increasing maps and satisfy $\pi_{++}(r) > r$, $\pi_{--}(r) > r$, $\lim_{r\searrow\alpha_1} \pi_{++}(r) = \infty$ and $\lim_{r\searrow\alpha_2} \pi_{--}(r) = \infty$. So it follows that the map g is defined for $r \in (a, b)$, where $a = \pi_{++}^{-1}(r^*)$ and $b = \pi_{++}^{-1}(f_1^{-1}(\alpha_2))$, and satisfies

$$\lim_{r\searrow a} g(r) = -h_1^{-1}(a) < 0,$$
$$\lim_{r\nearrow b} g(r) = \lim_{r\nearrow b} \pi_{--}(f_1(\pi_{++}(r))) - h_1^{-1}(b) = +\infty.$$

Then there exists $r_0 \in (a, b)$ such that $g(r_0) = 0$. Moreover, Propositions 10-(c) and 12-(c) imply that $g''(r) > 0$, thus we conclude that r_0 is unique. Therefore, by Lemma 28, the return map π has one fixed point r_0.

If $t < 0$ then there exist $\alpha_1 > 0$ and $\alpha_2 > 0$ such that

$$\pi_{++} : [0, \infty) \longrightarrow [0, \alpha_1)\ , \quad \pi_{--}\ [0, \infty) \longrightarrow [0, \alpha_2)\ ,$$

see Propositions 11 and 13. It follows that the map $g(r)$ is defined for $r \in (a, \infty)$, where $a = \pi_{++}^{-1}(r^*)$. Furthermore π_{++} and π_{--} are increasing maps which satisfy $\pi_{++}(r) < r$, $\pi_{--}(r) < r$ and, therefore

$$\pi_{--}(f_1(\pi_{++}(r))) < f_1(\pi_{++}(r)) < f_1(r) < h_1^{-1}(r).$$

On the last inequality we use the result of Lemma 25. Consequently $g(r) < 0$ for every $r \in (a, \infty)$, and the return map has not fixed point, see Lemma 28.

(ii) If $t = 0$ by Propositions 14 and 15 we have that π_{++} and π_{--} are the identity in $[0, +\infty)$. Then $g(r) = f_1(r) - h_1^{-1}(r)$ is defined for $r \in (r^*, +\infty)$ and, by Lemma 25, $g(r) < 0$. Therefore the return map π has no fixed point, see Lemma 28.

Assume that $t \neq 0$ then we have that the maps

$$\pi_{++} : [0, \infty) \longrightarrow [0, \infty)\ , \quad \pi_{--}\ [0, \infty) \longrightarrow [0, \infty)\ ,$$

are increasing, $\lim\limits_{r \nearrow +\infty} \pi_{++}(r) = +\infty$ and $\lim\limits_{r \nearrow +\infty} \pi_{--}(r) = +\infty$, see Propositions 14 and 15. Then it follows that the map $g(r)$ is defined for $r \in (a, \infty)$, where $a = \pi_{++}^{-1}(r^*)$.

If $t > 0$ Proposition 14 implies that $\pi_{++}(r) > r$, $\pi_{--}(r) > r$, $\pi'_{++}(r) > 1$, and $\pi'_{--}(r) > 1$. It follows that

$$(\pi_{--} \circ f_1 \circ \pi_{++})'(r) = \pi'_{--}(f_1(\pi_{++}(r)))f'_1(\pi_{++}(r))\pi'_{++}(r) > f'_1(\pi_{++}(r)),$$

where $f'_1(\pi_{++}(r)) = f'_1(r) = (h_1^{-1})'(r)$ is a constant, see equation (44). It follows that

$$\lim_{r \nearrow +\infty} g(r) = \lim_{r \nearrow +\infty} [\pi_{--}(f_1(\pi_{++}(r))) - h_1^{-1}(r)] = +\infty.$$

Furthermore we have that $\lim\limits_{r \searrow a} g(r) = -h_1^{-1}(a) < 0$ and, therefore there exists $r_0 \in (a, \infty)$ such that $g(r_0) = 0$. Since $g''(r) > 0$, see statement (c) of Proposition 14, we conclude that r_0 is unique. By Lemma 28 we conclude that the return map π have one fixed point.

Finally if $t < 0$ then $\pi_{++}(r) < r$ and $\pi_{--}(r) < r$ for every $r \in [0, \infty)$, see Proposition 15. Therefore

$$\pi_{--}(f_1(\pi_{++}(r))) < f_1(\pi_{++}(r)) < f_1(r) < h_1^{-1}(r),$$

where we use that f_1 is an increasing map and Lemma 25 on the second and third inequality, respectively. This implies that $g(r) < 0$ and, by Lemma 28 there is no fixed point of π. $\square$

Proof of Theorem B. Observe that if the contact point of X satisfy $p^- \in L_+^O$, by Theorem 30 we have that the discontinuous piecewise linear differential system (2) has one limit cycle if $t < 0$, and it has no limit cycles if $t \geq 0$. On the other hand, if the contact point of vector field X satisfies $p^- \in L_+^I$ then Theorem 31 implies that the discontinuous piecewise linear differential system (15) has one limit cycle if $t > 0$, and it has no limit cycles if $t \leq 0$, respectively. So Theorem B follows. $\square$

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