Universidade Federal de Goiás
Instituto de Matemática e Estatística

Jeidy Johana Jimenez Ruiz

On the crossing limit cycles for piecewise linear differential systems on the plane

Goiânia
2019

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Jeidy Johana Jimenez Ruiz

On the crossing limit cycles for piecewise linear differential systems on the plane

Tese apresentada ao Programa de Pós–Graduação do Instituto de Matemática e Estatística da Universidade Federal de Goiás, como requisito parcial para obtenção do título de Doutor em Matemática.

Área de concentração: Sistemas Dinâmicos.

Orientador: Prof. João Carlos da Rocha Medrado

Co–Orientador: Prof. Jaume Llibre Saló

Goiânia
2019

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Jimenez Ruiz, Jeidy Johana
On the crossing limit cycles for piecewise linear differential systems on the plane [manuscrito] / Jeidy Johana Jimenez Ruiz. - 2019.
162 f.: il.

Orientador: Prof. João Carlos da Rocha Medrado; co-orientador Jaume Llibre Saló.
Tese (Doutorado) - Universidade Federal de Goiás, Instituto de Matemática e Estatística (IME), Programa de Pós-Graduação em Matemática, Goiânia, 2019.
Bibliografia.
Inclui siglas, abreviaturas, símbolos, gráfico, lista de figuras.

1. Sistemas diferencias suaves por partes. 2. ciclos limite costurantes. 3. centros diferencias lineares. 4. cônicas. 5. cúbicas. I. da Rocha Medrado, João Carlos, orient. II. Título.

CDU 517.938

UFG

UNIVERSIDADE FEDERAL DE GOIÁS

INSTITUTO DE MATEMÁTICA E ESTATÍSTICA

ATA DE DEFESA DE TESE

Ata Nº 43 da sessão de Defesa de Tese de Jeidy Johana Jimenez Ruiz que confere o título de Doutor(a) em Matemática, na área de concentração em Sistemas Dinâmicos.

Ao quinto dia do mês de dezembro de dois mil e dezenove, a partir das quatorze horas e trinta minutos, no Auditório do Instituto de Matemática e Estatística, realizou-se a sessão pública de Defesa de Tese intitulada "On the crossing limit cycles for piecewise linear differential systems on the plane". Os trabalhos foram instalados pelo(a) Orientador, Professor Doutor Orientador: João Carlos da Rocha Medrado (IME/UFG) com a participação dos demais membros da Banca Examinadora: Professor Doutor Durval José Tonon (IME/UFG), membro titular interno; Professor Doutor Claudio Aguinaldo Buzzi (IBILCE/UNESP), membro titular externo; Professor Doutor Maurício Firmino Silva Lima (CMCC/UFABC), membro titular externo; Professor Doutor Ricardo Miranda Martins IMECC/UNICAMP, membro titular externo. Durante a argüição os membros da banca não fizeram sugestão de alteração do título do trabalho. A Banca Examinadora reuniu-se em sessão secreta a fim de concluir o julgamento da Tese tendo sido a candidata aprovada pelos seus membros. Proclamados os resultados pelo Professor Doutor João Carlos da Rocha Medrado, Presidente da Banca Examinadora, foram encerrados os trabalhos e, para constar, lavrou-se a presente ata que é assinada pelos Membros da Banca Examinadora, ao quinto dia do mês de dezembro de dois mil e dezenove.

TÍTULO SUGERIDO PELA BANCA

On the crossing limit cycles for piecewise linear differential systems on the plane

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Referência: Processo nº 23070.043660/2019-75 SEI nº 1019591

Jeidy Johana Jimenez Ruiz

Graduou-se em Matemática na Universidad del Cauca, Popayán, Colômbia.

A mi madre y a la memoria de mi padre.

Agradecimentos

En primer lugar agradezco a Dios por la fortaleza que me proporciona día a día. Agradezco a mi madre por ser mi más grande símbolo de fuerza y superación, a quien le debo todo lo que soy.

A mi padre por sus valiosas enseñanzas.

Agradezo a mi hermana por ser una de las motivaciones para ser alguién mejor y por darme mis dos maravillosos bebés.

A Emmanuel mi mini versión, que me coloca la gran responsabilidad de ser cada día mejor persona.

A Mathias quien llena mis días de amor y ternura con sus travesuras. Los amo y los amaré "hasta el infinito y más allá" mis peques.

A mi amigo y compañero de vida, Elismar, gracias por tu amor y apoyo incondiconal en cada una de mis decisiones aunque en muchos casos no las compartas.

A mi orientador, profesor João Carlos, por su paciencia, consejos y apoyo, sin su ayuda y comprensión dificilmente habría conseguido concluir esta etapa.

Al PPGIME por la gran oportunidad de realizar el doctorado−sanduiche en España.

Al departamento de Matemáticas de la UAB por el recibimiento.

Al Profesor Jaume Llibre, por su apoyo, paciencia y compresión.

À CAPES pelo apoio financeiro.

"Part of our human nature is that we don't learn the importance of anything until it's snatched from our hands"

Malala Yousafzai,
in interview.

Resumo

Jimenez, Jeidy Johana. **On the crossing limit cycles for piecewise linear differential systems on the plane**. Goiânia, 2019. 162p. Tese de doutorado. Instituto de Matemática e Estatística, Universidade Federal de Goiás.

Neste trabalho estudamos a versão do 16th problema de Hilbert para sistemas suaves por partes para um caso particular, mais precisamente no Capítulo 2 estudamos sobre o número máximo de ciclos que podem ter os sistemas lineares por partes separados por uma linha reta Σ e formados por dois sistemas lineares diferenciais X^{-}, X^{+} cujas singularidades são simétricas com relação à linha de descontinuidade Σ e estão sobre a linha reta $y = sx$, $s \in \mathbb{R}$. Em [24, 27] foi provado que os sistemas lineares suaves por partes formados por centros lineares separados por uma linha reta não têm ciclos limite costurantes no entanto em [20, 28] foram estudados estes mesmos sistemas quando a curva de descontinuidade não é uma linha reta e foi mostrado que o número de ciclos limite costurantes nesses sistemas é diferente de zero. Por esta razão é interesante estudar a influência da curva de descontinuidade no número de ciclos limite costurantes que sistemas suaves por partes formados por centros lineares podem possuir. No Capítulo 3 estudamos sobre as cotas superiores para o número máximo de ciclos limite costurantes com dois ou quatro pontos sobre a curva de descontinuidade Σ, quando Σ é uma cônica qualquer. Finalmente no Capítulo 4 estudamos sobre o número de ciclos limite costurantes com quatro pontos sobre a curva de descontinuidade Σ, quando Σ é uma cúbica redutível formada por um círculo e uma linha reta ou por uma parábola e uma linha reta.

Palavras–chave

Sistemas diferencias suaves por partes, ciclos limite costurantes, centros diferencias lineares, cônicas, cúbicas.

Abstract

Jimenez, Jeidy Johana. **On the crossing limit cycles for piecewise linear differential systems on the plane**. Goiânia, 2019. 162p. PhD. Thesis . Instituto de Matemática e Estatística, Universidade Federal de Goiás.

In this work we analyze the version of Hilbert's 16th problem for piecewise linear differential systems in the plane for a particular case, more precisely in Chapter 2 we study on the maximum numbers of crossing limit cycles that can have the planar piecewise linear differential systems separated by a straight line Σ and formed by two linear differential systems X^-, X^+ which singularities are symmetrical with respect to the straight line of discontinuity Σ and they are on the straight line $y = sx$, $s \in \mathbb{R}$. In [24, 27] it was proved that piecewise linear differential centers separated by a straight line have no crossing limit cycles nevertheless in [20, 28] were studied planar discontinuous piecewise linear differential centers where the curve of discontinuity is not a straight line, and it was shown that the number of crossing limit cycles in these systems is non-zero. For this reason it is interesting to study the role which plays the shape of the discontinuity curve in the number of crossing limit cycles that planar discontinuous piecewise linear differential centers can have. In Chapter 3 we study on the upper bounds for the maximum number of crossing limit cycles with either two or four points on the discontinuity curve Σ, when Σ is any conic. And finally in Chapter 4 we study on the numbers of crossing limit cycles with four points on the discontinuity curve Σ, when Σ is a reducible cubic curve formed either by a circle and a straight line, or by a parabola and a straight line.

Keywords

Piecewise linear differential systems, crossing limit cycles, linear differential centers, conics, cubic.

Introduction

The study of the discontinuous piecewise differential systems in the plane started with Andronov, Vitt and Khaikin in [1], and from there these systems have been a topic of great interest in the mathematical community due to their applications in various areas, because they are used for modeling real phenomena and different modern devices, see for instance the books [6, 37] and references quoted therein.

In the qualitative theory of differential systems in the plane, a *limit cycle* is a periodic orbit which is isolated in the set of all periodic orbits of the system. This concept was defined by Poincaré [31]. In several papers, as [3, 21, 30], it was shown that the limits cycles model many phenomena of the real world. Subsequently these works, the non-existence, existence, the maximum number and other properties of the limit cycles were extensively studied by mathematicians and physicists and more recently, by biologists, economist and engineers, see for instance [6, 38].

As for the general case of planar differential systems one of the main problems for the case of the piecewise differential systems is to determine the existence and the maximum number of crossing limits cycles that these systems can exhibit, that is the version of Hilbert's 16th problem for PWLS in the plane [15]. In this work, we study the *crossing limit cycles* which are periodic orbits isolated in the set of all periodic orbits of the piecewise differential system, which only have isolated points of intersection with the discontinuity curve.

The class of piecewise linear differential systems (PWLS for short) in $\mathbb{R}^2$ with two zones separated by a straight line Σ is the simplest class of piecewise differential systems. We can consider without loss of generality that the discontinuity straight line is $\Sigma = \left\{(x,y) \in \mathbb{R}^2 : x = 0\right\}$. It separates the plane into two regions, namely

$$\Sigma^- = \left\{(x,y) \in \mathbb{R}^2 : x < 0\right\} \text{ and } \Sigma^+ = \left\{(x,y) \in \mathbb{R}^2 : x > 0\right\}.$$

Therefore we obtain the PWLS

$$\dot{X} = \begin{cases} X^- = A^- X + B^-, & \text{if } (x,y) \in \Sigma^-, \\ X^+ = A^+ X + B^+, & \text{if } (x,y) \in \Sigma^+, \end{cases} \tag{0-1}$$

where

$$A^{\pm} = \begin{pmatrix} a_{11}^{\pm} & a_{12}^{\pm} \\ a_{21}^{\pm} & a_{22}^{\pm} \end{pmatrix}, \quad B^{\pm} = \begin{pmatrix} b_1^{\pm} \\ b_2^{\pm} \end{pmatrix} \text{ and } X = (x,y)^T \in \mathbb{R}^2$$

In [29] Lum and Chua conjectured that a continuous PWLS (0-1) has at most one crossing limit cycle. In [8] Freire et al. proved this conjecture. There are several papers tried to investigate the problem of Lum and Chua for the class of discontinuous PWLS in the plane. For instance in [14] Han and Zhang conjectured that discontinuous PWLS (0-1) have at most two crossing limit cycles. Via a numerical example with three crossing limit cycles in a discontinuous PWLS, Huan and Yang gave a negative answer to this conjecture, see [18]. Later on in [11, 26] were given analytical proofs for the existence of these three crossing limit cycles. Nevertheless until today it is an open problem to know if three is the upper bound for the maximum number of crossing limit cycles of discontinuous PWLS (0-1).

Due to the difficulty of this problem several researchers study the upper bounds of crossing limit cycles of system (0-1) under some special conditions, see [2, 7, 8, 9, 10, 12, 18, 16, 17, 35, 24, 26, 23, 33, 36]. In [10] the authors studied systems (0-1) such that have a maximal crossing set, and with a focus-focus dynamics, they proved that if $a_{12}^{-}a_{12}^{+} > 0$, then systems (0-1) have at most one crossing limit cycle. In [33] it was proved that systems (0-1) with focus-saddle type with $b_1^{+} = 0$ have at most one crossing limit cycle. Recently in [35] it was proved that systems (0-1) having a unique non-degenerated equilibrium can have at least three crossing limit cycles depending on the configurations of the equilibrium points for each linear differential system in (0-1). In [24] the authors proved that when one of linear differential systems of (0-1) has the equilibrium point on Σ, systems (0-1) have at most two crossing limit cycles and this upper bound is reached. In particular, in that paper it was proved that the class of planar discontinuous piecewise linear differential centers (PWLC for short) has no crossing limit cycles. However, recently in [20, 28] were studied planar discontinuous PWLC where the curve of discontinuity is not a straight line, and it was shown that the number of crossing limit cycles in these systems is non-zero. For this reason it is interesting to study the role which plays the shape of the discontinuity curve in the number of crossing limit cycles that planar discontinuous PWLC can have. For example, in this work, we study the planar discontinuous PWLC when the discontinuity curve is: either a conic; or a reducible cubic curve formed either by a circle and a straight line or by a parabola and a straight line.

This work consists of four parts, first one in Chapter 1 we briefly present some of the basic concepts, definitions and results used through this work.

In Chapter 2 we study the maximum number of crossing limit cycles that the planar PWLS (0-1) can have when the equilibrium points of the differential linear systems X^{-} and X^{+} are symmetric with respect to the line of discontinuity Σ and these singularities can be real or virtual.

Chapter 3 is devoted to provide an upper bound for the maximum number of crossing limit cycles of the planar discontinuous PWLC separated by a conic Σ.

And finally in Chapter 4 we study on the number of crossing limit cycles of the discontinuous PWLC in $\mathbb{R}^2$ separated by a reducible cubic curve formed either by a circle and a straight line, or by a parabola and a straight line.

Contents

List of Figures

CHAPTER 1

Preliminaries

Here some basic concepts, results, and tools necessary to the development of this work are presented. Most part of the results are given without proof, however references where they can be found, are included. In this work, we concern about planar discontinuous vector fields defined in two or more zones, for this reason, we present a generic definition for this type of vector fields.

1.1 Discontinuous Vector Fields

Definition 1.1 *A differential system defined in $\mathbb{R}^2$ is a piecewise linear differential system (PWLS) in $\mathbb{R}^2$ if there exists a set of $3-$tuples $\{(A_i, B_i, R_i)\}_{i\in I}$ where A_i is a 2×2 real matrix; $B_i \in \mathbb{R}^2$ and R_i are connected and open regions in $\mathbb{R}^2$ separated by a discontinuity manifold Σ. These regions satisfy $R_i \cap R_j = \emptyset$ for $i \neq j$ and $\cup_{i\in I} R_i \cup \Sigma = \mathbb{R}^2$; and $A_i\boldsymbol{x} + B_i$ is the vector field in R_i when $\boldsymbol{x} \in R_i$.*

The switching manifold or discontinuity manifold Σ is described as

$$\Sigma = \left\{(x,y) \in \mathbb{R}^2 : H(x,y) = 0\right\},$$

where $H : \mathbb{R}^2 \to \mathbb{R}$ is a C^r function, $r \geq 1$, and 0 is a regular value of H.

Under these conditions, a PWLS can be written as following

$$Z(\mathbf{p}) = X_i(\mathbf{p}) = A_i\mathbf{p} + B_i, \;\; \mathbf{p} \in R_i, \; i \in \{1, ..., n\}, \tag{1-1}$$

where each vector field X_i is smooth, and defines a smooth flow $\varphi_{X_i}(\mathbf{p}, t)$ within any open set $U \subset R_i$. In particular, each flow $\varphi_{X_i}(\mathbf{p}, t)$ is well defined on both sides of the boundary ∂R_i.

We note that Definition 1.1 does not specify a rule for the evolution of the dynamics within a discontinuity set. This depends basically of the behavior of the vector fields $X_{(i-1)}$ and X_i close to the discontinuity manifold for $i \in \{1, 2, 3, ..., n\}$, where we denote that $i-1 = n$ when $i = 1$. Here we namely $\Sigma_{(i-1)i} = \overline{R}_{(i-1)} \cap \overline{R}_i \subset \Sigma$. For to extend the evolution of the dynamics on the discontinuity manifold Σ, we divide each $\Sigma_{(i-1)i}$ in three regions. See Figure 1.1.

- Crossing region: $\Sigma^c_{(i-1)i} = \left\{\mathbf{p} \in \Sigma_{(i-1)i} | X_{(i-1)}H(\mathbf{p}) \cdot X_i H(\mathbf{p}) > 0\right\}$,
- Escaping region: $\Sigma^e_{(i-1)i} = \left\{\mathbf{p} \in \Sigma_{(i-1)i} | X_{(i-1)}H(\mathbf{p}) < 0 \text{ and } X_i H(\mathbf{p}) > 0\right\}$,
- Sliding region: $\Sigma^s_{(i-1)i} = \left\{\mathbf{p} \in \Sigma_{(i-1)i} | X_{(i-1)}H(\mathbf{p}) > 0 \text{ and } X_i H(\mathbf{p}) < 0\right\}$.

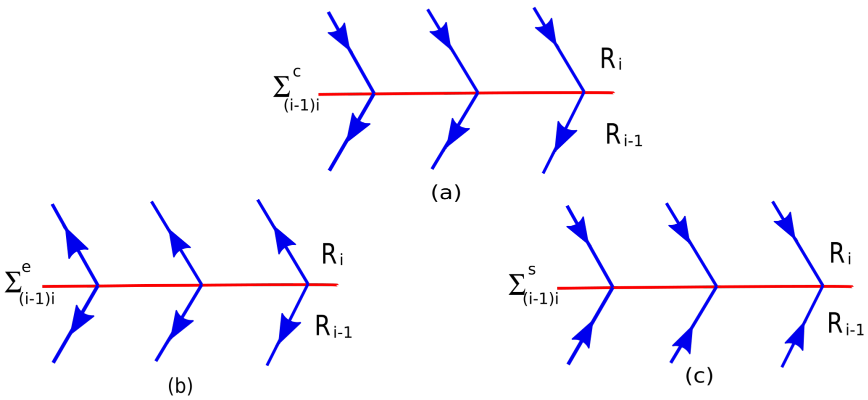

Figure 1.1: *The regions* $\Sigma^c_{(i-1)i}$ *in* (a), $\Sigma^e_{(i-1)i}$ *in* (b) *and* $\Sigma^s_{(i-1)i}$ *in* (c).

The regions $\Sigma^c_{(i-1)i}, \Sigma^e_{(i-1)i}$, and $\Sigma^s_{(i-1)i}$ are relatively open in Σ. The points $\mathbf{p} \in \Sigma_{(i-1)i}$ such that $X_i H(\mathbf{p}) = 0$ are called *tangency points* of the vector field X_i and $\mathbf{p} \in \Sigma_{ij}$ is said to be a tangency of order k of X_i if $X_i H(\mathbf{p}) = 0, \ldots, X_i^{k-1} H(\mathbf{p}) = 0$, and $X_i^k H(\mathbf{p}) \neq 0$. For instance if $k = 2$ we say that $\mathbf{p}$ is a quadratic tangency of X_i. On the other hand, if $X_i H(\mathbf{p}) \neq 0$ we say that X_i is transversal to Σ at $\mathbf{p}$.

When $\mathbf{p} \in \Sigma^c_{(i-1)i}$ it is natural to consider that the trajectories of $Z(\mathbf{p}) = X_{(i-1)}(\mathbf{p})$ are given by concatenations of trajectories of $X_{(i-1)}$ and X_i. So, for to determine all possible trajectories of such a vector field it is necessary to define the dynamics in regions $\Sigma^e_{(i-1)i}$ and $\Sigma^s_{(i-1)i}$. In $\Sigma^e_{(i-1)i} \cup \Sigma^s_{(i-1)i}$ we define a vector field $Z^s_{(i-1)i}$ given by the unique convex combination of $X_{(i-1)}$ and X_i which is tangent to $\Sigma_{(i-1)i}$ at $\mathbf{p}$, see Figure 1.2. This is if $\mathbf{p} \in \Sigma^e_{(i-1)i} \cup \Sigma^s_{(i-1)i}$, then

$$Z^s_{(i-1)i}(\mathbf{p}) = \frac{1}{X_i H(\mathbf{p}) - X_{(i-1)}H(\mathbf{p})} \left(X_i H(\mathbf{p}) X_{(i-1)}(\mathbf{p}) - X_{(i-1)}H(\mathbf{p}) X_i(\mathbf{p})\right).$$

Therefore we define the local trajectory for Filippov vector fields as follows. For more details see [13].

Definition 1.2 *The local trajectory of a Filippov vector field of the form* (1-1) *through a point* $\mathbf{p}$ *is defined as follows:*

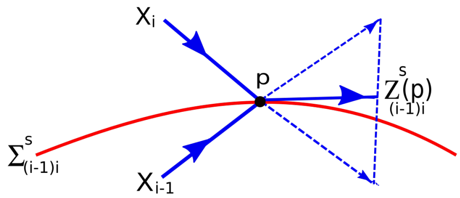

Figure 1.2: *Filippov vector field $Z^s_{(i-1)i}(\boldsymbol{p})$ when $\boldsymbol{p} \in \Sigma^s_{(i-1)i}$.*

- *For $\boldsymbol{p} \in R_i$ such that $X_i(\boldsymbol{p}) \neq 0$, $i = 1, \ldots, n$, the trajectory is given by $\varphi_Z(\boldsymbol{p},t) = \varphi_{X_i}(\boldsymbol{p},t)$ for $\boldsymbol{p} \in R_i$ and $t \in I \subset \mathbb{R}$.*
- *For $\boldsymbol{p} \in \Sigma^c_{(i-1)i}$ such that $X_{i-1}H(\boldsymbol{p}), X_iH(\boldsymbol{p}) > 0$ and taking the origin of time at $\boldsymbol{p}$, the trajectory is defined as $\varphi_Z(\boldsymbol{p},t) = \varphi_{X_{(i-1)}}(\boldsymbol{p},t)$ for $t \in I_p \cap (-\infty, 0]$ and $\varphi_Z(\boldsymbol{p},t) = \varphi_{X_i}(\boldsymbol{p},t)$ for $t \in I_p \cap [0,\infty)$. For the case $X_{i-1}H(\boldsymbol{p}), X_iH(\boldsymbol{p}) < 0$ the definition is the same reversing time.*
- *For $\boldsymbol{p} \in \Sigma^e_{(i-1)i} \cup \Sigma^s_{(i-1)i}$ such that $Z^s_{(i-1)i}(p) \neq 0$, $\varphi_Z(\boldsymbol{p},t) = \varphi_{Z^s_{(i-1)i}}(\boldsymbol{p},t)$ for $t \in I_p \subset \mathbb{R}$.*
- *For $\boldsymbol{p} \in \partial\Sigma^c_{(i-1)i} \cup \partial\Sigma^e_{(i-1)i} \cup \partial\Sigma^s_{(i-1)i}$ such that the definitions of trajectories for points in $\Sigma_{(i-1)i}$, in both sides of $\boldsymbol{p}$, can be extended to $\boldsymbol{p}$ and coincide, the trajectory through $\boldsymbol{p}$ is this trajectory. We will call these points regular tangency points.*
- *For any other point $\varphi_Z(\boldsymbol{p},t) = \boldsymbol{p}$ for all $t \in \mathbb{R}$. This is the case of the tangency points in $\Sigma_{(i-1)i}$ which are not regular and such that we will be called called the singular tangency points and the critical point of X_{i-1} in Σ_{i-1}, X_i in Σ_i and $Z^s_{(i-1)i}$ in $\Sigma^e_{(i-1)i} \cup \Sigma^s_{(i-1)i}$.*

In the particular case when the planar discontinuous vector field is defined in only two regions, we have that $Z(\mathbf{p}) = (X_1, X_2)$, then given a trajectory $\varphi_Z(\mathbf{q},t) \in R_1 \cup R_2$ and $\mathbf{p} \in \Sigma_{12}$, $\mathbf{p}$ is said to be a departing point (resp. arriving point) of $\varphi_Z(\mathbf{q},t)$ if there exists $t_0 < 0$ (resp. $t_0 > 0$) such that $\lim\limits_{t \to t_0^+} \varphi_Z(\mathbf{q},t) = \mathbf{p}$ (resp. $\lim\limits_{t \to t_0^-} \varphi_Z(\mathbf{q},t) = \mathbf{p}$). With these definitions if $\mathbf{p} \in \Sigma^c_{12}$, then it is a departing point (resp. arriving point) of $\varphi_{X_1}(\mathbf{q},t)$ for any $\mathbf{q} \in \gamma^1(\mathbf{p})$ (resp. $\mathbf{q} \in \gamma^2(\mathbf{p})$), where

$$\gamma^1(\mathbf{p}) = \{\varphi_Z(\mathbf{p},t); t \in I \cap t \geq 0\} \quad \text{and} \quad \gamma^2(\mathbf{p}) = \{\varphi_Z(\mathbf{p},t); t \in I \cap t \leq 0\},$$

are the trajectories of linear differential systems X_1 and X_2 in R_1 and R_2 through $\mathbf{p}$, respectively. These definitions are analogous if the piecewise linear differential system (1-1) is defined in n connected and open regions.

Definition 1.3 *A periodic orbit Γ of the discontinuous piecewise linear differential system* (1-1) *is a smooth piecewise curve which is formed by pieces of orbits of each linear differential system X_i, $\Gamma = \cup_{i \in I} \gamma^i$, contained in the regions R_i, respectively, and it is such that $\varphi_Z(\boldsymbol{p}, t+T) = \varphi_Z(\boldsymbol{p}, t)$, for some $T > 0$, where T is called of period of the orbit periodic Γ.*

If $\gamma^i \cap \Sigma \subset \Sigma^c$ for all $i = 1, \ldots, n$, then the periodic orbit Γ is called the crossing periodic orbit, otherwise is called sliding periodic orbit.

Definition 1.4 *If a periodic orbit Γ is isolated in the set of all periodic orbits of Z, then it is called limit cycle of piecewise linear differential system* (1-1).

Definition 1.5 *Consider two discontinuous vector fields Z and $\tilde{Z}$ defined in open sets, U and $\tilde{U}$, of $\mathbb{R}^2$ with discontinuity curves Σ and $\tilde{\Sigma}$, respectively. Then,*

- *Z and $\tilde{Z}$ are Σ-equivalent if there exists an orientation preserving homeomorphism $h : U \to \tilde{U}$ that sends Σ to $\tilde{\Sigma}$ and sends orbits of Z to orbits of $\tilde{Z}$;*
- *Z and $\tilde{Z}$ are topologically equivalent if there exists an orientation preserving homeomorphism $h : U \to \tilde{U}$ sends orbits of Z to orbits of $\tilde{Z}$.*

1.2 The Poincaré maps

First we collect the basic idea of the Poincaré map from the qualitative theory of the ordinary differential equations.

One of the most important tools in the study of flows in the neighborhood of periodic orbits is the so called Poincaré map. We consider a locally Lipschitz vector field $F : U \to \mathbb{R}^n$ and let $\varphi(\mathbf{x}, s)$ be the flow defined by the differential equation $\dot{\mathbf{x}} = F(\mathbf{x})$. Let L be a hypersurface in $\mathbb{R}^n$ and take a point $\mathbf{p} \in L \cap U$. The flow φ is said to be a transverse flow to L at point $\mathbf{p}$ if $F(\mathbf{p})$ is not contained in the tangent space to L at point $\mathbf{p}$. If $F(\mathbf{p}) \in T_{\mathbf{p}}L$, then the point $\mathbf{p}$ is called a contact point of the flow with L. Let V be an open subset of L. We say that the flow is transverse to L at V if the flow is transverse to L at every point in V.

Now we consider two open hypersurfaces L_1, L_2 and two points $\mathbf{p}_1 \in L_1 \cap U$, $\mathbf{p}_2 \in L_2 \cap U$ such that $\mathbf{p}_2 = \varphi(\mathbf{p}_1, s_1)$. There exist a neighborhood V_1 of $\mathbf{p}_1$ in $L_1 \cap U$, a neighborhood V_2 of $\mathbf{p}_2$ in $L_2 \cap U$ and a function $\tau : V_1 \to \mathbb{R}$ satisfying $\tau(\mathbf{p}_1) = s_1$ and $\varphi(\mathbf{q}, \tau(\mathbf{q})) \in V_2$ for every $\mathbf{q} \in V_1$. Moreover, if the vector field F is globally Lipschitz, C^r with $r \geq 1$, or analytic, then the function τ is also continuous, C^r with $r \geq 1$ or analytic, respectively. In this situation we define the Poincaré map as being the map $\pi : V_1 \to V_2$ such that

$$\pi(\mathbf{q}) = \varphi(\mathbf{q}, \tau(\mathbf{q})), \quad \text{for every } \mathbf{q} \in V_1.$$

When the vector field is globally Lipschitz, C^r with $r \geq 1$ or analytic, the Poincaré map π is also continuous, C^r with $r \geq 1$ or analytic, respectively. By reversing the sense of the flow it is possible to conclude that the Poincaré map is invertible and the inverse map π^{-1} is continuous, C^r with $r \geq 1$ or analytic, respectively. In the particular case of $L_1 = L_2$ the Poincaré map π will be called a *return map*.

Consider $\mathbf{p} \in L_1$ and let $\gamma(\mathbf{p})$ be a periodic orbit. From the continuous dependence of the flow on the initial conditions, it follows that a return map π can be defined in a neighborhood of $\mathbf{p}$, and $\mathbf{p}$ is a fixed point of π. Conversely if $\mathbf{p} \in L_1$ is a fixed point of a return map π then $\gamma(\mathbf{p})$ is a periodic orbit. Hence limit cycles are associated to isolated fixed points of return maps.

When we consider the PWLS (1-1) we can distinguish two different kind of periodic orbits Γ depending on their location in the phase plane. First the periodic orbits Γ which are contained in one of the open regions R_i where the systems are linear then periodic orbits in the class appear only inside of a linear center. And second the periodic orbit Γ that intersect the boundaries $\Sigma_{(i-1)i}$, this is $\Gamma = \cup_{i \in I} \gamma^i$, where $I = \{1, 2, 3, \ldots, n\}$, when $i = 1$ we denote $i - 1 = n$.

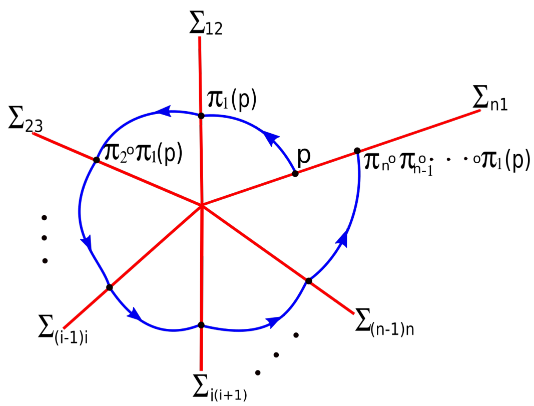

Figure 1.3: *Poincaré map of PWLS* (1-1).

Definition 1.6 *We define a section map π_i of system* (1-1) *in R_i as follows: for any $p \in \Sigma_{(i-1)i}$, a part from the origin, $\pi_i(\boldsymbol{p})$ is the first intersection of the flow of system* (1-1) *with $\Sigma_{i(i+1)}$, where the flow starting from $\boldsymbol{p}$ will always stay in R_i until it arrives at the point $\pi_i(\boldsymbol{p})$. This is $\pi_i : \Sigma_{(i-1)i} \to \Sigma_{i(i+1)}$ such that*

$$\pi_i(\boldsymbol{p}) = \varphi_{X_i}(\boldsymbol{p}, \tau_i(\boldsymbol{p})), \quad \textit{for every } \boldsymbol{p} \in \Sigma_{(i-1)i}.$$

The complete return map π associated to system (1-1) is given by the composition of these section map π_i, see Figure 1.3. This is $\pi : \Sigma_{(i-1)i} \to \Sigma_{(i-1)i}$ and

$$\pi(\mathbf{p}) = \pi_n \circ \pi_{n-1} \circ \cdots \pi_1(\mathbf{p}), \quad \text{for every } \mathbf{p} \in \Sigma_{(i-1)i}.$$

When $i = 2$ and the discontinuity curve Σ is a straight line, we have that Σ split the phase plane into the half–planes $\Sigma^{\pm}$ and we obtain the PWLS (0-1), then $\pi : \Sigma \to \Sigma$ and

$$\pi(\mathbf{p}) = \pi_{+}(\pi_{-}(\mathbf{p})), \quad \text{for every } \mathbf{p} \in \Sigma.$$

In order to have limit cycles for PWLS (0-1) we must determine the fixed points of the function π.

1.3 Extended Complete Chebyshev systems

The functions $f_0, f_1, ..., f_n$, defined on an open set $U \subset \mathbb{R}$ are *linearly independent* functions if

$$\text{for every } t \in U, \ \sum_{i=0}^{n} \alpha_i f_i(t) = 0 \text{ implies that } \alpha_0 = \alpha_1 = ... = \alpha_n = 0.$$

Proposition 1.7 *Let $f_0, f_1, ..., f_n$ be analytic functions defined on an open interval $U \subset \mathbb{R}$. If the functions $f_0, f_1, ..., f_n$ are linearly independent then there exists $\tilde{t}_1, ..., \tilde{t}_n \in U$ and $\tilde{\alpha}_0, \tilde{\alpha}_1, ..., \tilde{\alpha}_n \in \mathbb{R}$ such that $\sum_{i=0}^{n} \tilde{\alpha}_i f_i(\tilde{t}_j) = 0$, for every $j \in \{1, ..., n\}$.*

For a proof of Proposition 1.7 see [19] or [22].

Now we recall the concept of Chebyshev systems. For more details see [19].

Definition 1.8 *Let $\mathcal{F} = \{f_0, f_1, ..., f_n\}$ be an ordered set of smooth real functions defined on an interval $I \subset \mathbb{R}$. The set $\mathcal{F}$ is an Extended Chebyshev system (ET-system) on I if and only if the maximum number of zeros counting multiplicities by any non-trivial linear combination of functions in $\mathcal{F}$ is at most n, and this number is reached. The family $\mathcal{F}$ is an Extended Complete Chebyshev system (ECT-system) on I if and only if for any $k \in \{0, 1, ..., n\}$ the set $\mathcal{F}_k = \{f_0, f_1, ..., f_k\}$ is an Extended Chebyshev system.*

We recall the definition of the Wronskian of a set of functions:

$$W_k(f_0, ..., f_k)(s) = \det M(f_0, ..., f_k)(s),$$

where

$$M(f_0, f_1, ..., f_k)(s) = \begin{pmatrix} f_0(s) & f_1(s) & \cdots & f_k(s) \\ f_0'(s) & f_1'(s) & \cdots & f_k'(s) \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(k)}(s) & f_1^{(k)}(s) & \cdots & f_k^{(k)}(s) \end{pmatrix}.$$

Proposition 1.9 *The ordered set of functions $\mathcal{F}$ is an ECT-system on I if and only if the Wronskians $W_k(f_0, f_1, ..., f_k)(t) \neq 0$, on I for each $k \in \{0, 1, ..., n\}$.*

For a proof see [19].

1.4 Lambert function

In mathematics, the Lambert $\mathcal{W}$ function, also called the omega function or product logarithm, is a set of functions, namely the branches of the inverse relation of the function $f(z) = ze^z$, this is

$$z = f^{-1}(ze^z) = \mathcal{W}(ze^z).$$

It is called in this way due to mathematical Johann Heinrich Lambert. The relation $\mathcal{W}$

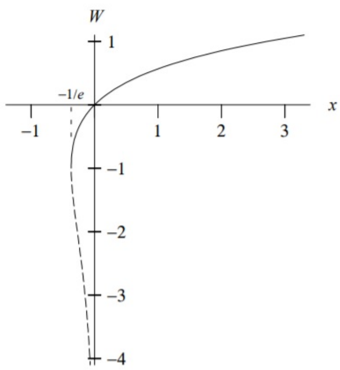

Figure 1.4: *The Lambert function.*

is multivalued (except at 0) and it decreases if $z < -1$ and creases if $z > -1$. The global minimum of function $f(z) = ze^z$ is $f(-1) = -1/e$. Then $\mathcal{W}(z)$ defines a single-valued function in $(-1/e, \infty)$.

For more details see [4].

1.5 First integrals

The aim of this section is to introduce the terminology of the Darboux theory of integrability for real planar polynomial differential systems. For a detailed discussion of this theory see [5]. A real planar polynomial differential system or simply a polynomial system will be a differential system of the form

$$\frac{dx}{ds} = \dot{x} = P(x,y), \quad \frac{dy}{ds} = \dot{y} = Q(x,y), \tag{1-2}$$

where x and y are real variables, the independent one (the time) s is real, and P and Q are polynomials in the variables x and y with real coefficients. The degree of polynomial system (1-2) is defined as $m = \max\{\deg P, \deg Q\}$. The vector field X associated to system (1-2) is defined by

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}.$$

System (1-2) is integrable on an open subset U of $\mathbb{R}^2$ if there exists a non constant analytic function $H : U \to \mathbb{C}$, called a *first integral* of (1-2) on U, which is constant on all orbits of system (1-2) contained on U.

1.6 Bezout inequality

In this section we consider the intersection of two algebraic curves. Suppose the curves are given by $F(x,y) = 0$ and $G(x,y) = 0$, then we shall be interested in their common solutions. In order to solve this system of equations we need to introduce some elimination theory for polynomial equations. In general this is done using so-called Grobner basis techniques or with the resultant of the two polynomials. Consider that R is an integral domain and consider two polynomials $F(x), G(x) \in R[x]$ and write them in the form $F(x) = p_m x^m + p_{m-1}x^{m-1} + \ldots + p_1 x + p_0$, $G(x) = q_n x^n + \ldots + q_1 x + q_0$. The *resultant* of F and G is given by the determinant of the so called *Sylvester matrix*

$$Res(F,G) = \begin{pmatrix} p_0 & p_1 & \cdots & p_m & 0 & \cdots & 0 \\ 0 & p_0 & \cdots & p_{m-1} & p_m & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & \cdots & p_0 & p_1 & \cdots & p_m \\ q_0 & q_1 & \cdots & q_n & 0 & \cdots & 0 \\ 0 & q_0 & \cdots & q_{n-1} & q_n & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & \cdots & q_0 & \cdots & q_{n-1} & q_n \end{pmatrix}$$

Theorem 1.10 *Let $F, G \in k[x,y]$ and suppose the total degree of F is m and the total degree of G is n. Then $Res_x(F,G)$ is either zero or a polynomial of degree $\leq mn$ in y.*

Let $F, G \in k[x,y]$ be two polynomials without common non-constant factor. Let x_0, y_0 be such that $F(x_0,y_0) = G(x_0,y_0) = 0$. Then $Res_x(F,G)(y_0) = 0$ as well. Since $Res_x(F,G)$ has degree at most mn we see that at most mn values of y_0 are possible. Similarly at most mn values of x_0 are possible. As a consequence the set of points x_0, y_0 satisfying $F(x_0,y_0) = G(x_0,y_0) = 0$ is at most finite. We can phrase this alternatively. Consider two algebraic curves C, D given by the equations $F(x,y) = 0$ and $G(x,y) = 0$. We shall say that C, D have a common component if F, G have non-constant common divisor

$H \in k[x,y]$. The common component is then the curve given by the equation $H(x,y) = 0$. If F, G do not have a non-constant common factor we say that the curves do not have a common component. Any point x_0, y_0 satisfying $F(x_0, y_0) = G(x_0, y_0) = 0$ can be seen as an intersection point of C and D. So we see that two algebraic curves without common component intersect in finitely many points. We can say a bit more though.

Theorem 1.11 (Bezout Theorem) *Let C, D be two algebraic curves of degree m, n respectively. Suppose that the curves have no common component. Then the number of intersection points of C, D is at most mn.*

For more details, see [34].

CHAPTER 2

Crossing limit cycles for PWLS separated by a straight line and having symmetric equilibrium points

2.1 Introduction

The study to provide a sharp upper bound for the maximum number of crossing limit cycles for discontinuous PWLS separated by a curve is a very difficult problem, even when this curve is a straight line. And there are two reasons that make difficult the analysis of this problem. First, even one can easily integrate the solution of every linear differential system X^- and X^+, respectively, it is difficult to determine explicitly the time that an orbit expends in each region governed by each linear differential system. And second, the number of parameters needed to analyze all possible cases is in general not small.

In order to simplify the determination of the time that each orbit expends in each region of a discontinuous PWLS separated by a straight line, several researchers restrict these systems to some special cases and study the upper bounds for maximum number of crossing limit cycles for them. For instance, in [24] it was studied the case when one of the linear differential systems in (0-1) has its equilibrium point on the straight line of discontinuity, [10] the authors studied systems (0-1) such that have a maximal crossing set and in [35] it was studied the case when the discontinuous PWLS (0-1) has a unique non-degenerated equilibrium.

In this chapter we study the maximum number of crossing limit cycles that can have the planar PWLS (0-1) when the equilibrium points of the differential linear systems X^- and X^+ are symmetric with respect to the line of discontinuity Σ.

The singularities considered in these cases can be real or virtual. We say that $P^r \in \mathbb{R}^2$ is a real singularity of system (0-1) if $P^r = (x_1, x_2)$ is such that either $x_1 < 0$ and $X^-(P^r) = 0$, or $x_1 > 0$ and $X^+(P^r) = 0$. On the other hand $P^v = (x_1, x_2) \in \mathbb{R}^2$ is a virtual singularity if either $x_1 > 0$ and $X^-(P^v) = 0$, or $x_1 < 0$ and $X^+(P^v) = 0$. It follows that the singularities of system (0-1) can be of type virtual-virtual (P_-^v, P_+^v), virtual-real (P_-^v, P_+^r) and real-real (P_-^r, P_+^r) depending on the singularities of the systems X^- and X^+ are either virtual or real.

In order to reduce the number of parameters on which the PWLS (0-1) depends we use the canonical forms in the Propositions 2.1 and 2.2.

2.1.1 Canonical forms

We observe that the PWLS (0-1) depends on twelve parameters. Then in order to simplify the analysis of these systems we consider the following canonical forms.

Proposition 2.1 *There exists a topological equivalence between the phase portrait of the discontinuous PWLS (0-1) and the phase portrait of the discontinuous PWLS (2-1) for all the orbits not having points in common with the sliding set.*

$$\dot{X}(x,y) = \begin{cases} X^-(x,y) = \begin{pmatrix} 2l & -1 \\ l^2-\alpha^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ a \end{pmatrix}, & \text{if } (x,y) \in \Sigma^-, \\ X^+(x,y) = \begin{pmatrix} 2r & -1 \\ r^2-\beta^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b \\ c \end{pmatrix}, & \text{if } (x,y) \in \Sigma^+, \end{cases} \tag{2-1}$$

where $\alpha, \beta \in \{i, 0, 1\}$. If $\alpha = i$, we have that the equilibrium point of a linear differential system X^- has eigenvalues $\lambda_{1,2}^- = l \pm i$, so it is a focus if $l \neq 0$ or a center if $l = 0$. When $\alpha = 0$, then the equilibrium point of a linear differential system X^- has one eigenvalue of multiplicity 2, namely $\lambda^- = l \neq 0$, so it is a non-diagonalizable node. If $\alpha = 1$ the equilibrium point of a linear differential system X^- has eigenvalues $\lambda_1^- = l-1$ and $\lambda_2^- = l+1$, then we have that the equilibrium point of X^- is a saddle if $|l| < 1$ or it is a diagonalizable node if $|l| > 1$. Analogously for the linear differential system X^+.

For a proof of Proposition 2.1 see [8].

Other normal form which is independent of the change of coordinates it is provide in the following proposition.

Proposition 2.2 *Consider the linear differential system*

$$\dot{X}(x,y) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{2-2}$$

it has a singularity

(a) of type focus(F) (resp. a center(C)) if

$$\dot{X}(x,y) = \begin{pmatrix} A & B \\ \dfrac{-(A-\tilde{C})^2 - d^2}{B} & 2\tilde{C} - A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{2-3}$$

with $B < 0$ and $\tilde{C} \neq 0$ (resp. $\tilde{C} = 0$ and $B < 0$);

(b) of type diagonalizable node(N) (resp. an improper node(iN)) if

$$\dot{X}(x,y) = \begin{pmatrix} A & B \\ \dfrac{-(A-\tilde{C})^2 + d^2}{B} & 2\tilde{C} - A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{2-4}$$

with $\tilde{C}^2 > d^2 > 0$ and $B < 0$ (resp. $d = 0$ and $B < 0$);

(c) of type saddle(S) if

$$\dot{X}(x,y) = \begin{pmatrix} A & B \\ \dfrac{-(A-\tilde{C})^2 + d^2}{B} & 2\tilde{C} - A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{2-5}$$

with $0 < \tilde{C}^2 < d^2$ and $B < 0$.

Where the parameters $\tilde{C}$,A and B in (2-3), (2-4) and (2-5) are such that $2\tilde{C} = a_{11} + a_{22}$, $A = a_{11}$ and $B = a_{12}$.

Proof. We know that the eigenvalues of linear differential system (2-2) are

$$\lambda_{1,2} = \frac{a_{11} + a_{22} \pm \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}}{2}. \tag{2-6}$$

(a) If we consider $a_{11} + a_{22} = 2\tilde{C}$, this is $a_{22} = 2\tilde{C} - a_{11}$, with $\tilde{C}, a_{11} \in \mathbb{R}$ and $(a_{11} - a_{22})^2 + 4a_{12}a_{21} = -4d^2$, this is $a_{21} = (-(a_{11} - \tilde{C})^2 - d^2)/a_{12}$, with $d, a_{12} \in \mathbb{R}$. Then the eigenvalues (2-6) are $\lambda_{1,2} = \tilde{C} \pm id$, therefore the singularity of linear differential system (2-2) is a focus (F) if $\tilde{C} \neq 0$, and a center (C) if $\tilde{C} \neq 0$. Considering $a_{11} = A$ and $a_{12} = B$, we obtain system (2-3).

(b) We consider $a_{11} + a_{22} = 2\tilde{C}$, then analogously to the above case $a_{22} = 2\tilde{C} - A$, and we assume that $(a_{11} - a_{22})^2 + 4a_{12}a_{21} = 4d^2$, then $a_{21} = (-(A - \tilde{C})^2 + d^2)/B$. Then the eigenvalues (2-6) are $\lambda_{1,2} = \tilde{C} \pm d$, therefore the singularity of linear differential system (2-2) is a diagonalizable node (N), if $\tilde{C}^2 > d^2 > 0$ and $B < 0$, because the two eigenvalues would have the same sign, and it is a improper node (iN), if $d = 0$, because the two eigenvalues would be equals. Therefore we obtain system (2-4).

(c) Analogously to the previous case we consider $a_{22} = 2\tilde{C} - A$ and $a_{21} = (-(A - \tilde{C})^2 + d^2)/B$. Then the eigenvalues (2-6) are $\lambda_{1,2} = \tilde{C} \pm d$, therefore the singularity of linear differential system (2-2) is a saddle(S), if $0 < \tilde{C}^2 < d^2$ and $B < 0$, because with this condition we have that $\lambda_1 \lambda_2 < 0$. Therefore we obtain system (2-5). □

2.1.2 Closing equations method

Usually the name *closing equations method*, means a method for determining periodic orbits in piecewise linear dynamical systems. The main idea of the method is to integrate the corresponding system in each linear zone and obtain a system of equations, called *closing equations*, whose solutions correspond to the periodic orbits of the initial PWLS. This method was used for instance in the papers [1, 32].

We consider the method for the PWLS (0-1) where $\Sigma = \{(x,y) : x = 0\}$ is the discontinuity straight line. Our interest is to analyze the properties of periodic orbits that intersect both zones of $\mathbb{R}^2$, namely $\Sigma^{\pm}$; on the contrary there should be periodic orbits totally contained in one of the half-spaces Σ^- or Σ^+, thereby being purely linear periodic orbits belonging to a linear center. If we assume the existence of one periodic orbit Γ which intersects Σ at the two points $\mathbf{p} = (0, y_0)$ and $\mathbf{q} = (0, y_1)$. If $t_1 > 0$ is the finite time that an orbit of linear differential system X^- expends inside Σ^- starting at the point $\mathbf{p}$ and entering in Σ^- in forward time, and let $t_2 > 0$ be the finite time that an orbit of linear differential system X^+ expends inside Σ^+ starting at the point $\mathbf{q}$ and entering in Σ^+ in forward time. We consider that the point $\mathbf{p}$ is mapped into $\mathbf{q}$ by the flow on the left region. Since the PWLS (0-1) is formed by two linear differential systems in each region, we have that the solution of linear differential system X^- starting at $\mathbf{p}$ is

$$\varphi_{X^-}(\mathbf{p},t) = (x^-(t), y^-(t)) = e^{A^- t}\mathbf{p} + \int_0^t e^{A^-(t-s)} B^- ds.$$

Therefore

$$\mathbf{q} = (0, y_1) = e^{A^- t_1}\mathbf{p} + \int_0^{t_1} e^{A^-(t_1-s)} B^- ds. \tag{2-7}$$

Analogously in the region Σ^+, we conclude that

$$\mathbf{p} = (0, y_0) = e^{A^+ t_2}\mathbf{q} + \int_0^{t_2} e^{A^+(t_2-s)} B^+ ds. \tag{2-8}$$

Then we obtain that (2-7) and (2-8) form a nonlinear system with 4 equations and 4 unknowns. Namely y_0, y_1 and the the flight times t_1 and t_2. As we have that $\mathbf{q} = \pi_-(\mathbf{p})$, this is $y_1 = y^-(t_1)$ we obtain that system (2-7) is equivalent to equation $x^-(t_1) = 0$. In order to reduce the unknowns in the system formed by (2-7) and (2-8) we consider $-t_2$ be the finite time that an orbit of linear differential system X^+ expends inside Σ^+ starting at the point $\mathbf{p}$ and entering in Σ^+ in backward time. With these conditions system (2-8) is

$$\mathbf{q} = (0, y_1) = e^{-A^+ t_2}\mathbf{p} - \int_{-t_2}^{0} e^{-A^+(t_2+s)} B^+ ds. \tag{2-9}$$

We summarize the above ideas in the following result.

Proposition 2.3 *Assume that the PWLS (0-1) has a crossing periodic orbit that transversely intersecting the straight line Σ in the points $\boldsymbol{p} = (0, y_0)$ and $\boldsymbol{q} = (0, y_1)$ where $y_1 = y^-(t_1)$ and $y_0 > y_1$, with flight times $t_1 > 0$ and $t_2 > 0$ in the zones Σ^- and Σ^+, respectively. Then (t_1, t_2, y_0) are real solutions of the closing equations:*

$$\begin{aligned} e_1 :&\quad x^-(t_1) = 0, \\ e_2 :&\quad x^+(-t_2) = 0, \\ e_3 :&\quad y^+(-t_2) - y^-(t_1) = 0. \end{aligned} \tag{2-10}$$

2.2 Statement of the main results

We analyze the possible configurations that to arise when the equilibrium points of the linear differential systems X^- and X^+ are symmetric with respect to the straight line Σ. We denote those configurations like (P_-, P_+) depending of type and the position of the equilibrium points, $P_-, P_+ \in \{C^r, C^v, F^r, F^v, N^r, N^v, iN^r, iN^v, S^r, S^v\}$. Where (C^r) denotes a singularity of real center type; (C^v) denotes a singularity of virtual center type; (F^r) denotes a singularity of real focus type; (F^v) denotes a singularity of virtual focus type; (N^r) denotes a singularity of real diagonal node type; (N^v) denotes a singularity of virtual diagonal node type; (iN^r) denotes a singularity of real improper node type; (iN^v) denotes a singularity of virtual improper node type; (S^r) denotes a singularity of real saddle type and (S^v) denotes a singularity of virtual saddle type.

We observed that the equilibrium points P_- and P_+ can not be a saddle S^v, a diagonalizable node N^r or an improper node iN^r because the first return map for the linear differential systems X^- or X^+ is not defined on the discontinuity straight line Σ.

We assume that the equilibrium points P_- and P_+ of linear differential systems X^- and X^+, respectively are symmetric with respect to the line of discontinuity Σ. Then we obtain two options, first the case when the singularities of X^- and X^+ are symmetric with respect Σ and they are on the straight line $y = \varepsilon$, $\varepsilon \in \mathbb{R}$, this is, the singularities are $(-k, \varepsilon)$ or (k, ε), with $k \in \mathbb{R}^+$. Second we have the case when the singularities of linear differential systems X^- and X^+ are symmetric with respect Σ and they are on the straight line $y = sx$, with $s \in \mathbb{R}$, this is, the equilibrium points are $(-k, -sk)$ and (k, sk).

In Theorem A we assume that the singularities P_- and P_+ are on the straight line $y = sx$, with $s \in \mathbb{R}$ and we observe that this condition is sufficient to analyze the above two cases because when $\varepsilon = 0$ the equilibrium points are $(-k,0)$ and $(k,0)$ which are on the straight line $y = sx$, with $s = 0$ and it is possible to verify that the number of crossing limit cycles when the equilibrium point are on the straight line $y = \varepsilon$ independent of the epsilon.

If the linear differential system X^- has a center (C) we have the following options of configurations: $(C^r,C^r),(C^r,F^r),(C^r,S^r),(C^v,C^v),(C^v,F^v),(C^v,N^v)$ and (C^v,iN^v). In the paper [24] it was proved that if the planar PWLS (0-1) has the configuration (C^r,C^r) or (C^v,C^v), then there are no crossing limit cycles. Therefore in statement (i) of Theorem A we study the remaining five cases.

When the singularity P_- of the linear differential system X^- is a focus (F) we have the following options: $(F^r,C^r),(F^r,F^r),(F^r,S^r),(F^v,C^v),(F^v,F^v),(F^v,N^v)$ and (F^v,iN^v), here we observed that due to that having symmetric equilibrium points with respect the discontinuity straight line Σ, the configurations (F^r,C^r) and (C^r,F^r); (F^v,C^v) and (C^v,F^v) are equivalent. Then we study the remaining five cases in statement (ii) of Theorem A.

If P_- is a saddle (S) we have the configurations $(S^r,C^r),(S^r,F^r)$ and (S^r,S^r), but the configurations (S^r,C^r) and (C^r,S^r) are equivalent, and the configurations (S^r,F^r) and (F^r,S^r) are equivalent, then in this case we only have one possible new configuration (S^r,S^r) which is analyzed in statement (iii) of Theorem A.

When P_- is a diagonalizable node (N), we have the following configurations: (N^v,C^v), (N^v,F^v), (N^v,N^v) and (N^v,iN^v), since the previous two cases have been already studied, we only need to study the cases (N^v,N^v) and (N^v,iN^v) in statement (iv) of Theorem A. The configuration (N^v,F^v) is in the statement (ii) of Theorem because it is equivalent to the configuration (F^v,N^v) due to that having symmetric equilibrium points with respect the discontinuity straight line Σ.

When the singularity P_- is an improper node (iN), we only study the configuration (iN^v,iN^v) in statement (v) of Theorem A, because having symmetric equilibrium points with respect to discontinuity straight line Σ, the configurations $(iN^v,C^v),(iN^v,F^v),(iN^v,N^v)$ are considered in the above cases.

We denote the maximum number of crossing limit cycles of planar PWLS (0-1) by $\mathcal{N}(P_-,P_+)$.

Theorem A *Consider that the linear differential systems X^- and X^+ in (0-1) have symmetric equilibrium points with respect the discontinuity straight line Σ and they are on the straight line $y = sx$, $s \in \mathbb{R}$. Then the following statements hold.*

(i) $\mathcal{N}(C^r,F^r) = \mathcal{N}(C^r,S^r) = \mathcal{N}(C^v,F^v) = \mathcal{N}(C^v,N^v) = \mathcal{N}(C^v,iN^v) = 1$. *Moreover these upper bounds are reached and the crossing limit cycles are stables.*

(ii) $\mathcal{N}(F^r,F^r)\geq 2$, $\mathcal{N}(F^r,S^r)\geq 2$, $\mathcal{N}(F^v,F^v)\geq 2$, $\mathcal{N}(F^v,N^v)\geq 1$ *and* $\mathcal{N}(F^v,iN^v)\geq 2$. *See Figures* 2.6, 2.7, 2.8, 2.9 *and* 2.11, *respectively.*

(iii) $\mathcal{N}(S^r,S^r)\geq 1$. *See Figure* 2.12.

(iv) $\mathcal{N}(N^v,N^v)\geq 2$ *and* $\mathcal{N}(N^v,iN^v)\geq 2$. *See Figures* 2.13, 2.15.

(v) $\mathcal{N}(iN^v,iN^v)\geq 1$. *See Figure* 2.16.

Theorem A is proved in Section 2.3.

Proposition 2.4 *The upper bound for the maximum number of crossing limit cycles provided in statement (i) of Theorem A is reached and the crossing limit cycle in each configuration of statement (i) it is hyperbolic. See Figures* 2.1 − 2.5.

2.3 Proof of the mains results

In this section we provide the proofs of Proposition 2.4 and Theorem *A*. The proof of Proposition 2.4 is provide by the following examples, where we prove that the upper bound provided in statement (i) of Theorem A is reached in each case.

Example 2.5 *We consider PWLS* (0-1) *with the configuration (C^r,F^r) formed by the linear differential systems* (2-18) *and* (2-22), *with $A=-2$, $B=-8/10$, $d=7/10$, $r=-2/10$, $k=1$ and $s=0$ then we obtain that*

$$X^-(x,y)=\begin{pmatrix}-2 & -\frac{4}{5}\\ \frac{449}{80} & 2\end{pmatrix}X+\begin{pmatrix}-2\\ \frac{449}{80}\end{pmatrix},\quad X^+(x,y)=\begin{pmatrix}-\frac{2}{5} & -1\\ \frac{26}{25} & 0\end{pmatrix}X+\begin{pmatrix}\frac{2}{5}\\ -\frac{26}{25}\end{pmatrix}. \tag{2-11}$$

For this PWLS we have that closing equations (2-23) *are*

$$\begin{aligned}
-1+\cos\left(\frac{7t_1}{10}\right)-\frac{4}{7}(5+2y_0)\sin\left(\frac{7t_1}{10}\right)&=0,\\
1+e^{t_2/5}\left(-\cos(t_2)+\left(-\frac{1}{5}+y_0\right)\sin(t_2)\right)&=0,\\
-y_0\cos\left(\frac{7t_1}{10}\right)-\frac{1}{56}(449+160y_0)\sin\left(\frac{7t_1}{10}\right)&\\
+\frac{1}{25}e^{t_2/5}\left(25y_0\cos(t_2)+(26-5y_0)\sin(t_2)\right)&=0.
\end{aligned} \tag{2-12}$$

Taking into account that $t_1t_2>0$ and that $t_1,t_2\in(0,2\pi)$ it is possible verify computationally that the system (2-12) *has two real solutions, namely $(t_1^1,t_2^1,y_0^1)=(4.796799..,3.418539..,5.564042..)$ and $(t_1^2,t_2^2,y_0^2)=(5.859455..,5.731792..,-0.819335..)$. Nevertheless the orbit of linear differential system X^+ starting at the point $(x,y)=(0,y_0^2)=(0,-0.819335..)$ and with flight* *time* $t_2^2 = 5.731792..$ *it is such that intersects the region* Σ^- *which cannot happen to obtain a crossing limit cycle of PWLS* (2-11)*, therefore we have the unique real solution that generates one crossing limit cycle* Γ_1 *of the PWLS* (2-11) *is* $(t_1^1, t_2^1, y_0^1) = (4.796799.., 3.418539.., 5.564042..)$*, and that crossing limit cycle starts at the point* $(0, y_0^1) = (0, 5.564042..)$*, enters in the half-plane* Σ^- *and after a time* $t_1^1 = 4.796799..$ *reaches the discontinuity line* Σ *at the point* $(0, y_1^1) = (0, -10.564042..)$*, enters in the half-plane* Σ^+ *and after a time* $t_2^1 = 3.418539..$ *reaches the point* $(0, y_0^1)$*. See Figure* 2.1*.*

Now we analyze the stability of the crossing limit cycle Γ_1*. We consider the PWLS* (2-11) *and we analyze the flow of PWLS around of the crossing limit cycle* Γ_1 *which intersects the discontinuity straight line* Σ *at the points* $y_0 = 5.564042..$ *and* $y_1 = -10.564042...$

We consider a point $W_0 \in \Sigma$ *and within the region limited by the crossing limit cycle* Γ_1*, this is,* $W_0 = (0, w_0)$ *with* $-10.564042.. < w_0 < 5.564042...$ *For example we consider that* $w_0 = 5$*, then the solution of linear differential system* X^- *in* (2-11) *starting at the point* $W_0 = (0, 5) \in \Sigma$ *is*

$$x^-(t) = -1 + \cos\left(\frac{7t}{10}\right) - \frac{60}{7}\sin\left(\frac{7t}{10}\right), \quad y^-(t) = 5\cos\left(\frac{7t}{10}\right) + \frac{1249}{56}\sin\left(\frac{7t}{10}\right),$$

and the flight time in the region Σ^- *is*

$$t^- = \frac{10}{7}\left(-\pi + \arctan\left(\frac{840}{3551}\right) + 2\pi\right),$$

then the intersection point with Σ *is* $W_1 = (0, w_1) = (0, y^-(t^-))$*, where* $y^-(t^-) = -10$*. Now the solution of linear differential system* X^+ *in* (2-11) *starting at the point* $W_1 = (0, -10)$ *is*

$$x^+(t) = 1 + \frac{e^{-t/5}}{5}(-5\cos(t) + 51\sin(t)), \quad y^+(t) = -\frac{2}{25}e^{-t/5}(125\cos(t) + 38\sin(t)),$$

the flight time in the region Σ^+ *is* $t^+ = 3.434483..$ *and the intersection point of this orbit with the discontinuity straight line is the point* $W_2 = (0, w_2) = (0, y^+(t^+)) = (0, 5.258689..)$*, then* $5 = w_0 < w_2 = 5.258689$*. Therefore we obtain that the flow of PWLS* (2-11) *spirals in the counterclockwise outward for points* $W_0 = (0, w_0)$ *with* $-10.564042.. < w_0 < 5.564042...$ *Now we consider a point on* Σ *and outside the region limited by* Γ_1*, namely* $Z_0 = (0, z_0)$ *with* $z_0 > y_0$*. We consider* $Z_0 = (0, 6)$ *and similarly to above case we determine the solution* $(x^-(t), y^-(t))$ *of linear differential system* X^- *in* (2-11) *starting at the point* $Z_0 = (0, 6) \in \Sigma$ *and we get the flight time in the region* Σ^-*, namely* $T^- = 10/7(-\pi + \arctan(952/4575) + 2\pi)$*, and the intersection point of this orbit*

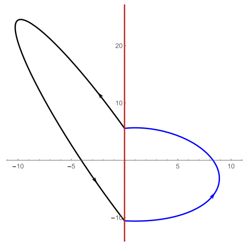

Figure 2.1: *The crossing limit cycle of the discontinuous PWLS (2-11) with configuration (C^r, F^r).*

with Σ is $Z_1 = (0, z_1) = (0, y^-(T^-)) = (0, -11)$. We determine the solution $(x^+(t), y^+(t))$ of linear differential system X^+ in (2-11) starting at the point $Z_1 = (0, -11) \in \Sigma$ and we get the flight time in the region Σ^+, $T^+ = 3.407359..$ and finally we obtain the intersection point of this orbit with Σ, $Z_2 = (0, z_2) = (0, y^+(T^+)) = (0, 5.799713..)$, then $6 = z_0 > z_2 = 5.799713...$ Therefore obtain that the flow of PWLS (2-11) spirals in the counterclockwise inward for points $Z_0 = (0, z_0)$with $z_0 > y_0$. Therefore we can conclude that the crossing limit cycle Γ_1 is a crossing limit cycle stable. $\square$

Example 2.6 *We consider PWLS (0-1) with the configuration (C^r, S^r) formed by the linear differential systems (2-18) and (2-26), with $A = -7/2$, $B = -8/3$, $r = 79/100$, $d = -28/10$, $k = 1$ and $s = 0$, then we obtain the piecewise linear differential system formed by*

$$X^-(x,y) = \begin{pmatrix} -\frac{7}{2} & -\frac{8}{3} \\ \frac{6027}{800} & \frac{7}{2} \end{pmatrix} X + \begin{pmatrix} -\frac{7}{2} \\ \frac{6027}{800} \end{pmatrix}, \quad X^+(x,y) = \begin{pmatrix} \frac{79}{50} & -1 \\ -\frac{3759}{10000} & 0 \end{pmatrix} X + \begin{pmatrix} -\frac{79}{50} \\ \frac{3759}{10000} \end{pmatrix}. \tag{2-13}$$

For this PWLS it is possible verify computationally that the closing equations (2-27) have two real solutions for $t_1, t_2 \in (0, 2\pi)$, namely $(t_1^1, t_2^1, y_0^1) = (1.941361.., 3.063722.., -0.838949..)$ and $(t_1^2, t_2^2, y_0^2) = (4.185356.., 3.063722.., -0.838949..)$. Nevertheless we have that the orbit of linear differential system X^- started at point $(0, y_0^2) = (0, -0.838949..)$ and with flight time $t_1^2 = 4.185356..$ it intersects the region Σ^- which cannot happen to obtain a crossing limit cycle of PWLS (2-13), therefore we have that the unique real solution that generates one crossing limit cycle of the PWLS (2-13) is (t_1^1, t_2^1, y_0^1)=(1.941361.., 3.063722.., −0.838949..), and that crossing limit cycle Γ starts at the point $(0, y_0^1) = (0, -0.838949..)$, enters in the half-plane Σ^- and after a time $t_1^1 = 1.941361..$ reaches the discontinuity line Σ at the point $(0, y_1^1) = (0, -1.786050..)$, enters in the half-plane Σ^+ and after a time $t_2^1 = 3.063722..$ reaches the point $(0, y_0^1)$.

Now we analyze the stability of the crossing limit cycle Γ. We consider the PWLS (2-13) and we analyze the flow of PWLS around of the crossing limit cycle Γ which intersects the discontinuity straight line Σ at the points $y_0 = -0.838949..$ and

$y_1 = -1.786050...$

We consider a point $W_0 \in \Sigma$ *and within the region limited by the crossing limit cycle* Γ, *this is,* $W_0 = (0, w_0)$ *with* $-1.786050.. < w_0 < -0.838949...$ *For example we consider that* $w_0 = -9/10$, *then the solution of linear differential system* X^- *in* (2-13) *starting at the point* $W_0 = (0, -9/10) \in \Sigma$ *is*

$$\begin{aligned} x^-(t) = & \; -1 + \cos\left(\frac{14t}{5}\right) - \frac{11}{28}\sin\left(\frac{14t}{5}\right), \\ y^-(t) = & \; \frac{3}{320}\left(-96\cos\left(\frac{14t}{5}\right) + 167\sin\left(\frac{14t}{5}\right)\right), \end{aligned}$$

and the flight time in the region Σ^- *is*

$$t^- = \frac{5}{14}\left(-\arctan\left(\frac{616}{663}\right) + 2\pi\right),$$

then the intersection point with Σ *is* $W_1 = (0, w_1) = (0, y^-(t^-))$, *where* $y^-(t^-) = -69/40$. *Now the solution of linear differential system* X^+ *in* (2-13) *starting at the point* $W_1 = (0, -69/40)$ *is*

$$x^+(t) = 1 + \frac{e^{-21t/100}}{400}\left(-387 - 13e^{2t}\right), \;\; y^+(t) = \frac{3e^{-21t/100}(-23091 + 91e^{2t})}{40000},$$

the flight time in the region Σ^+ *is* $t^+ = 1.097023..$ *and the intersection point of this orbit with the discontinuity straight line is the point* $W_2 = (0, w_2) = (0, y^+(t^+)) = (0, -1.326846..)$, *then* $-9/10 = w_0 > w_2 = -1.326846...$ *Therefore we obtain that the flow of PWLS* (2-13) *spirals in the counterclockwise inward for points* $W_0 = (0, w_0)$ *with* $-1.786050.. < w_0 < -0.838949...$ *Now we consider a point on* Σ *and outside the region limited by* Γ, *namely* $Z_0 = (0, z_0)$ *with* $z_0 > y_0$. *We consider* $Z_0 = (0, -209/250)$ *and similarly to above case we determine the solution* $(x^-(t), y^-(t))$ *of linear differential system* X^- *in* (2-13) *starting at the point* $Z_0 = (0, -209/250) \in \Sigma$ *and we get the flight time in the region* Σ^-, *namely* $T^- = -1.939696..$, *and the intersection point of this orbit with* Σ *is* $Z_1 = (0, z_1) = (0, y^-(T^-)) = (0, -1789/1000)$. *We determine the solution* $(x^+(t), y^+(t))$ *of linear differential system* X^+ *in* (2-11) *starting at the point* $Z_1 = (0, -1789/1000) \in \Sigma$ *and we get the flight time in the region* Σ^+, $T^+ = 3.923945..$ *and finally we obtain the intersection point of this orbit with* Σ, $Z_2 = (0, z_2) = (0, y^+(T^+)) = (0, -0.666883)$, *then* $-209/250 = z_0 > z_2 = -0.666883..$ *Therefore obtain that the flow of PWLS* (2-11) *spirals in the counterclockwise outward for points* $Z_0 = (0, z_0)$*with* $z_0 > y_0$. *Therefore we can conclude that the crossing limit cycle* Γ *is an unstable crossing limit cycle. See Figure* 2.2. □

Example 2.7 *We consider PWLS* (0-1) *with the configuration* (C^v, F^v) *formed by the*

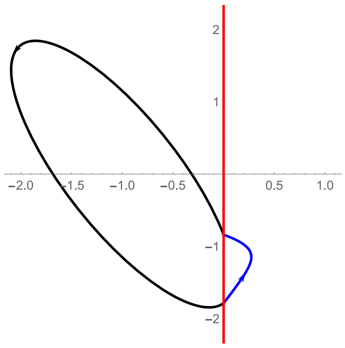

Figure 2.2: *The crossing limit cycle of the discontinuous PWLS (2-13) system with configuration (C^r, S^r).*

linear differential systems (2-29) and (2-30), with $A=-3$, $B=-1$, $r=4/5$, $d=-4$, $k=1$ and $s=0$, then we obtain the PWLS formed by

$$X^-(x,y)=\begin{pmatrix}-3 & -1\\ 25 & 3\end{pmatrix}X+\begin{pmatrix}3\\ -25\end{pmatrix},\quad X^+(x,y)=\begin{pmatrix}\frac{8}{5} & -1\\ \frac{41}{25} & 0\end{pmatrix}X+\begin{pmatrix}\frac{8}{5}\\ \frac{41}{25}\end{pmatrix}. \tag{2-14}$$

For this PWLS it is possible verify computationally that closing equations (2-31) have four real solution for $t_1, t_2 \in (0, 2\pi)$, namely (t_1^1, t_2^1, y_0^1)=(0.299957.., 1.862980.., 5.736049..), (t_1^2, t_2^2, y_0^2)=(1.870753.., 1.862980.., 5.736049..), (t_1^3, t_2^3, y_0^3)=(3.441550.., 1.862980.., 5.736049..), (t_1^4, t_2^4, y_0^4)=(5.012346.., 1.862980.., 5.736049..). *Nevertheless the orbit of the linear differential system X^- started at the point y_0^i and with flight time t_1^i is such that intersects the region Σ^+ for $i=2,3,4$ which cannot happen to obtain a crossing limit cycle of PWLS (2-13), therefore we have that the unique real solution that generates one crossing limit cycle Γ of the PWLS (2-14) is $(t_1^1, t_2^1, y_0^1) =$* (0.299957..,1.862980..,5.736049..) *which intersects Σ in $(0, y_0^1) = (0, 5.736049..)$ and $(0, y_1^1) = (0, 0.263950..)$. Analogously to above case (C^r, S^r), it is possible verify numerically that Γ is an unstable crossing limit cycle. See Figure 2.3.*

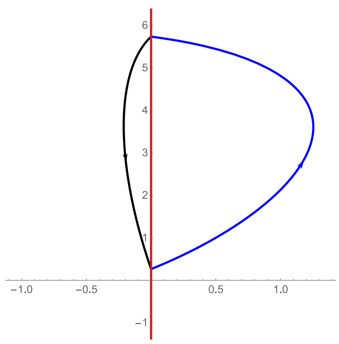

Figure 2.3: *The crossing limit cycle of the discontinuous PWLS (2-14) with configuration (C^v, F^v).*

Example 2.8 *We consider PWLS* (0-1) *with the configuration* (C^v, N^v) *formed by the linear differential systems* (2-29) *and* (2-32)*, with* $A = -5$*,* $B = -18/10$*,* $r = 13/10$*,* $d = -3/2$ *and* $s = 0$*, we obtain the PWLS formed by*

$$X^-(x,y) = \begin{pmatrix} -5 & -\dfrac{9}{5} \\ \dfrac{545}{36} & 5 \end{pmatrix} X + \begin{pmatrix} 5 \\ -\dfrac{545}{36} \end{pmatrix}, \quad X^+(x,y) = \begin{pmatrix} \dfrac{13}{5} & -1 \\ \dfrac{69}{100} & 0 \end{pmatrix} X + \begin{pmatrix} \dfrac{13}{5} \\ \dfrac{69}{100} \end{pmatrix}. \tag{2-15}$$

For this PWLS it is possible verify computationally that closing equations (2-33) *have two real solutions with* $t_1, t_2 \in (0, 2\pi)$*, namely* $(t_1^1, t_2^1, y_0^1) = (0.608026.., 1.109920.., 3.186528..)$*,* $(t_1^2, t_2^2, y_0^2) = (4.796816.., 1.109920.., 3.186528..)$*. But the orbit of the linear differential system* X^- *intersect the region* Σ^+ *when started at the point* $(0, y_0^2) = (0, 3.186528..)$ *with flight time* $t_1^2 = 4.796816..$ *therefore this real solution cannot generates a crossing limit cycle of PWLS* (2-15) *and we only have one crossing limit cycle* Γ *which intersects* Σ *in* $(0, y_0^1) = (0, 3.186528..)$ *and* $(0, y_1^1) = (0, 2.369026..)$ *with flight times* $t_1^1 = 0.608026..$ *and* $t_2^1 = 1.109920..$ *in the regions* Σ^- *and* Σ^+*, respectively. Analogously to above cases, it is possible verify numerically that* Γ *is an unstable crossing limit cycle. See Figure* 2.4.

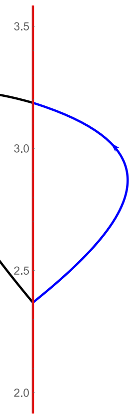

Figure 2.4: *The crossing limit cycle of the discontinuous PWLS* (2-15) *with configuration* (C^v, N^v).

Example 2.9 *We consider PWLS* (0-1) *formed by the linear differential systems* (2-29) *and* (2-34)*, with* $A = -1/2$*,* $B = -1/10$*,* $r = 17/10$*,* $d = -4/10$ *and* $s = 0$*, we obtain the PWLS formed by*

$$X^-(x,y) = \begin{pmatrix} -\dfrac{1}{2} & -\dfrac{1}{10} \\ \dfrac{41}{10} & \dfrac{1}{2} \end{pmatrix} X + \begin{pmatrix} \dfrac{1}{2} \\ -\dfrac{41}{10} \end{pmatrix}, \quad X^+(x,y) = \begin{pmatrix} \dfrac{17}{5} & -1 \\ \dfrac{289}{100} & 0 \end{pmatrix} X + \begin{pmatrix} \dfrac{17}{5} \\ \dfrac{289}{100} \end{pmatrix}. \tag{2-16}$$

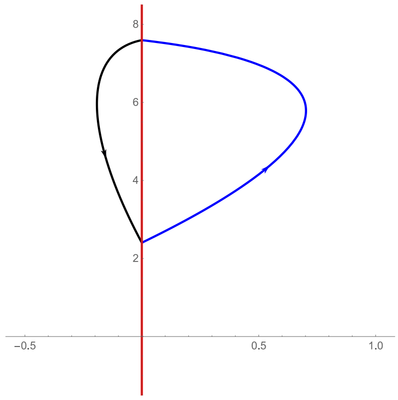

Figure 2.5: *The crossing limit cycle of the discontinuous PWLS* (2-16) *with configuration* (C^v, iN^v).

For this PWLS it is possible verify computationally that closing equations (2-36) *have one real solution, namely* $(t_1,t_2,y_0) = (2.877804..,1.249557..,7.595368..)$, *then the PWLS* (2-16) *has one crossing limit cycle* Γ *which intersects* Σ *in* $(0,7.595368..)$ *and* $(0,2.404631..)$. *Analogously to above cases, it is possible verify numerically that* Γ *is an unstable crossing limit cycle. See Figure* 2.5.

In the proof of statement (i) of Theorem A we use the following lemma.

Lemma 2.10 *We consider the functions*

$$f_0(t_2) = \sin(t_2), \quad f_1(t_2) = \sinh(rt_2), \quad f_2(t_2) = \sinh(t_2), \quad f_3(t_2) = t_2.$$

The following statements hold.

(a) The set of functions $\mathcal{F}^1 = \{f_0, f_1\}$ *is an ECT-system on the intervals* $(0,2\pi)\setminus\{\pi\}$ *for every* $r \neq 0$;

(b) The set of functions $\mathcal{F}^2 = \{f_2, f_1\}$ *is an ECT-system for every* $t_2 \neq 0$ *and* $r \neq 1$;

(c) The set of functions $\mathcal{F}^3 = \{f_3, f_1\}$ *is an ECT-system for every* $t_2 \neq 0$ *and* $r \neq 0$.

Proof.

(a) Considering the functions f_0 and f_1 the Wronskian is

$$W(t_2) = r\cosh(rt_2)\sin(t_2) - \cos(t_2)\sinh(rt_2).$$

Since $W(0) = 0$ and $W'(t_2) = (1+r^2)\sin(t_2)\sinh(rt_2)$ does not vanish for any $t_2 \in (0,2\pi)\setminus\{\pi\}$ and $r \neq 0$. Then $W(t_2) \neq 0$ for $t_2 \in (0,2\pi)\setminus\{\pi\}$ and $r \neq 0$, therefore by Proposition 1.9, statement (a) is proved.

(b) The Wronskian of the functions f_1 and f_2 is

$$W(t_2) = r\cosh(rt_2)\sinh(t_2) - \cosh(t_2)\sinh(rt_2),$$

and we observed that $W(0) = 0$ and $W'(t_2) = (-1+r^2)\sinh(t_2)\sinh(rt_2)$, then $W'(t_2)$ does not vanish for every $t_2 \neq 0$ and $r \neq 1$. Therefore $W(t_2) \neq 0$ for $t_2 \neq 0$ and $r \neq 1$, then by Proposition 1.9, statement (b) is proved.

(c) The Wronskian of the functions f_1 and f_3 is

$$W(t_2) = rt_2\cosh(rt_2) - \sinh(rt_2),$$

and we observed that $W(0) = 0$ and $W'(t_2) = r^2 t_2 \sinh(rt_2)$, we have that $W'(t_2)$ does not vanish if $t_2 \neq 0$ and $r \neq 0$, then $W(t_2) \neq 0$ for $t_2 \neq 0$ and $r \neq 0$. Therefore by Proposition 1.9, statement (c) is proved. □

In what follows we prove Theorem A.

Proof of statement (i) ***of Theorem A.*** We have that the equilibrium point of linear differential system X^- is a center, then using Proposition 2.2, we consider that the linear differential system X^- is in the canonical form (2-3) with $\tilde{C} = 0$. Then the equilibrium point of linear differential system X^- is

$$P_- = (x_0, y_0) = \left(\frac{Ab_1 + Bb_2}{d^2}, -\frac{A^2 b_1 + ABb_2 + b_1 d^2}{Bd^2}\right). \tag{2-17}$$

We separate the proof of statement (i) of Theorem A in two cases.

Case 1: P_- is a real singularity of X^-. We assume that $P_- = (-k, -sk)$, for this we must consider $b_1 = Ak + Bsk$ and $b_2 = -k(A^2 + d^2 + ABs)/B$. Therefore, linear differential system X^- is

$$X^-(x,y) = \begin{pmatrix} A(x+k) + B(y+sk) \\ -\dfrac{(A^2+d^2)(k+x) + AB(y+sk)}{B} \end{pmatrix}. \tag{2-18}$$

When we have an equilibrium point a C^r for the linear differential system X^-, by hypothesis, we have two possible configurations for the equilibrium points of the PWLS (0-1), namely, we can have the configurations (C^r, F^r) and (C^r, S^r).

We consider that linear differential system X^+ is in the canonical form (2-1) which has the equilibrium point

$$P_+ = (x_1, y_1) = \left(-\frac{c}{r^2 - \beta^2}, \frac{-2cr + b(r^2 - \beta^2)}{r^2 - \beta^2}\right). \tag{2-19}$$

Therefore the equilibrium point P_+ is a real singularity of X^+ if

$$b = -k(2r-s),\ c = -k(r^2-\beta^2); \tag{2-20}$$

and P_+ is a virtual singularity of X^+ if

$$b = k(2r-s),\ c = k(r^2-\beta^2). \tag{2-21}$$

Configuration (C^r, F^r)**:** For the linear differential system X^+, we consider the condition (2-20) with $\beta = i$ and $r \neq 0$.

The linear differential system X^+ in this case is

$$X^+(x,y) = \begin{pmatrix} -y+2r(x-k)+sk \\ (1+r^2)(x-k) \end{pmatrix}. \tag{2-22}$$

With those conditions the solution of system (2-18) starting at the point $(x,y) = (0,y_0) \in \Sigma$ is

$$\begin{aligned} x^-(t) &= k(-1+\cos(dt)) + \frac{(Ak+B(y_0+sk))\sin dt}{d}, \\ y^-(t) &= -sk + (y_0+sk)\cos(dt) - \frac{((A^2+d^2+ABs)k+ABy_0)\sin(dt)}{Bd}, \end{aligned}$$

and the solution of system (2-22) starting at the point $(x,y) = (0,y_0) \in \Sigma$ is

$$\begin{aligned} x^+(t) &= k - e^{rt}(k\cos(t) + ((r-s)k+y_0)\sin(t)), \\ y^+(t) &= sk + e^{rt}((y_0-sk)\cos(t) - (k+r((r-s)k+y_0))\sin(t)). \end{aligned}$$

Considering that there exists $t_1, t_2 > 0$ the finite times defined in Proposition 2.3. We have that system (2-10) is equivalent to system

$$\begin{aligned} e_1:\ & kd(-1+\cos(dt_1)) + (Ak+B(y_0+sk))\sin(dt_1) = 0, \\ e_2:\ & k + e^{-rt_2}(-k\cos(t_2) + ((r-s)k+y_0)\sin(t_2)) = 0, \\ e_3:\ & 2sk - (y_0+sk)\cos(dt_1) + \frac{((A^2+d^2+ABs)k+ABy_0)\sin(dt_1)}{Bd} \\ & + e^{-rt_2}((y_0-sk)\cos(t_2) + (k+r((r-s)k+y_0))\sin(t_2)) = 0. \end{aligned} \tag{2-23}$$

From the first equation we obtain

$$\begin{aligned} \cos(dt_1) &= \frac{(-A^2+d^2)k^2 - 2ABk(y_0+ks) - B^2(y_0+sk)^2}{(A^2+d^2)k^2 + 2ABk(y_0+ks) + B^2(y_0+ks)^2}, \\ \sin(dt_1) &= \frac{2kd(Ak+B(y_0+ks))}{(A^2+d^2)k^2 + 2ABk(y_0+ks) + B^2(y_0+ks)^2}, \end{aligned} \tag{2-24}$$

from equation e_2 we get $y_0 = -k(r-s-\cot(t_2) + e^{rt_2}\csc(t_2))$. Substituting y_0 in equation

e_3 we have $e_3 = 2k(A/B - r + 2s - \csc(t_2)\sinh(rt_2))$, and to determine the solutions for this equation is equivalent to determine the solutions for the following equation

$$\frac{2k}{B\sin(t_2)}((A - rB + 2Bs)f_0(t_2) - Bf_1(t_2)) = 0, \quad \text{with } t_2 \in (0, 2\pi) \setminus \{\pi\}, \tag{2-25}$$

and we can conclude that equation (2-25) has at most one real solution for $t_2 \in (0, 2\pi) \setminus \{\pi\}$, because by statement (a) of Lemma 2.10 the set of functions $\mathcal{F}^1 = \{f_0, f_1\}$ is an extended complete Chebyshev system for $t_2 \in (0, 2\pi) \setminus \{\pi\}$ for every $r \neq 0$ and even more the coefficients $A - rB + 2Bs$ and B can be chosen arbitrarily. Therefore we have proved that a PWLS (0-1) with the configuration (C^r, F^r) formed by the linear differential systems (2-18) and (2-22) has at most one crossing limit cycle. □

Configuration (C^r, S^r)**:** The equilibrium point P_+ of system X^+ satisfies the condition (2-20) with $\beta = 1$ and $|r| < 1$. Therefore

$$X^+(x,y) = \begin{pmatrix} -y + 2r(x-k) + sk \\ (-1 + r^2)(x-k) \end{pmatrix}. \tag{2-26}$$

The solution of system (2-26) starting at the point $(x, y) = (0, y_0) \in \Sigma$ is

$$\begin{aligned} x^+(t) &= \frac{e^{-t}}{2}\left(2ke^t + e^{rt}((-1 + r - s)k + y_0) - e^{(2+r)t}(k(1 + r - s) + y_0)\right), \\ y^+(t) &= \frac{e^{-t}}{2}\left(2e^t sk + e^{rt}(1 + r)(y_0 + (r - 1 - s)k - e^{(2+r)t}(r - 1)(y_0 + (1 + r - s)k)\right). \end{aligned}$$

Let t_1 and t_2 be the finite times defined in Proposition 2.3. In this case we have that system (2-10) is equivalent to system

$$\begin{aligned} e_1 :&\quad kd(-1 + \cos(dt_1)) + (Ak + B(y_0 - sk))\sin(dt_1) = 0, \\ e_2 :&\quad 2ke^{-t_2} + e^{-rt_2}((r - 1 - s)k + y_0) - e^{-(2+r)t_2}(k(1 + r - s) + y_0)) = 0, \\ e_3 :&\quad sk + (y_0 + sk)\cos(dt_1) + \frac{((A^2 + d^2 + ABs)k + ABy_0)\sin(dt_1)}{Bd} + \frac{e^{t_2}}{2}(2e^{-t_2}sk \\ &\quad + e^{-rt_2}(1 + r)(y_0 + (r - 1 - s)k) - e^{-(2+r)t_2}(r - 1)(y_0 + (1 + r - s)k)\Big) = 0. \end{aligned} \tag{2-27}$$

Then the real solutions of system (2-27) generate crossing limit cycles of PWLS (0-1) formed by the linear differential systems (2-18) and (2-26). Similar to Case (C^r, Fr), from equation e_1 we obtain equations (2-24), from e_2 we get

$$y_0 = -k\frac{-1 + 2e^{t_2 + rt_2} - r + e^{2t_2}(-1 + r - s) + s}{-1 + e^{2t_2}},$$

then $e_3 = 2k(A/B - r + 2s - \operatorname{csch}(t_2)\sinh(rt_2))$. To determine the solutions for equation

e_3 is equivalent to determine the solutions for the following equation

$$\frac{2k}{B\sinh(t_2)}((A-rB+2sB)f_2(t_2)-Bf_1(t_2))=0, \text{ with } t_2\neq 0. \tag{2-28}$$

By statement (b) of Lemma 2.10 the set of functions $\mathcal{F}^2=\{f_2,f_1\}$ is an extended complete Chebyshev system for $t_2\neq 0$ and $r\neq 1$ and moreover the coefficients $A-rB+2sB$ and B can be chosen arbitrarily. Then we can conclude that equation (2-28) has at most one real solution for $t_2\neq 0$ and $|r|<1$. Therefore PWLS (0-1) with the configuration (C^r,S^r) formed by the linear differential systems (2-18) and (2-26) has at most one crossing limit cycle. □

Case 2: P_- is a virtual singularity of X^-. We consider that the equilibrium point P_- in (2-17) is a center C^v, this is $P_-=(k,sk)$, for this we must consider $b_1=-Ak-Bks$ and $b_2=k(A^2+d^2+ABs)/B$. Therefore linear differential system X^- is

$$X^-(x,y)=\begin{pmatrix} A(x-k)+B(y-ks) \\ \dfrac{(A^2+d^2)(-x+k)+AB(-y+sk)}{B}\end{pmatrix}. \tag{2-29}$$

When the equilibrium point P_- is a C^v for the linear differential system X^-, then we have three possible configurations for the equilibrium points (P_-^-,P_+) of the PWLS (0-1), namely, we have the configurations (C^v,F^v), (C^v,N^v) and (C^v,iN^v). **Configuration (C^v,F^v):** We consider that the configuration of the equilibrium points of the linear differential systems X^- and X^+ in (0-1) is (C^v,F^v), then the equilibrium point P_+ satisfies (2-21) with $\beta=i$ and $r\neq 0$. Therefore

$$X^+(x,y)=\begin{pmatrix} -y+2r(k+x)-ks \\ (1+r^2)(k+x)\end{pmatrix}. \tag{2-30}$$

The solutions of systems (2-29) and (2-30) starting at the point $(x,y)=(0,y_0)\in\Sigma$ are

$$\begin{aligned}
x^-(t)&=k(1-\cos(dt))+\frac{(By_0-(A+Bs)k)\sin dt}{d},\\
y^-(t)&=\frac{Bdks+Bd(y_0-ks)\cos(dt)+((A^2+d^2+ABs)k-ABy_0)\sin(dt)}{Bd},\\
x^+(t)&=-k-e^{rt}(k\cos(t)-(k(s-r)+y_0)\sin(t)),\\
y^+(t)&=-ks+e^{rt}((y_0+ks)\cos(t)+(k+r^2k-r(y_0+ks))\sin(t)).
\end{aligned}$$

Let t_1 and t_2 be the finite times defined in Proposition 2.3. Here we have that system (2-10) is equivalent to system

$$
\begin{aligned}
e_1:&\quad kd(1-\cos(dt_1))+(-(A+bs)k+By_0)\sin(dt_1)=0,\\
e_2:&\quad -k+e^{-rt_2}(k\cos(t_2)+((-r+s)k+y_0)\sin(t_2))=0,\\
e_3:&\quad -2ks+(-y_0+ks)\cos(dt_1)-\frac{((A^2+d^2+ABs)k-ABy_0)\sin(dt_1)}{Bd}\\
&\quad +e^{-rt_2}((y_0+ks)\cos(t_2)+(-k+r(y_0+(-r+s)k))\sin(t_2))=0.
\end{aligned}
\tag{2-31}
$$

Similar to case (C^r,F^r), we obtain that e_3 is equivalent to equation (2-25) then we can conclude that PWLS (0-1) with the configuration (C^v,F^v) formed by the linear differential systems (2-29) and (2-30) has at most one crossing limit cycle. □

We observe that in the previous cases the constant k does not influence the number of solutions of system (2-10) and in the following cases the same thing happens, therefore without loss of generality we can assume that $k=1$, this is the singularities of systems X^- and X^+ are in $(-1,-s)$ or $(1,s)$, with $s\in\mathbb{R}$.

Configuration (C^v,N^v)**:** We consider that the configuration of the equilibrium points of the linear differential systems X^- and X^+ in (0-1) is (C^v,N^v), then the equilibrium point P_+ satisfies (2-21) with $\beta=1$ and $|r|>1$. Therefore the linear differential system X^+ is

$$
X^+(x,y)=\begin{pmatrix}-y+2r(1+x)-s\\(-1+r^2)(1+x)\end{pmatrix}.
\tag{2-32}
$$

The solution of system (2-32) starting at the point $(x,y)=(0,y_0)\in\Sigma$ is

$$
\begin{aligned}
x^+(t)&=\frac{e^{-t}}{2}\left(-2e^t+e^{(2+r)t}(1+r-s-y_0)+e^{rt}(1-r+s+y_0)\right),\\
y^+(t)&=\frac{e^{-t}}{2}\left(-2e^ts+e^{(2+r)t}(-1+r)(1+r-s-y_0)+e^{rt}(1+r)(1-r+s+y_0)\right).
\end{aligned}
$$

Considering t_1 and t_2 the finite times defined in Proposition 2.3, we obtain that system (2-10) is equivalent to system

$$
\begin{aligned}
e_1:&\quad d(1-\cos(dt_1))-(A+Bs-By_0)\sin(dt_1)=0,\\
e_2:&\quad -2e^{-t_2}+e^{-(2+r)t_2}(1+r-s-y_0)+e^{-rt_2}(1-r+s+y_0)=0,\\
e_3:&\quad -s+(s-y_0)\cos(dt_1)-(A^2+d^2+AB(s-y_0))\sin(dt_1)-\frac{e^{t_2}}{2}(-2e^{-t_2}s\\
&\quad +e^{-(2+r)t_2}(-1+r)(1+r-s-y_0)+e^{-rt_2}(1+r)(1-r+s+y_0)\Big)=0.
\end{aligned}
\tag{2-33}
$$

From equation e_1 we obtain that

$$
\cos(dt_1)=\frac{-A^2+d^2+2AB(-s+y_0)-B^2(s-y_0)^2}{A^2+d^2+2AB(s-y_0)+B^2(s-y_0)^2},
$$

$$\sin(dt_1) \quad = \frac{2d(A+B(s-y_0))}{A^2+d^2+2AB(s-y_0)+B^2(s-y_0)^2},$$

and from e_2 we get

$$y_0 = \frac{-1+2e^{t_2+rt_2}-r+e^{2t_2}(-1+r-s)+s}{-1+e^{2t_2}},$$

then substituting in e_3 we obtain that e_3 is equivalent to equation (2-28), therefore we can conclude that PWLS (0-1) with the configuration (C^v, N^v) formed by the linear differential systems (2-29) and (2-32) has at most one crossing limit cycle. □

Configuration (C^v, iN^v)**:** We consider that the configuration of the equilibrium points of the linear differential systems X^- and X^+ in (0-1) is (C^v, iN^v). We consider that equilibrium point P_+ satisfies (2-21) with $\beta = 0$ and $r \neq 0$. Then

$$X^+(x,y) = \begin{pmatrix} -y+2r(1+x)-s \\ r^2(1+x) \end{pmatrix}. \tag{2-34}$$

The solution of system (2-34) starting at the point $(x,y) = (0, y_0) \in \Sigma$ is

$$x^+(t) = -1+e^{rt}(1-t(y_0-r+s)), \qquad y^+(t) = -s+e^{rt}(y_0-rty_0+s+r(r-s)t), \tag{2-35}$$

Considering t_1 and t_2 the finite times defined in Proposition 2.3, we obtain that system (2-10) is equivalent to system

$$\begin{aligned} e_1: &\quad d(1-\cos(dt_1))+(By_0-(A+Bs))\sin(dt_1) = 0, \\ e_2: &\quad -1+e^{-rt_2}(1+t_2(y_0-r+s)) = 0, \\ e_3: &\quad -2s+e^{-rt_2}(y_0+rt_2y_0+(s+r(-r+s)t_2)) \\ &\quad +\frac{Bd(-y_0+s)\cos(dt_1)-(-ABy_0+(A^2+d^2+ABs))\sin(dt_1)}{Bd} = 0. \end{aligned} \tag{2-36}$$

From equation e_1 we obtain the expression (2.3) and from e_2 we get

$$y_0 = \frac{-1+e^{rt_2}+(r-s)t_2}{t_2},$$

then

$$e_3 = 2\left(-\frac{A}{B}+r-2s+\frac{\sinh(rt_2)}{t_2}\right) = 0,$$

and to determine the solutions for equation e_3 is equivalent to determine the solutions for the equation

$$\frac{-2}{Bt_2}((A-rB-2sA)f_3(t_2)-Bf_1(t_2)) = 0, \text{ with } t_2 \neq 0. \tag{2-37}$$

By statement (c) of Lemma 2.10 the set of functions $\mathcal{F}^3 = \{f_3, f_1\}$ is an extended complete Chebyshev system for $t_2 \neq 0$ and $r \neq 0$ and moreover the coefficients $A - rB - 2sA$ and B can be chosen arbitrarily. Then we can conclude that equation (2-37) has at most one real solution for $t_2 \neq 0$ and $r \neq 0$. Therefore the PWLS (0-1) with configuration (C^v, iN^v) formed by the linear differential systems (2-29) and (2-34) has at most one crossing limit cycle. □

Moreover the upper bound provided in the above cases is reached, see the examples in the proof of Proposition 2.4.

Proof of statement (ii) ***of Theorem*** **A.** Here we analyze the number of crossing limit cycles of PWLS (0-1) when the equilibrium point of linear differential system X^- is a real or virtual focus (F^r) or (F^v). We consider that system X^- is in the canonical form (2-3) with $\tilde{C} \neq 0$. Then the equilibrium point of system X^- is

$$P_- = (x_0, y_0) = \left(\frac{Ab_1 + Bb_2 - 2b_1\tilde{C}}{\tilde{C}^2 + d^2}, -\frac{A^2b_1 + ABb_2 - 2Ab_1\tilde{C} + b_1\tilde{C}^2 + b_1d^2}{B\tilde{C}^2 + Bd^2} \right).$$

We separate the proof of statement (ii) of Theorem A in two cases, first we study the case when P_- is a real focus and second we assume that P_- is a virtual focus. We consider that linear differential system X^+ is in canonical form (2-1) then the equilibrium point is (2-19).

Case 1: P_- is a real focus of X^-. We assume that $P_- = (-1, -s)$, for this we must consider that

$$b_1 = A + Bs, \quad b_2 = -\frac{A^2 - 2A\tilde{C} + \tilde{C}^2 + d^2 + ABs - 2B\tilde{C}s}{B}. \tag{2-38}$$

Then linear differential system X^- is

$$X^-(x,y) = \begin{pmatrix} A(x+1) + B(y+s) \\ -\dfrac{(A^2 + c^2 + d^2)(x+1) - 2Bc(y+s) + A(-2c(x+1) + B(y+s))}{B} \end{pmatrix}. \tag{2-39}$$

The solution of linear differential system (2-39) starting at the point $(x,y) = (0, y_0) \in \Sigma$ is

$$\begin{aligned} x^-(t) = &\ -1 + \frac{e^{\tilde{C}t}(d\cos(dt) + (By_0 + A - \tilde{C} + Bs)\sin(dt))}{d}, \\ y^-(t) = &\ -s + \frac{e^{\tilde{C}t}}{Bd}\left((-(B(A-\tilde{C})y_0 + (d^2 + (A-\tilde{C})(A - \tilde{C} + Bs))\sin(dt)) \right. \\ &\left. + (Bd(y_0 + s)\cos(dt))\right)\right). \end{aligned} \tag{2-40}$$

When P_- is a real focus then we have two possible configurations for the equilibrium points of the PWLS (0-1), namely we obtain the configurations (F^r, F^r) and (F^r, S^r).

Configuration (F^r, F^r)**:** We assume that the equilibrium point P_- satisfies the conditions (2-38) and the equilibrium point P_+ satisfies the conditions (2-20) with $\beta = i,\ r \neq 0$, then we have the configuration (F^r, F^r).

In the following example we provide a PWLS having two crossing limit cycles. We consider that $A = 1/2,\ B = -1/2,\ \tilde{C} = -67/500,\ d = 123/100,\ r = 2/5$ and $s = 0$

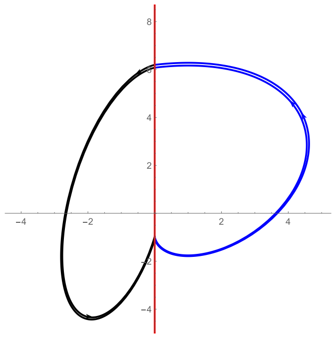

Figure 2.6: *The two crossing limit cycles Γ_1 and Γ_2 of the discontinuous PWLS (2-41) with configuration (F^r, F^r).*

then we obtain the PWLS formed by

$$X^-(x,y) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{239357}{62500} & -\frac{96}{125} \end{pmatrix} X + \begin{pmatrix} \frac{1}{2} \\ \frac{239357}{62500} \end{pmatrix}, \quad X^+(x,y) = \begin{pmatrix} \frac{4}{5} & -1 \\ \frac{29}{25} & 0 \end{pmatrix} X + \begin{pmatrix} -\frac{4}{5} \\ -\frac{29}{25} \end{pmatrix}. \tag{2-41}$$

For this PWLS we have that system (2-10) is equivalent to system

$$\begin{aligned} -1 + \frac{1}{615} e^{-67t_1/500}(615\cos(123t_1/100) + (317 - 250y_0)\sin(123t_1/100)) &= 0, \\ 1 + e^{-2t_2/5}\left(-\cos(t_2) + \left(\frac{2}{5} + y_0\right)\sin(t_2)\right) &= 0, \\ e^{-67t_1/500}(-76875y_0\cos(123t_1/100) + (-239357 + 39625y_0)\sin(123t_1/100)) & \\ +3075e^{-2t_2/5}(25y_0\cos(t_2) + (29 + 10y_0)\sin(t_2)) &= 0. \end{aligned} \tag{2-42}$$

Which has two real solutions with $t_1, t_2 \in (0, 2\pi)$, namely $(t_1^1, t_2^1, y_0^1) = (3.586636..,$ $4.260216.., 6.196201..)$ and $(t_1^2, t_2^2, y_0^2) = (3.614645.., 4.344295.., 6.078132..)$. Therefore the PWLS (2-41) has two crossing limit cycles Γ_1 and Γ_2 which intersect Σ in $(0, y_0^1) = (0, 6.196201..)$ and $(0, y_1^1) = (0, y_{11}^-(t_1^1)) = (0, -1.088003..)$ with flight times $t_1^1 = 3.586636..$ and $t_2^1 = 4.260216..$ in the regions Σ^- and Σ^+, respectively; and $(0, y_0^2) = (0, 6.078132..)$ and $(0, y_1^2) = (0, y_{11}^-(t_1^2)) = (0, -0.974222..)$ with flight times $t_1^2 = 3.614645..$ and $t_2^2 = 4.344295..$ in the regions Σ^- and Σ^+, respectively. See Figure 2.6. □

Configuration (F^r, S^r)**:** If the equilibrium point P_- is a focus F^r and the equilibrium point P_+ satisfies the conditions (2-20) with $\beta = 1,\ |r| < 1$, then we have the configuration (F^r, S^r). In what follows we provide a PWLS having two crossing limit cycles. Considering $A = -2/5$, $B = -7/2$, $\tilde{C} = 1/20$, $d = -1$, $r = 1/100$ and $s = 0$, we obtain

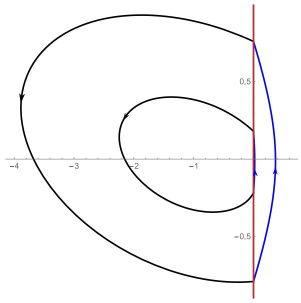

Figure 2.7: *The two crossing limit cycles Γ_1 and Γ_2 of the discontinuous PWLS (2-43) with configuration (F^r, S^r).*

the PWLS formed by

$$X^-(x,y) = \begin{pmatrix} -\dfrac{2}{5} & -\dfrac{7}{2} \\ \dfrac{449}{1400} & \dfrac{3}{10} \end{pmatrix} X + \begin{pmatrix} -\dfrac{2}{5} \\ \dfrac{449}{1400} \end{pmatrix}, \quad X^+(x,y) = \begin{pmatrix} \dfrac{1}{50} & -1 \\ -\dfrac{9999}{10000} & 0 \end{pmatrix} X + \begin{pmatrix} -\dfrac{1}{50} \\ \dfrac{9999}{10000} \end{pmatrix}. \tag{2-43}$$

For this PWLS we have that system (2-10) has two real solutions with $t_1, t_2 \in (0, 2\pi)$, namely $(t_1^1, t_2^1, y_0^1) = (3.854989.., \ 2.065073.., \ 0.759545..)$ and $(t_1^2, t_2^2, y_0^2) = (\ 5.114523.., \ 0.403781.., \ 0.1794388..)$. Therefore the PWLS (2-43) has two crossing limit cycle Γ_1 and Γ_2 which intersect Σ in $(0, y_0^1) = (0, 0.759545..)$ and $(0, y_1^1) = (0, -0.790192..)$; and $(0, y_0^2) = (0, 0.1794388..)$ and $(0, y_1^2) = (0, -0.218905..)$, respectively. See Figure 2.7. $\square$

Case 2: P_- is virtual focus of X^-. We consider that P_- is a focus (F^v), this is, $P_- = (1, s)$, therefore

$$b_1 = -A - Bs, \quad b_2 = -\frac{-A^2 + 2A\tilde{C} - \tilde{C}^2 - d^2 - ABs + 2B\tilde{C}s}{B}.$$

Then linear differential system X^- is

$$X^-(x,y) = \begin{pmatrix} A(x-1) + B(y-s) \\ -\dfrac{A^2x - 2A\tilde{C}x + \tilde{C}^2x + d^2x + ABy - 2B\tilde{C}y - ((A-\tilde{C})^2 + d^2 + B(A - 2\tilde{C})s)}{B} \end{pmatrix}. \tag{2-44}$$

The solution of linear differential system (2-44) starting at the point $(x,y)=(0,y_0)\in\Sigma$ is

$$
\begin{aligned}
x^-(t) = &\ 1+\frac{e^{\tilde{C}t}(-d\cos(dt)+(By_0-(A-\tilde{C}+Bs)\sin(dt))}{d},\\
y^-(t) = &\ s+\frac{e^{\tilde{C}t}}{Bd}\left(((B(-A+\tilde{C})y_0+(d^2+(A-\tilde{C})(A-\tilde{C}+Bs))\sin(dt))\right.\\
&\left.+(Bd(y_0-s)\cos(dt)))\right).
\end{aligned}
\tag{2-45}
$$

When P_- is a focus (F^v) we have three possible configurations for the equilibrium point of PWLS (0-1), namely we have the configurations (F^v,F^v), (F^v,N^v) and (F^v,iN^v).

Configuration (F^v,F^v)**:** The equilibrium point P_+ is a focus F^v and the equilibrium point P_+ satisfies the condition (2-21) with $\beta=i$ and $r\neq 0$, then we have the configuration (F^v,F^v). We provide a PWLS with two crossing limit cycles. Considering $A=-7/10$, $B=-1/2$, $\tilde{C}=-2$, $d=-1$, $r=6/10$ and $s=0$, we obtain the PWLS formed by

$$
X^-(x,y)=\begin{pmatrix} -\dfrac{7}{10} & -\dfrac{1}{2}\\ \dfrac{269}{50} & -\dfrac{33}{10}\end{pmatrix}X+\begin{pmatrix}\dfrac{7}{10}\\ -\dfrac{269}{50}\end{pmatrix},\quad X^+(x,y)=\begin{pmatrix}\dfrac{6}{5} & -1\\ \dfrac{34}{25} & 0\end{pmatrix}X+\begin{pmatrix}\dfrac{6}{5}\\ \dfrac{34}{25}\end{pmatrix}.
\tag{2-46}
$$

For this PWLS we have that system (2-10) has two real solution with $t_1,t_2\in(0,2\pi)$, namely $(t_1^1,t_2^1,y_0^1)=(0.903052..,\ 2.593104..,\ 11.325957..)$ and $(t_1^2,t_2^2,y_0^2)=(\ 0.276244..,\ 1.538684..,3.086535..)$. Therefore the PWLS (2-43) has two crossing limit cycle Γ_1 and Γ_2 which intersect Σ in $(0,y_0^1)=(0,11.325957..)$ and $(0,y_1^1)=(0,-1.441285..)$; and $(0,y_0^2)=(0,3.086535..)$ and $(0,y_1^2)=(0,0.234677..)$, respectively. See Figure 2.8. Therefore we have that a PWLS (0-1) with the configuration (F^v,F^v) it

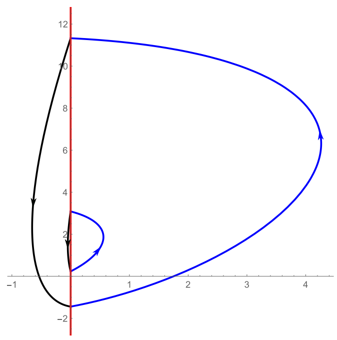

Figure 2.8: *The two crossing limit cycles Γ_1 and Γ_2 of the discontinuous PWLS (2-46) with configuration (F^v,F^v).*

has at least two crossing limit cycles. □

Configuration (F^v,N^v)**:** The equilibrium point P_- is a focus F^v and the equilibrium point P_+ satisfies the conditions (2-21) with $\beta=1$ and $|r|>1$, then we have the configuration (F^v,N^v). We provide a PWLS with this configuration and with one crossing limit cycle.

If $A=-3,\ B=-1/2,\ \tilde{C}=-3/10,\ d=1,\ r=2$ and $s=0$, we obtain the PWLS formed by

$$X^-(x,y)=\begin{pmatrix} -3 & -\dfrac{1}{2} \\ \dfrac{829}{50} & \dfrac{12}{5}\end{pmatrix}X+\begin{pmatrix} 3 \\ -\dfrac{829}{50}\end{pmatrix},\quad X^+(x,y)=\begin{pmatrix} 4 & -1 \\ 3 & 0\end{pmatrix}X+\begin{pmatrix} 4 \\ 3\end{pmatrix}. \tag{2-47}$$

For this PWLS we have that system (2-10) has one real solution with $t_1,t_2\in(0,2\pi)$, namely $(t_1,t_2,y_0)=(2.073656..,\ 1.547693..,\ 10.752069..)$. Then the PWLS (2-47) has one crossing limit cycle which intersects Σ in $(0,y_0)=(0,10.752069..)$ and $(0,y_1)=(0,3.074636..)$. See Figure 2.9. Therefore we can conclude that a PWLS (0-1) with the

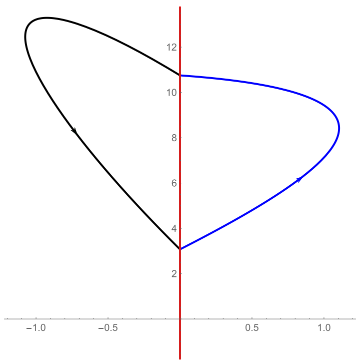

Figure 2.9: *The crossing limit cycle of the discontinuous PWLS (2-47) with configuration (F^v,N^v).*

configuration (F^v,N^v) it has at least one crossing limit cycles. $\square$

Configuration (F^v,iN^v)**:** The equilibrium point P_- is a focus F^v and the equilibrium point P_+ satisfies the conditions (2-21) with $\beta=0$ and $r\neq 0$, then we have the configuration (F^v,iN^v). Then considering t_1 and t_2 as in Proposition 2.3 and from equations (2-35) and (2-45), system (2-10) is equivalent to system

$$\begin{aligned}
e_1:\ \ & d+e^{\tilde{C}t_1}(-d\cos(dt_1)+(By_0-A+\tilde{C}-Bs)\sin(dt_1))=0,\\
e_2:\ \ & -1+e^{-rt_2}(1+t_2(y_0-r+s))=0,\\
e_3:\ \ & Bde^{-rt_2}(y_0+rt_2y_0+(s+r(-r+s)t_2))-e^{ct_1}(Bd(y_0-s)\cos(dt_1)\\
& +(B(-A+\tilde{C})y_0+(d^2+(A-\tilde{C})(A-\tilde{C}+Bs)))\sin(dt_1))-2Bds=0.
\end{aligned} \tag{2-48}$$

From equation e_1 we get

$$y_0=\frac{A-\tilde{C}+Bs+d\cot(dt_1)-de^{-\tilde{C}t_1}\csc(dt_1)}{B},$$

and from e_2 we get

$$t_2 = -\frac{1}{y_0 - r + s} - \frac{\mathcal{W}\left(-\frac{e^{-\frac{r}{y_0-r+s}}r}{y_0-r+s}\right)}{r}, \tag{2-49}$$

then substituting y_0 and t_2 in e_3 we obtain that

$$e_3 = \frac{1}{B}\left((-A+\tilde{C}+Br-2Bs)\tilde{f}_0(t_1) + d\tilde{f}_1(t_1) - rB\tilde{f}_2(t_1)\right) = 0. \tag{2-50}$$

Here $\tilde{f}_0(t_1) = 1,\ \tilde{f}_1(t_1) = \cot(dt_1) - e^{\tilde{C}t_1}\csc(dt_1),$ and

$$\tilde{f}_2(t_1) = \frac{1}{\mathcal{W}\left(\frac{Bre^{\tilde{C}t_1 + \frac{Bre^{\tilde{C}t_1}}{e^{\tilde{C}t_1}(-A+\tilde{C}+Br-2Bs-d\cot(dt_1))+d\csc(dt_1)}}}{e^{\tilde{C}t_1}(-A+\tilde{C}+Br-2Bs-d\cot(dt_1))+d\csc(dt_1)}\right)},$$

where $\mathcal{W}$ is the *Lambert Function*. When

$$t_1 \in (0, \pi/d) \text{ and } \eta(t_1) = e^{\tilde{C}t_1}(-A+\tilde{C}+Br-2Bs-d\cot(dt_1)) + d\csc(dt_1) \neq 0, \tag{2-51}$$

we can conclude that equation (2-50) has at least two real solutions by Proposition 1.7. Thus system (2-48) has at least two real solutions, that is, a PWLS with the configuration (F^v, iN^v) has at least two crossing limit cycles.

In what follows we provide a PWLS with configuration (F^v, iN^v) having two crossing limit cycles. Considering $A = -25/2,\ B = -13/10,\ \tilde{C} = -6/5,\ d = 13/10,$ $r = 5$ and $s = 0$, we have that condition (2-51) is not empty.

$$\eta(t_1) = \frac{1}{10}\left(e^{-6t_1/5}\left(48 - 13\cot\left(\frac{13t_1}{10}\right)\right) + 13\csc\left(\frac{13t_1}{10}\right)\right),\quad t_1 \in \left(0, \frac{10\pi}{13}\right).$$

It is possible verify that in the interval $\left(0, \frac{10\pi}{13}\right)$ the unique critical value is $t_1^* =$ 1.501574.., and it is a minimum value of the function $\eta(t_1)$ for $t_1 \in \left(0, \frac{10\pi}{13}\right)$, moreover $\eta(t_1^*) = 2.278475.. > 0$, therefore $\eta(t_1) > 0$ for $t_1 \in \left(0, \frac{10\pi}{13}\right)$. See Figure 2.10. With

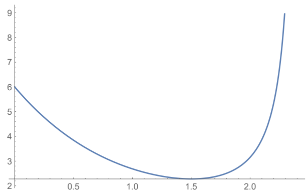

Figure 2.10: *The graphic of the function* $\eta(t_1)$ *in the interval* $(0, 10\pi/13)$.

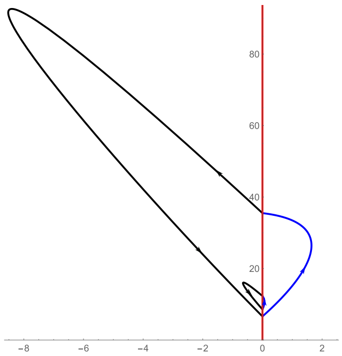

Figure 2.11: *Two crossing limit cycles of the discontinuous PWLS* (2-52) *with configuration* (F^v, iN^v).

these parameters we obtain the PWLS formed by

$$X^-(x,y) = \begin{pmatrix} -\dfrac{25}{2} & -\dfrac{13}{10} \\ \dfrac{6469}{65} & \dfrac{101}{10} \end{pmatrix} X + \begin{pmatrix} \dfrac{25}{2} \\ -\dfrac{6469}{65} \end{pmatrix}, \quad X^+(x,y) = \begin{pmatrix} 10 & -1 \\ 25 & 0 \end{pmatrix} X + \begin{pmatrix} 10 \\ 25 \end{pmatrix}. \tag{2-52}$$

For this PWLS we have that system (2-48) has two real solutions, namely $(t_1^1, t_2^1, y_0^1) = (1.096629.., \; 0.143589.., \; 12.314051..)$; and $(t_1^2, t_2^2, y_0^2) = (2.043521.., \; 0.588292.., 35.501071..)$. Then the PWLS (2-52) has two crossing limit cycles Γ_1 and Γ_2 which intersect Σ in $(0, y_0^1) = (0, 12.314051..)$ and $(0, y_1^1) = (0, 2.476508..)$, $(0, y_0^2) = (0, 35.501071..)$ and $(0, y_1^2) = (0, 6.610102..)$, respectively. See Figure 2.11. □

Proof of statement (iii) ***of Theorem A.*** In this case we analyze the maximum number of crossing limit cycles of PWLS (0-1) when the equilibrium point of linear differential system X^- is a real saddle (S^r). We consider that system X^- is in the canonical form (2-5), then

$$P_- = (x_0, y_0) = \left(\frac{Ab_1 + Bb_2 - 2b_1\tilde{C}}{\tilde{C}^2 - d^2}, -\frac{A^2 b_1 + ABb_2 - 2Ab_1\tilde{C} + b_1\tilde{C}^2 - b_1 d^2}{B\tilde{C}^2 - Bd^2} \right), \tag{2-53}$$

with $0 < \tilde{C}^2 < d^2$ and $B < 0$. This equilibrium point is a $S^r = (-1, -s)$, if $b_1 = A + Bs$, $b_2 = -(A^2 - 2A\tilde{C} + \tilde{C}^2 - d^2 + ABs - 2B\tilde{C}s)/B$. When system X^- is a S^r we have that linear differential system X^+ must be a saddle S^r, then we consider that system X^+ is in the canonical form (2-1) and the equilibrium point P_+ satisfies (2-20) with $\beta = 1, \ |r| < 1$. Therefore we obtain the configuration (S^r, S^r). In the following example we provide a PWLS (0-1) such that the equilibrium points of the linear differential systems X^- and X^+ have the configuration (S^r, S^r) and it has one crossing limit cycle.

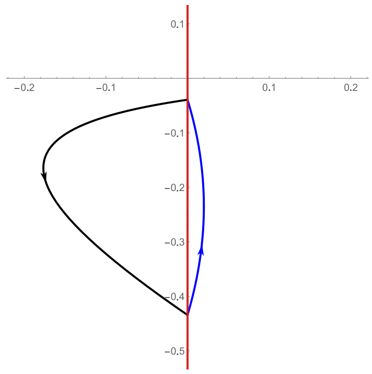

Figure 2.12: *The crossing limit cycle of the discontinuous PWLS* (2-54) *with configuration* (S^r, S^r).

Considering the parameters $A = -1, \ B = -5, \ \tilde{C} = 4/5, \ d = -19/10, \ r = 6/50$ and $s = 0$, we obtain the PWLS formed by

$$X^-(x,y) = \begin{pmatrix} -1 & -5 \\ -\dfrac{37}{500} & \dfrac{12}{5} \end{pmatrix} X + \begin{pmatrix} -1 \\ -\dfrac{37}{500} \end{pmatrix}, \quad X^+(x,y) = \begin{pmatrix} \dfrac{6}{25} & -1 \\ -\dfrac{616}{625} & 0 \end{pmatrix} X + \begin{pmatrix} \dfrac{6}{25} \\ \dfrac{616}{625} \end{pmatrix}. \tag{2-54}$$

For this PWLS we have that system (2-10) has one real solution, namely $(t_1, t_2, y_0) = (0.754087.., \ 0.406189.., \ -0.039307..)$. Then the PWLS (2-54) has one crossing limit cycle which intersects Σ in $(0, y_0) = (0, -0.039307..)$ and $(0, y_1) = (0, -0.434309..)$. See Figure 2.12. □

Proof of statement (iv) ***of Theorem A.*** In this case we analyze the maximum number of crossing limit cycles of PWLS (0-1) when the equilibrium point P_- is a virtual diagonalizable node (N^v). We consider that system X^- is in the canonical form (2-4), then P_- is equal to (2-53) with $0 > \tilde{C}^2 > d^2$ and $B < 0$. This equilibrium point is a N^v if $b_1 = -A - Bs, \ b_2 = -(-A^2 + 2A\tilde{C} - \tilde{C}^2 + d^2 - ABs + 2B\tilde{C}s)/B$. We consider that system X^+ is in the canonical form (2-1) and the equilibrium point P_+ can be a diagonalizable node N^v or an improper node iN^v. Then we have two possible configurations (N^v, N^v) and (N^v, iN^v).

Configuration (N^v, N^v)**:** We assume that P_- is a diagonalizable node N^v and that P_+ satisfies (2-21) with $\beta = 1$ and $|r| > 1$. Then we obtain the configuration (N^v, N^v).

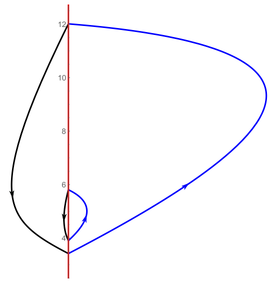

Figure 2.13: *Two crossing limit cycles of the discontinuous PWLS* (2-55) *with configuration* (N^v, N^v).

Considering the parameters $A = -23/10$, $B = -1/2$, $\tilde{C} = -41/10$, $d = 7/2$, $r = 57/25$ and $s = 0$, we obtain the PWLS formed by

$$X^-(x,y) = \begin{pmatrix} -\dfrac{23}{10} & -\dfrac{1}{2} \\ -\dfrac{901}{50} & -\dfrac{59}{10} \end{pmatrix} X + \begin{pmatrix} \dfrac{23}{10} \\ \dfrac{59}{10} \end{pmatrix}, \quad X^+(x,y) = \begin{pmatrix} \dfrac{114}{25} & -1 \\ \dfrac{2624}{625} & 0 \end{pmatrix} X + \begin{pmatrix} \dfrac{114}{25} \\ \dfrac{2624}{625} \end{pmatrix}. \tag{2-55}$$

For this PWLS we have that system (2-10) has two real solutions, namely $(t_1^1, t_2^1, y_0^1) = (0.796618.., \ 1.259611.., \ 12.011789..)$; and $(t_1^2, t_2^2, y_0^2) = (0.205065.., \ 0.425140.., \ 5.805536..)$. Then the PWLS (2-55) has two crossing limit cycles which intersect Σ in $(0, y_0^1) = (0, 12.011789..)$ and $(0, y_1^1) = (0, 3.420218..)$; and $(0, y_0^2) = (0, 5.805536..)$ and $(0, y_1^2) = (0, 3.906249..)$, respectively. See Figure 2.13. Therefore we have that PWLS with the configuration (N^v, N^v) have at least two crossing limit cycles. $\square$

Configuration (N^v, iN^v)**:** The equilibrium point P_- is a diagonalizable node N^v and P_+ satisfies (2-21) with $\beta = 0$ and $r \neq 0$. Then we obtain the configuration (N^v, iN^v). The solution of system X^- starting in $(0, y_0) \in \Sigma$ is

$$\begin{aligned} x^-(t) = & \ \frac{d + e^{\tilde{C}t}(-d\cosh(dt) + (By_0 - (A - \tilde{C}) + Bs))\sinh(dt)}{d}, \\ y^-(t) = & \ s + \frac{e^{\tilde{C}t}}{Bd}(Bd(y_0 - s)\cosh(dt) + (B(-A + \tilde{C})y_0 + (-d^2 + (A - \tilde{C}) \\ & (A - \tilde{C} + Bs)))\sinh(dt)). \end{aligned} \tag{2-56}$$

By (2-35) and (2-56) we obtain that system (2-10) is equivalent to system

$$\begin{aligned} e_1 : \ & d + e^{\tilde{C}t_1}(-d\cosh(dt_1) + (By_0 - (A - \tilde{C} + Bs))\sinh(dt_1)) = 0, \\ e_2 : \ & -1 + e^{-rt_2}(1 + t_2(y_0 - r + s)) = 0, \\ e_3 : \ & -2Bds + Bde^{-rt_2}(y_0 + rt_2y_0 + (s + r(-r + s)t_2)) - e^{\tilde{C}t_1}(Bd(y_0 - s)\cosh(dt_1) \\ & + (B(-A + \tilde{C})y_0 + (-d^2 + (A - \tilde{C})(A - \tilde{C} + Bs)))\sinh(dt_1)) = 0. \end{aligned} \tag{2-57}$$

From equation e_1 we get $y_0 = (A - \tilde{C} + Bs + d\coth(dt_1) - de^{-\tilde{C}t_1}\operatorname{csch}(dt_1))/B$, and from e_2 we get the expression (2-49) for t_2, then substituting y_0 and t_2 in e_3 we obtain that

$$e_3 = \frac{1}{B}\left((-A+\tilde{C}+Br-2Bs)\tilde{f}_0(t_1) + d\tilde{f}_3(t_1) - rB\tilde{f}_4(t_1)\right) = 0,$$

where $\tilde{f}_0(t_1) = 1,\ \tilde{f}_3(t_1) = \coth(dt_1) - e^{\tilde{C}t_1}\operatorname{csch}(dt_1)$ and

$$\tilde{f}_4(t_1) = \frac{1}{\mathcal{W}\left(\dfrac{Bre^{\tilde{C}t_1 + \frac{Bre^{\tilde{C}t_1}}{e^{\tilde{C}t_1}(-A+\tilde{C}+Br-2Bs-d\coth(dt_1))+d\operatorname{csch}(dt_1)}}}{e^{\tilde{C}t_1}(-A+\tilde{C}+Br-2Bs-d\coth(dt_1))+d\operatorname{csch}(dt_1)}\right)}.$$

If

$$t_1 \in (0,\infty) \ \text{ and } \ \tilde{\eta}(t_1) = e^{\tilde{C}t_1}(-A+\tilde{C}+Br-2Bs-d\coth(dt_1)) + d\operatorname{csch}(dt_1) \neq 0, \quad \text{(2-58)}$$

by Proposition 1.7 we can conclude that a system (2-57) has at least two real solutions therefore a PWLS with the configuration (N^v, iN^v) has at least two crossing limit cycles. In what follows we provide a PWLS with configuration (N^v, iN^v) and having two crossing limit cycles.

Considering $A = -23/10,\ B = -8/5,\ \tilde{C} = -24/5,\ d = 37/10,\ r = 3/5$ and $s = 0$, we have that

$$\tilde{\eta}(t_1) = e^{-24t_1/5}\left(-\frac{173}{50} - \frac{37}{10}\coth\left(\frac{37t_1}{10}\right)\right) + \frac{37}{10}\operatorname{csch}\left(\frac{37t_1}{10}\right).$$

Substituting

$$\coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \ \text{ and } \ \operatorname{csch}(x) = \frac{2}{e^x - e^{-x}},$$

in the equation $\tilde{\eta}(t_1)$ we obtain that

$$\tilde{\eta}(t_1) = \frac{e^{37t_1/10}(-370 + 12e^{-17t_1/2} + 358e^{-11t_1/10})}{50(1 - e^{37t_1/5})} > 0, \ \text{ for } \ t_1 > 0.$$

Therefore the condition (2-58) is satisfied. See the graphic of this function in Figure 2.14. Moreover we obtain the PWLS formed by

$$X^-(x,y) = \begin{pmatrix} -\frac{23}{10} & -\frac{8}{5} \\ -\frac{93}{20} & -\frac{73}{10} \end{pmatrix} X + \begin{pmatrix} \frac{23}{10} \\ \frac{93}{20} \end{pmatrix}, \quad X^+(x,y) = \begin{pmatrix} \frac{6}{5} & -1 \\ \frac{9}{25} & 0 \end{pmatrix} X + \begin{pmatrix} \frac{6}{5} \\ \frac{9}{25} \end{pmatrix}. \quad \text{(2-59)}$$

For this PWLS we have that system (2-57) has two real solutions, namely

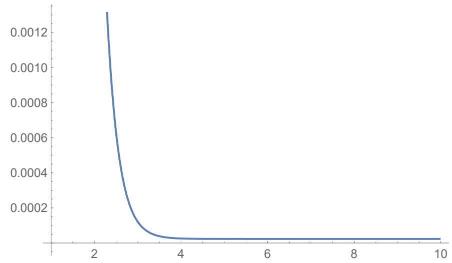

Figure 2.14: *The graphic of the function* (2-58) *for* $t_1 > 0$.

$(t_1^1, t_2^1, y_0^1) = (0.564675.., 5.217342.., 4.794330..)$; and $(t_1^2, t_2^2, y_0^2) = (0.763740.., 6.119198.., 6.860880..)$. Then the PWLS (2-59) has two crossing limit cycles which intersect Σ in $(0, y_0^1) = (0, 4.794330..)$ and $(0, y_1^1) = (0, 0.783292..)$ and $(0, y_0^2) = (0, 6.860880..)$ and $(0, y_1^2) = (0, 0.759263..)$, respectively. See Figure 2.15. $\square$

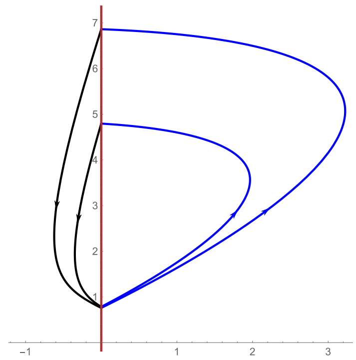

Figure 2.15: *Two crossing limit cycles of the discontinuous PWLS* (2-59) *with configuration* (N^v, iN^v).

Proof of statement (v) ***of Theorem A.*** In this case we analyze the maximum number of crossing limit cycles of PWLS (0-1) when the equilibrium point of linear differential system X^- is a virtual improper node (iN^v). We consider that system X^- is in the canonical form (2-4) with $d = 0$ and $B < 0$, then equilibrium point P_- is

$$P_- = (x_0, y_0) = \left(\frac{Ab_1 + Bb_2 - 2b_1\tilde{C}}{\tilde{C}^2}, -\frac{A^2b_1 + ABb_2 - 2Ab_1\tilde{C} + b_1\tilde{C}^2}{B\tilde{C}^2} \right).$$

This equilibrium point is a virtual improper node iN^v if $P_- = (1, s)$, then $b_1 = -A - Bs$, $b_2 = -(-A^2 + 2A\tilde{C} - \tilde{C}^2 - ABs + 2B\tilde{C}s)/B$. With these condition the solution of system X^- starting in $(0, y_0) \in \Sigma$ is

$$\begin{aligned} x^-(t) &= 1 + e^{\tilde{C}t}(Bty_0 - (1 + (A - \tilde{C} + Bs)t)), \\ y^-(t) &= \frac{Bs + (A - \tilde{C})^2 e^{\tilde{C}t} t + Be^{\tilde{C}t}(1 - At + \tilde{C}t)(y_0 - s)}{B}. \end{aligned} \tag{2-60}$$

We consider that linear differential system X^+ is an improper node iN^v, then we consider that system X^+ is in the canonical form (2-1) and the equilibrium point P_+ satisfies (2-21) with $\beta = 0$ and $r \neq 0$. Therefore we obtain the configuration (iN^v, iN^v).

Now considering t_1 and t_2 as in Proposition 2.3 and by equations (2-60) and (2-35), system (2-10) is equivalent to

$$\begin{aligned} e_1: &\quad 1+e^{\tilde{C}t_1}(Bt_1y_0-(1+(A-\tilde{C}+Bs)t_1))=0,\\ e_2: &\quad -1+e^{-rt_2}(1+t_2(y_0-r+s))=0,\\ e_3: &\quad 2Bs+(A-\tilde{C})^2e^{\tilde{C}t_1}t_1+Be^{\tilde{C}t_1}(1-At_1+\tilde{C}t_1)(y_0-s)\\ &\quad -Be^{-rt_2}(y_0+rt_2y_0+(s+r(-r+s)t_2))=0. \end{aligned} \tag{2-61}$$

By the equation e_1, we get $y_0=(1-e^{-\tilde{C}t_1}+(A-\tilde{C}+Bs)t_1)/Bt_1$, and from e_2 we obtain the expression (2-49) for t_2. Substituting these expressions in e_3, we get

$$e_3=\frac{1}{B}\left((-A+\tilde{C}+Br-2Bs)\tilde{f}_0(t_1)+\tilde{f}_5(t_1)-Br\tilde{f}_6(t_1)\right)=0,$$

where $\tilde{f}_0(t_1)=1,\ \tilde{f}_5(t_1)=\dfrac{1-e^{\tilde{C}t_1}}{t_1}$ and

$$\tilde{f}_6(t_1)=\frac{1}{\mathcal{W}\left(\dfrac{Brt_1e^{t_1\left(\tilde{C}+\frac{Bre^{\tilde{C}t_1}}{1+e^{\tilde{C}t_1}(-1+(-A+\tilde{C}+Br-2Bs)t_1)}\right)}}{1+e^{\tilde{C}t_1}(-1+(-A+\tilde{C}+Br-2Bs)t_1)}\right)}.$$

Therefore by Proposition 1.7 we can conclude that system (2-61) has at least two real solutions for

$$t_1\in(0,\infty)\ \text{ and }\ \bar{\eta}(t_1)=1+e^{\tilde{C}t_1}(-1+(-A+\tilde{C}+Br-2Bs)t_1)\neq 0. \tag{2-62}$$

Due to symmetry we have that if (t_1,t_2,y_0) is a real solution of system (2-61) then $(-t_1,-t_2,y_1)$ also it is a real solution of system (2-61), where $y_1=y^-(t_1)=y^+(-t_2)$, we observed that the real solutions (t_1,t_2,y_0) and $(-t_1,-t_2,y_1)$ of system (2-61) provide the same crossing limit cycle of PWLS with the configuration (iN^v,iN^v). Therefore a PWLS with the configuration (iN^v,iN^v) has at least one crossing limit cycle.

In what follows we provide a example of a PWLS with the configuration (iN^v,iN^v) having one crossing limit cycle.

Considering $A=-6,\ B=-14/5,\ \tilde{C}=-6/5,\ r=11/10$ and $s=0$, we have that

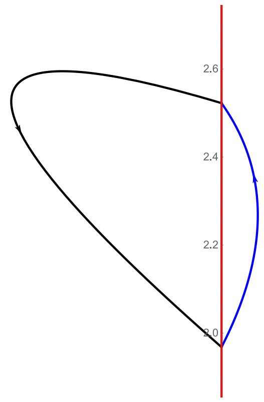

Figure 2.16: *One crossing limit cycle of the discontinuous PWLS* (2-63) *with configuration* (iN^v, iN^v).

$$\bar{\eta}(t_1) = 1 + e^{-6t_1/5}\left(-1 + \frac{43}{25}t_1\right) > 0, \quad \text{for } t_1 > 0.$$

Therefore the condition (2-62) is satisfied.

Moreover we obtain the PWLS formed by

$$X^-(x,y) = \begin{pmatrix} -6 & -\frac{14}{5} \\ \frac{288}{35} & \frac{126}{35} \end{pmatrix} X + \begin{pmatrix} 6 \\ -\frac{288}{35} \end{pmatrix}, \quad X^+(x,y) = \begin{pmatrix} \frac{11}{5} & -1 \\ \frac{121}{100} & 0 \end{pmatrix} X + \begin{pmatrix} \frac{11}{5} \\ \frac{121}{100} \end{pmatrix}. \tag{2-63}$$

For this PWLS we have that system (2-57) has two real solutions, namely $(t_1,t_2,y_0) = (0.964798.., \; 0.448780.., \; 2.522296..)$ and $(-t_1,-t_2,y_1) = (-0.964798.., \; -0.448780.., \; 1.968154..)$, which provide one crossing limit cycle such that intersects Σ in $(0,y_0) = (0,2.522296..)$ and $(0,y_1) = (0,1.968154..)$. See Figure 2.16. □

2.4 Discussions and conclusions

In this chapter we studied the number of crossing limit cycles that the PWLS (0-1) can have when the equilibrium points of linear differential systems X^- and X^+ are symmetric with respect to the line of discontinuity Σ. We observe that having that symmetry to arise fourteen configurations for the singular points of X^- and X^+ which were denoted by (P_-,P_+) depending of type and the position of the equilibrium points. In particular in statement (i) of Theorem A we achieved to provide an upper bound for the maximum number of crossing limit cycles for PWLS (0-1) when the linear differential system X^- has an equilibrium point of type either real or virtual center and in Proposition 2.4 we proved that this upper bound is reached in each case.

For the configurations of the equilibrium points of the linear differential system $X^{\pm}$ of PWLS (0-1) considered in the statements $(ii),(iii),(iv)$ and (v) of Theorem A we can only determine a lower bound for the maximum number of crossing limit cycles and the main drawbacks to provide an upper bound were: first the difficulty of to determine explicitly the time that an orbit expends in each region governed by each linear differential system $X^{\pm}$ even when we can integrate the solution of every linear differential system system; and second the lack of techniques to determine zeros of nonlinear equations. For instance the case obtained in the configuration (F^v, iN^v) where we needed to determine the zeros of the equation

$$e_3 = \frac{1}{B}\left((-A+\tilde{C}+Br-2Bs)\tilde{f}_0(t_1)+d\tilde{f}_1(t_1)-rB\tilde{f}_2(t_1)\right)=0. \qquad (2\text{-}64)$$

With $\tilde{f}_0(t_1)=1,\ \tilde{f}_1(t_1)=\cot(dt_1)-e^{\tilde{C}t_1}\csc(dt_1),$ and

$$\tilde{f}_2(t_1) = \frac{1}{\mathcal{W}\left(\dfrac{Bre^{\tilde{C}t_1+\dfrac{Bre^{\tilde{C}t_1}}{e^{\tilde{C}t_1}(-A+\tilde{C}+Br-2Bs-d\cot(dt_1))+d\csc(dt_1)}}}{e^{\tilde{C}t_1}(-A+\tilde{C}+Br-2Bs-d\cot(dt_1))+d\csc(dt_1)}\right)}.$$

We do not get to verify that the family $\mathcal{F}=\left\{\tilde{f}_0,\tilde{f}_1,\tilde{f}_2\right\}$ is an *Extended Complete Chebyshev* system to know the maximum number of zeros that the equation (2-64) can have and for this reason we use the Proposition 1.7 which only provides the a lower bound for the number of zeros that any non-trivial linear combination of functions in $\mathcal{F}$ can have and thus we only obtain a lower bound for the maximum number of crossing limit cycles for the PWLS (0-1) with the configuration (F^v, iN^v). Similarly it happens in cases considered in the statements $(ii),(iii),(iv)$ and (v).

CHAPTER 3

Crossing limit cycles for a class of PWLC separated by a conic

3.1 Introduction

These last years the study of the version of Hilbert's 16th problem for PWLS in the plane, has increased strongly and there are many papers studying the maximum number of crossing limit cycles when the differential system is defined in two zones separated by a straight line, in particular in [24, 27] it was proved that piecewise linear differential centers (PWLC for short) separated by a straight line have no crossing limit cycles, however recently in [20, 28] were studied planar discontinuous PWLC where the curve of discontinuity is not a straight line, and it was shown that the number of crossing limit cycles in these systems is non-zero. For this reason it is interesting to study the role which plays the shape of the discontinuity curve in the number of crossing limit cycles that planar discontinuous PWLC can have.

We remark that in general it is not easy to provide an explicit upper bound for the maximum number of limit cycles in a class of differential systems, and such that this bound is reached. Therefore in this chapter we study on the either upper or lower bounds for the maximum number of crossing limit cycles of the planar discontinuous PWLC which discontinuity curve Σ is any conic. We denote by R_Σ^i the components of $\mathbb{R}^2 \setminus \Sigma$.

3.1.1 Canonical form

In this chapter we consider that the PWLS are formed by linear differential systems of type centers, and we use the normal form for any arbitrary linear differential center provided in the paper [27], since the proof is short we present it here for completeness.

Lemma 3.1 *Through a linear change of variables and a rescaling of the independent variable every center in $\mathbb{R}^2$ can be written*

$$\dot{x} = -bx - \frac{4b^2 + \omega^2}{4a} y + d, \qquad \dot{y} = ax + by + c, \tag{3-1}$$

with $a \neq 0$ and $\omega > 0$. This system has the first integral

$$H_1(x,y) = 4(ax+by)^2 + 8a(cx-dy) + y^2\omega^2. \tag{3-2}$$

Proof. Consider an arbitrary linear differential system in the plane

$$\dot{x} = Ax + By + d, \qquad \dot{y} = ax + by + c,$$

and suppose that it has a center. Then the eigenvalues of this system are

$$\lambda_{1,2} = \frac{A + b \pm \sqrt{4aB + (A-b)^2}}{2}.$$

If this system has a center then $A + b = 0$ and $4aB + (A-b)^2 = -\omega^2$ for some $\omega > 0$ and $aB < 0$, this is, $A = -b$, $B = -(4b^2+\omega^2)/(4a)$ and $a > 0$. And $\lambda_{1,2} = \pm i\omega$. □
We remark that the normal form in Lemma 3.1 is independent of the change of coordinates, so we can use this normal form in each region R^i_Σ.

We know that using an affine change of coordinates, any conic can be written in one of following nine canonical forms:

- (p): $x^2 + y^2 = 0$ two complex straight lines intersecting at a real point;
- (CL): $x^2 + 1 = 0$ two complex parallel straight lines;
- (CE): $x^2 + y^2 + 1 = 0$ complex ellipse;
- (DL): $x^2 = 0$ one double real straight line;
- (PL): $x^2 - 1 = 0$ two real parallel straight lines;
- (LV): $xy = 0$ two real straight lines intersecting at a real point;
- (E): $x^2 + y^2 - 1 = 0$ ellipse;
- (H): $x^2 - y^2 - 1 = 0$, hyperbola;
- (P): $y - x^2 = 0$ parabola.

Here we do not consider conics of type (p), (CL) or (CE) because they do not separate the plane in connected regions.

We observe that we have three options for limit cycles of discontinuous PWLC separated by a conic Σ, first we have the crossing limit cycles such that intersect the discontinuity curve in exactly two points, we study this case in Section 3.2, second we have the limit cycles which intersect the discontinuity curve Σ in three points such that two points intersect the discontinuity curve transversely and the third point of intersection is a tangency point of limit cycle with the discontinuity curve Σ, then these limit cycles are not crossing limit cycles, therefore we do not consider them in this work, we study them in future works. And finally we have crossing limit cycles which intersect the discontinuity curve Σ in four points, we study this case in Section 3.3. Finally in Sections 3.4 and 3.5 we verify that the crossing limit cycles having two intersection points with the conic Σ can coexist with crossing limit cycles having four intersection points with Σ.

We denote by $\mathcal{N}_{\Sigma}^{m}$ the maximum number of crossing limit cycles of a PWLC when the discontinuity curve is the conic Σ, with $\Sigma \in \{(DL),(PL),(LV),(E),(H),(P)\}$. And the crossing limit cycles intersect the discontinuity curve Σ in m points, where $m \in \{2,4\}$.

3.1.2 Closing equations

We know that the problem of to determine crossing limit cycles in PWLS systems it is a tedious problem because the main technique to determine periodic solutions in these systems is the Poincare map which it is difficult to use because to determine explicitly the time that an orbit expends in each region governed by each linear differential center it is difficult, even when we can integrate the solution of every system in each region. Therefore in [25] it was provided an alternative way to the Poincaré map for analyzing the limit cycles of planar PWLS using the first integrals of different planar linear differential systems in each region and studying the closing equations of PWLS.

If we consider that the PWLS (1-1) is formed by two linear differential systems, X^1 and X^2, and $H_1(x,y)$ and $H_2(x,y)$ are the first integrals of linear differential systems. The *closing equations* of PWLS (1-1) with two points on Σ are:

$$\begin{aligned} H_1(x_1,y_1) &= H_1(x_2,y_2), \\ H_2(x_1,y_1) &= H_2(x_2,y_2), \end{aligned} \tag{3-3}$$

where $(x_1,y_1),(x_2,y_2) \in \Sigma$. We observe that if there is a crossing periodic orbit of PWLS (1-1) which intersects the discontinuity curve Σ in the two points $(\tilde{x}_1,\tilde{y}_1)$ and $(\tilde{x}_2,\tilde{y}_2)$, the point $(\tilde{x}_1,\tilde{y}_1,\tilde{x}_2,\tilde{y}_2)$ must be a real solution of system (3-3). Therefore in order to study the number $\mathcal{N}_{\Sigma}^{2}$ of PWLS (1-1), we must determine the real solutions (x_1,y_1,x_2,y_2) of system (3-3).

By the Implicit Function Theorem, each equation in (3-3) defines a differentiable function $\phi_1(x_1,y_1)$ and $\phi_2(x_1,y_1)$, respectively in such a way that $\pi = \phi_1 \circ \phi_2$ is the Poincaré map of PWLS (1-1).

In order to study the number $\mathcal{N}_{\Sigma}^{4}$ of PWLS (1-1), with $(x_i,y_i) \in \Sigma$ the intersection points of the crossing limit cycle and Σ, for $i = 1,2,3,4$. We must determine the real solutions $(x_1,y_1,x_2,y_2,x_3,y_3,x_4,y_4)$ of system

$$
\begin{aligned}
H_1(x_1,y_1) &= H_1(x_2,y_2),\\
H_2(x_2,y_2) &= H_2(x_3,y_3),\\
H_1(x_3,y_3) &= H_1(x_4,y_4),\\
H_2(x_4,y_4) &= H_2(x_1,y_1).
\end{aligned}
\tag{3-4}
$$

It is possible to define the closing equations analogously when the PWLS (1-1) is formed by three or more linear differential systems.

3.2 Crossing limit cycles intersecting the discontinuity curve Σ in two points

The goal in this section is to study the maximum number of crossing limit cycles of PWLS formed by linear differential centers and separated by any conic which only intersect the discontinuity curve in two points, this is, $\mathcal{N}^2_\Sigma$. For this we must study the real solutions of system (3-3).

The problem of to determine $\mathcal{N}^2_\Sigma$ with $\Sigma \in \{(DL),(PL),(LV),(P),(E)\}$ was studied in the papers [24, 27, 28, 20]. More precisely, in [24, 27] it was proved that the discontinuous PWLC separated by one straight line have no crossing limit cycles with two points on Σ. And we observe that to determine the numbers $\mathcal{N}^2_{(LV)}$, $\mathcal{N}^2_{(PL)}$ and $\mathcal{N}^2_{(DL)}$, it is equivalent to determine the number of crossing limit cycles with two points on the discontinuity curve for PWLC separated by a single straight line. Therefore it is possible to conclude that $\mathcal{N}^2_{(LV)} = \mathcal{N}^2_{(PL)} = \mathcal{N}^2_{(DL)} = 0$.

In the paper [28] the authors considered discontinuous PWLC separated by the parabola $y = x^2$ and proved that they have at most three crossing limit cycles that intersect Σ in two points, i.e., $\mathcal{N}^2_{(P)} = 3$.

With regard to the discontinuous PWLC separated by an ellipse, in [20] the authors shown that the class of planar discontinuous PWLC separated by the circle $\mathbb{S}^1$ has at most two crossing limit cycles that intersect $\mathbb{S}^1$ in two points, this is $\mathcal{N}^2_{(E)} = 2$. Moreover, in [20] the authors provided a PWLC which reach the upper bound of 2 crossing limit cycles, see Example 3.2.

Example 3.2 *We consider the discontinuous PWLS in $\mathbb{R}^2$ separated by the ellipse (E) and both linear differential centers are defined as follows:*

$$
\dot{x} = -2x - 2y - \sqrt{2} - 1, \qquad \dot{y} = 4x + 2y + \sqrt{2},
$$

in the unbounded region limited by the ellipse (E), and in the bounded region with *boundary the ellipse (E) we have the linear differential center*

$$\dot{x} = x - \frac{5}{4}y + \frac{1}{\sqrt{2}} + \frac{1}{8}, \qquad \dot{y} = x - y - \frac{1}{\sqrt{2}}.$$

This discontinuous PWLC has exactly two crossing limit cycles, see Figure 3.1.

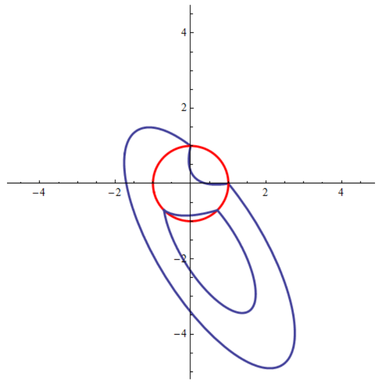

Figure 3.1: *The two limit cycles of the discontinuous PWLC of Example 3.2.*

We summarize the above results in the following theorem.

Theorem 3.3 *Consider a planar discontinuous PWLC where the discontinuity curve Σ is such that, $\Sigma \in \{(DL),(PL),(LV),(P),(E)\}$. The following statements holds.*

(a) The numbers $\mathcal{N}^2_{(LV)}$, $\mathcal{N}^2_{(PL)}$, $\mathcal{N}^2_{(DL)}$ are equals to zero;

(b) The number $\mathcal{N}^2_{(P)}$ is three;

(c) The number $\mathcal{N}^2_{(E)}$ is two.

We observe that in the cases considered in Theorem 3.3, there is no a result determining the maximum number of crossing limit cycles for discontinuous PWLC when Σ is a hyperbola (H), this is, the number $\mathcal{N}^2_{(H)}$. This is the main result in this section.

For the systems of the class $\mathcal{F}_H$ we have the following regions in the plane

$$\begin{aligned} R^1_H &= \{(x,y) \in \mathbb{R}^2 : x^2 - y^2 > 1, x > 0\}, \\ R^2_H &= \{(x,y) \in \mathbb{R}^2 : x^2 - y^2 < 1\}, \\ R^3_H &= \{(x,y) \in \mathbb{R}^2 : x^2 - y^2 > 1, x < 0\}. \end{aligned} \tag{3-5}$$

And we have that the PWLC in the family $\mathcal{F}_H$ are formed by three linear differential centers, namely one in each region R^i_H, $i = 1,2,3$. Nevertheless a crossing limit cycle which intersects the hyperbola $x^2 - y^2 = 1$ in only two different points $p = (x_1, y_1)$ and $q = (x_2, y_2)$ it is formed by parts of orbits of only two linear differential centers, namely either the linear differential centers in R_H^1 and R_H^2 or the linear differential centers in the regions R_H^3 and R_H^2, then without lost of generality we consider that the crossing limit cycles are formed by parts of orbits of linear differential systems in the regions R_H^1 and R_H^2. In order to have crossing limit cycles that intersect (H) in the points $p=(x_1,y_1)$ and $q=(x_2,y_2)$, these points must satisfy the *closing equations* given in (3-3), this is

$$\begin{aligned} H_1(x_1,y_1) &= H_1(x_2,y_2), \\ H_2(x_2,y_2) &= H_2(x_1,y_1), \\ x_1^2-y_1^2 &= 1, \\ x_2^2-y_2^2 &= 1. \end{aligned} \tag{3-6}$$

3.2.1 Statement of the main results

Theorem B *Let $\mathcal{F}_H$ be the family of planar discontinuous PWLC where Σ is a hyperbola (H). Then the maximum number of crossing limit cycles that intersect Σ in two points is two, this is, $\mathcal{N}^2_{(H)}=2$. Moreover this upper bound is reached.*

Theorem B is proved in Section 3.2.2.

Proposition 3.4 *If the parameters b,c,d and ω of the linear differential center in the region R_H^2 satisfy that $b=c=d=0$ and $\omega=2$. Then the PWLC in $\mathcal{F}_H$ have no crossing limit cycles.*

Proposition 3.5 *There are PWLC in $\mathcal{F}_H$ having exactly one crossing limit cycle that intersects Σ in two points, see Figure 3.2.*

Proposition 3.6 *The upper bound for the maximum number of crossing limit cycles provided in Theorem B is reached. See Figure 3.3.*

3.2.2 Proof of the main results

***Proof of Proposition* 3.4.** We consider a discontinuous PWLC formed by two linear differential centers as in Lemma 3.1, such that the linear differential center in the region R_H^2 satisfies that $b=c=d=0$ and $\omega=2$, this is the linear differential center in R_H^2 is

$$\dot{x}=-y, \qquad \dot{y}=x. \tag{3-7}$$

We have that the orbits of this linear center intersect the hyperbola (H) in two or in four points, when it intersects (H) in exactly two points these are $(\pm 1,0)$, which are points of tangency between the hyperbola and the solution curves of the center (3-7), then it is impossible that there are crossing periodic orbits independent of the linear differential center that can be considered in the region R_H^1. So the orbits which can produce a crossing limit cycle intersect the hyperbola in four points and clearly these orbits cannot be crossing limit cycles with exactly two points on the discontinuity curve (H). □

Proof of Proposition 3.5. In the region R_H^1 we consider the arbitrary linear differential center (3-1) which has first integral (3-2). In the region R_H^2 we consider the arbitrary linear differential center

$$\dot{x} = -Bx - \frac{4B^2+\Omega^2}{4A}y + D, \qquad \dot{y} = Ax + By + C, \tag{3-8}$$

with $A \neq 0$ and $\Omega > 0$. Which has the first integral

$$H_2(x,y) = 4(Ax+By)^2 + 8A(Cx-Dy) + y^2\Omega^2. \tag{3-9}$$

It is possible to do a rescaling of time in the two above systems and to assume without loss of generality that $a = A = 1$. For to build the example of a PWLC in $\mathcal{F}_H$ formed by the linear differential centers (3-1) and (3-8) with one crossing limit cycle, we will impose the existence of a periodic solution and we will determine the parameters for each linear differential center (3-1) and (3-8) with the established conditions.

In order to have a periodic solution of the PWLC formed by the linear differential centers (3-1) and (3-8) that intersect (H) in the points $p = (x_1, y_1)$ and $q = (x_2, y_2)$, these points must satisfy the *closing equations* given in (3-6), that is,

$$\begin{aligned}
e_1 &: \ 8cx_1 - 8cx_2 - 8dy_1 + 4(x_1+by_1)^2 + 8dy_2 - 4(x_2+by_2)^2 + y_1^2\omega^2 - y_2^2\omega^2 = 0,\\
e_2 &: \ -8Cx_1 + 8Cx_2 + 8Dy_1 - 4(x_1+By_1)^2 - 8Dy_2 + 4(x_2+By_2)^2 - y_1^2\Omega^2 + y_2^2\Omega^2 = 0,\\
e_3 &: \ x_1^2 - y_1^2 = 1,\\
e_4 &: \ x_2^2 - y_2^2 = 1.
\end{aligned} \tag{3-10}$$

We assume that there is a real solution of system (3-10), namely $q^1 = (x_1, y_1, x_2, y_2) = (1, 0, \sqrt{5}, 2)$, then the equations e_3 and e_4 are satisfied and by equation e_1 we get the following expression for the parameter d

$$d = \frac{1}{4}\left(4 + 4\sqrt{5}b + 4b^2 - 2c + 2\sqrt{5}c + \omega^2\right),$$

and the equation e_2 we obtain the following expression for the parameter D:

$$D = \frac{1}{4}\left(4 + 4\sqrt{5}B + 4B^2 - 2C + 2\sqrt{5}C + \Omega^2\right).$$

Now we fix the remaining parameters, namely we consider that $b = 0; B = 0; c = -2; C = -3/2; \omega = 10; \Omega = 1$, with these condition we obtain a PWLC such that in the region R_H^1 it has the linear differential center

$$\dot{x} = 27 - \sqrt{5} - 25y, \qquad \dot{y} = -2 + x, \tag{3-11}$$

this linear differential center has the first integral $H_1(x,y) = 4(-4+x)x + 4y(-54 + 2\sqrt{5} + 25y)$. In the region R_H^2 we obtain the linear differential center

$$\dot{x} = 2 - \frac{3\sqrt{5}}{4} - \frac{y}{4}, \qquad \dot{y} = -\frac{3}{2} + x, \tag{3-12}$$

which has the first integral $H_2(x,y) = 4(-3+x)x + y(-16+6\sqrt{5}+y)$. With these linear differential centers the *closing equations* (3-10) are equivalent to system

$$\begin{aligned} 4(-4x_1 + x_1^2 + 2(-27+\sqrt{5})y_1 + 25y_1^2 + 4x_2 - x_2^2 + 54y_2 \\ -2\sqrt{5}y_2 - 25y_2^2) = 0, \\ -12x_1 + 4x_1^2 + 2(-8+3\sqrt{5})y_1 + y_1^2 + 12x_2 \\ -4x_2^2 + 16y_2 - 6\sqrt{5}y_2 - y_2^2 = 0, \\ x_1^2 - y_1^2 = 1, \\ x_2^2 - y_2^2 = 1. \end{aligned} \tag{3-13}$$

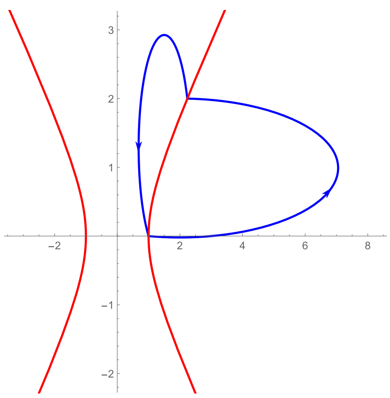

Figure 3.2: *The crossing limit cycle of the discontinuous PWLC formed by the centers (3-11) and (3-12).*

Taking into account that we are only interested in the solutions $(p,q) = (x_1,y_1,x_2,y_2)$ satisfying $x_1x_2 > 0$ and $p \neq q$, this discontinuous PWLC formed by the linear differential centers (3-11) and (3-12) has one crossing limit cycle, because the unique real solution with those conditions (p,q) is $p = (1,0)$ and $q = (\sqrt{5},2)$.

The first linear differential system (3-11) has the solution

$$\begin{aligned} x_1(t) &= 2 - \cos(5t) - \frac{1}{5}(-27+\sqrt{5})\sin(5t), \\ y_1(t) &= -\frac{2}{25}\sin\left(\frac{5t}{2}\right)\left(5\cos\left(\frac{5t}{2}\right) + (-27+\sqrt{5})\sin\left(\frac{5t}{2}\right)\right), \end{aligned} \tag{3-14}$$

satisfying the initial conditions $x_1(0) = 1$ and $y_1(0) = 0$. The linear differential system (3-12) has the solution

$$\begin{aligned} x_2(t) &= \frac{1}{2}\left(3+(-3+2\sqrt{5})\cos\left(\frac{t}{2}\right)-3(-2+\sqrt{5})\sin\left(\frac{t}{2}\right)\right), \\ y_2(t) &= 8-3\sqrt{5}+3(-2+\sqrt{5})\cos\left(\frac{t}{2}\right)+(-3+2\sqrt{5})\sin\left(\frac{t}{2}\right), \end{aligned} \tag{3-15}$$

satisfying the initial conditions $x_2(0) = \sqrt{5}$ and $y_2(0) = 2$. The time that the solution $(x_1(t), y_1)$ contained in R_H^1 needs to reach the point $q = (\sqrt{5}, 2)$ is $t_1 = 0.658816...$ The flying time of the solution $(x_2(t), y_2(t))$ in the region R_H^2 is $t_2 = 7.210481...$ Drawing the orbits $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ with the times t_1 and t_2, respectively, we obtain the crossing limit cycle of Figure 3.2. □

The examples illustrated throughout of this work are similarly constructed.

***Proof of Proposition* 3.6.** In the region R_H^1 we consider the linear differential center

$$\begin{aligned} \dot{x} &= \frac{289-305\sqrt{6}+\sqrt{3(85057-9248\sqrt{6})}+32\sqrt{3}x-169y}{768}, \\ \dot{y} &= \frac{-(289\sqrt{2}-32\sqrt{3})(1+\sqrt{2+\sqrt{3}})+768x-32\sqrt{3}y}{768}, \end{aligned} \tag{3-16}$$

which has the first integral

$$\begin{aligned} H_1(x,y) =& 384x^2 + x\left(\left(32\sqrt{3}-289\sqrt{2}\right)\left(1+\sqrt{2+\sqrt{3}}\right)-32\sqrt{3}y\right) \\ &+ y\left(98y-\sqrt{3\left(85057-9248\sqrt{6}\right)}+305\sqrt{6}-289\right). \end{aligned}$$

In the region R_H^2 we have the linear differential center

$$\dot{x} = \frac{1}{8}\left(-3+8\sqrt{2}+\sqrt{3}-\sqrt{6}\right)-\frac{x}{2}-\frac{y}{2}, \quad \dot{y} = \frac{1}{8}\left(-1-5\sqrt{2}-\sqrt{3}\right)+x+\frac{y}{2}, \tag{3-17}$$

this system has the first integral

$$H_2(x,y) = 4x^2 - x\left(1+5\sqrt{2}+\sqrt{3}-4y\right)+y\left(3-8\sqrt{2}-\sqrt{3}+\sqrt{6}+2y\right).$$

The discontinuous PWLC formed by the linear differential centers (3-16) and (3-17) has two crossing limit cycles, because the unique real solutions (p,q) of system (3-6) are $\left(1,0,\sqrt{2},1\right)$ and $\left(\sqrt{2},-1,\sqrt{3},\sqrt{2}\right)$, therefore the intersection points of the two crossing limit cycles with the hyperbola are the pairs $(1,0)$, $\left(\sqrt{2},1\right)$ and $\left(\sqrt{2},-1\right)$, $\left(\sqrt{3},\sqrt{2}\right)$. See these two crossing limit cycles in Figure 3.3.

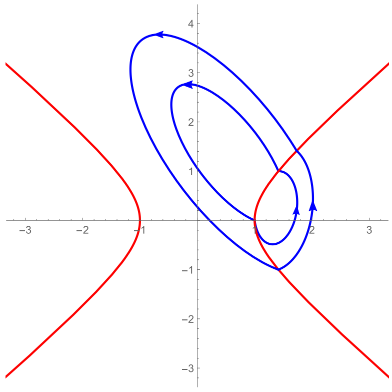

Figure 3.3: *The two limit cycles of the discontinuous PWLC formed by the centers (3-16) and (3-17).*

With this example we get to prove that the upper bound provided in Theorem B is reached. □

Proof of Theorem B. In the region R_H^1 we consider the arbitrary linear differential center (3-1) which has first integral (3-2). In the region R_H^2 we consider the arbitrary linear differential center (3-8) which has the first integral (3-9).

It is possible to do a rescaling of time in the two above systems. Suppose $\tau = at$ in R_H^1 and $s = At$ in R_H^2. These two rescaling change the velocity in which the orbits of systems (3-1) and (3-8) travel, nevertheless they do not change the orbits, therefore they will not change the crossing limit cycles that the discontinuous PWLC may have. After these rescalings of the time we can assume without loss of generality that $a = A = 1$, and the dot in system (3-1) (resp. (3-8)) denotes derivative with respect to the new time τ (resp. s).

We assume that the discontinuous PWLC formed by the two linear differential centers (3-1) and (3-8) has at least three crossing periodic solutions. For this we must impose that the system of equations (3-6) has three pairs of points as solution, namely (p_i, q_i), $i = 1,2,3$, since these solutions provide crossing periodic solutions. We consider

$$p_i = (\cosh r_i, \sinh r_i) \quad \text{and} \quad q_i = (\cosh s_i, \sinh s_i), \text{ for } i = 1,2,3. \tag{3-18}$$

These points are the points where the three crossing periodic solutions intersect the hyperbola (H). Now we consider that the point (p_1, q_1) satisfies system (3-6) and with this condition we obtain the following expression

$$\begin{aligned} d =& \frac{1}{8(\sinh r_1 - \sinh s_1)} \big(4\cosh^2 r_1 - 4\cosh^2 s_1 + 8\cosh r_1 (c + b\sinh r_1) \\ &- 8\cosh s_1 (c + b\sinh s_1) + (4b^2 + \omega^2)(\sinh^2 r_1 - \sinh^2 s_1)\big), \end{aligned}$$

and D has the same expression that d changing (b, c, ω) by (B, C, Ω).

We assume that the point (p_2,q_2) satisfies system (3-6) and we get the expression

$$
\begin{aligned}
c = &\frac{-1}{8(\sinh(r_1-r_2)+\sinh(r_2-s_1)-\sinh(r_1-s_2)+\sinh(s_1-s_2))}\left((\sinh r_2-\sinh s_2)\right.\\
&\left(4\cosh^2 s_1+4b\sinh(2s_1)-4\cosh^2 r_1-4b\sinh(2r_1)\right)+(\sinh r_1-\sinh s_1)\left(4\cosh^2 r_2\right.\\
&-4\cosh^2 s_2+8b\cosh r_2\sinh r_2-8b\cosh s_2\sinh s_2+(4b^2+\omega^2)(\sinh r_2-\sinh s_2)\\
&\left.\left.(-\sinh r_1+\sinh r_2-\sinh s_1+\sinh s_2)\right)\right),
\end{aligned}
$$

and C has the same expression that c changing (b,ω) by (B,Ω).

Finally we are going to impose that the point (p_3,q_3) satisfies system (3-6) and we get an expression for ω^2. In this case $\omega^2=K/L$, where the expression of K is

$$
\begin{aligned}
&4\left((1+b^2)\operatorname{csch}\left(\frac{r_1-r_2+s_1-s_2}{2}\right)\sinh\left(\frac{r_3-s_3}{2}\right)\left(\cosh\left(\frac{r_1-r_2-r_3+s_1-s_2-3s_3}{2}\right)\right.\right.\\
&-\cosh\left(\frac{r_1-r_2-r_3+s_1-3s_2-s_3}{2}\right)+\cosh\left(\frac{r_1-r_2-3r_3+s_1-s_2-s_3}{2}\right)\\
&-\cosh\left(\frac{r_1-3r_2-r_3+s_1-s_2-s_3}{2}\right)-\cosh\left(\frac{3r_1+r_2-r_3+s_1+s_2-s_3}{2}\right)\\
&+\cosh\left(\frac{r_1+3r_2-r_3+s_1+s_2-s_3}{2}\right)-\cosh\left(\frac{r_1+r_2-r_3+3s_1+s_2-s_3}{2}\right)\\
&\left.+\cosh\left(\frac{r_1+r_2-r_3+s_1+3s_2-s_3}{2}\right)\right)+b^2\left(\cosh\left(\frac{3r_1-r_2+r_3+s_1-s_2+s_3}{2}\right)\right.\\
&\left.\left.-\cosh\left(\frac{r_1-r_2+3r_3+s_1-s_2+s_3}{2}\right)\right)\right)-2b\left(\sinh\left(\frac{r_1-r_2-r_3+s_1-s_2-3s_3}{2}\right)\right.\\
&-\sinh\left(\frac{r_1-r_2-r_3+s_1-3s_2-s_3}{2}\right)-\sinh\left(\frac{2r_1-r_2+2r_3+s_1-s_2+s_3}{2}\right)\\
&\sinh\left(\frac{r_1-r_3}{2}\right)+2\cosh\left(\frac{2r_1+2r_2-r_3+s_1+s_2-s_3}{2}\right)\sinh\left(\frac{r_1-r_2}{2}\right)\\
&-2\cosh\left(\frac{2r_1-r_2+2r_3+s_1-s_2+s_3}{2}\right)\sinh\left(\frac{r_1-r_3}{2}\right)+2\sinh\left(\frac{r_2-r_3}{2}\right)\\
&\cosh\left(\frac{r_1-2r_2-2r_3+s_1-s_2-s_3}{2}\right)+2\cosh\left(\frac{r_1+r_2-r_3+2(s_1+s_2)-s_3}{2}\right)\\
&\left.\sinh\left(\frac{s_1-s_2}{2}\right)-2\cosh\left(\frac{r_1-r_2+r_3+2s_1-s_2}{2}+s_3\right)\sinh\left(\frac{s_1-s_3}{2}\right)\right)\\
&\left.+2(1+b^2)\sinh\left(\frac{r_1-r_2+r_3+2s_1-s_2}{2}+s_3\right)\sinh\left(\frac{s_1-s_3}{2}\right)\right)
\end{aligned}
$$

and the expression of L is

$$
\begin{aligned}
&\operatorname{csch}\left(\frac{r_1-r_2+s_1-s_2}{2}\right)\sinh\left(\frac{r_3-s_3}{2}\right)\left(-\cosh\left(\frac{r_1-r_2-r_3+s_1-s_2-3s_3}{2}\right)\right.\\
&+\cosh\left(\frac{r_1-r_2-r_3+s_1-3s_2-s_3}{2}\right)-\cosh\left(\frac{r_1-r_2-3r_3+s_1-s_2-s_3}{2}\right)\\
&+\cosh\left(\frac{r_1-3r_2-r_3+s_1-s_2-s_3}{2}\right)+\cosh\left(\frac{3r_1+r_2-r_3+s_1+s_2-s_3}{2}\right)\\
&-\cosh\left(\frac{r_1+3r_2-r_3+s_1+s_2-s_3}{2}\right)+\cosh\left(\frac{r_1+r_2-r_3+3s_1+s_2-s_3}{2}\right)\\
&-\cosh\left(\frac{r_1+r_2-r_3+s_1+3s_2-s_3}{2}\right)-\cosh\left(\frac{3r_1-r_2+r_3+s_1-s_2+s_3}{2}\right)\\
&+\cosh\left(\frac{r_1-r_2+3r_3+s_1-s_2+s_3}{2}\right)-\sinh\left(\frac{r_1-r_2+r_3+2s_1-s_2}{2}+s_3\right)\\
&\left.2\sinh\left(\frac{s_1-s_3}{2}\right)\right),
\end{aligned}
$$

and the expression of Ω^2 is the same than the expression of ω^2 changing b by B.

Now we replace d, c, ω^2 in the expression of the first integral $H_1(x,y)$ and we have

$$
H_1(x,y) = 4(x^2-y^2) + h(x,y,r_1,r_2,r_3,s_1,s_2,s_3)b, \tag{3-19}
$$

and analogously we have

$$
H_2(x,y) = 4(x^2-y^2) + h(x,y,r_1,r_2,r_3,s_1,s_2,s_3)B. \tag{3-20}
$$

Now we are going to analyze if the discontinuous PWLC formed by (3-1) and (3-8) has more crossing periodic solutions than the three supposed in (3-18). Taking into account (3-19) and (3-20) the *closing equations* (3-6) becomes

$$
\begin{aligned}
h(x_1,y_1,r_1,r_2,r_3,s_1,s_2,s_3) &= h(x_2,y_2,r_1,r_2,r_3,s_1,s_2,s_3),\\
x_1^2-y_1^2 &= 1,\\
x_2^2-y_2^2 &= 1.
\end{aligned} \tag{3-21}
$$

That is, we must solve a system with three equations and four unknowns x_1, y_1, x_2, y_2 variables. We know that in (3-18) we have at least the three solutions, so system (3-21) has a continuum of solutions which produce a continuum of crossing periodic solutions, so such systems cannot have crossing limit cycles. Since in Proposition 3.6, we have proved that there are systems in $\mathcal{F}_H$ with two crossing limit cycles, it follows that the maximum number of crossing limit cycles that intersect Σ in two points is two. This upper bound is reached as it was proved in Proposition 3.6 and this completes the proof of Theorem B. □

The study of upper bound for the maximum number of crossing limit cycles with two points on the discontinuity curve Σ for PWLC formed by linear differential centers when Σ is any conic it is completed with the above two Theorems 3.3 and B.

In the following section we study the number of crossing limit cycles for PWLC formed by linear differential centers and separated by any conic which intersect the discontinuity curve in four points.

3.3 Crossing limit cycles intersecting the discontinuity curve Σ in four points

In this section we study the number of crossing limit cycles for PWLC and separated by any conic which intersect the discontinuity curve in four points, this is, we determine the numbers $\mathcal{N}_{\Sigma}^{4}$. We do not consider the case where the discontinuity curve is the conic (DL), because first in [24, 27] it was proved that discontinuous PWLC separated by a straight line have no crossing limit cycles and second because the crossing limit cycles of these discontinuous PWLC cannot have four points on the discontinuity curve.

3.3.1 Statements of the main results

In the following theorems we determine the maximum number of crossing limit cycles for planar discontinuous PWLC with four points on discontinuity curve, $\mathcal{N}_{\Sigma}^{4}$ where the plane is divided by the curve of discontinuity $\Sigma \in \{(PL),(LV),(P),(E),(H)\}$.

Theorem C *Let $\mathcal{F}_{\Sigma}$ be the family of planar discontinuous PWLC with $\Sigma \in \{(PL),(P),(E),(H)\}$. Then the number $\mathcal{N}_{\Sigma}^{4}$ is equal to one. Moreover this upper bound is reached.*

In the paper [27] it was proved Theorem C for the family $\mathcal{F}_{PL}$, for a particular linear center between the two parallel straight lines. In Section 3.3.2 we prove it for any linear center.

Proposition 3.7 *The upper bound provided in Theorem C is reached for each family $\mathcal{F}_{\Sigma}$ with $\Sigma \in \{(PL),(P),(E),(H)\}$. See Figures 3.4-3.7.*

When the discontinuity curve Σ is of the type (LV), then we have the following regions in the plane:

$$\begin{aligned}
R_{LV}^{1} &= \{(x,y) \in \mathbb{R}^2 : x > 0 \text{ and } y > 0\},\\
R_{LV}^{2} &= \{(x,y) \in \mathbb{R}^2 : x < 0 \text{ and } y > 0\},\\
R_{LV}^{3} &= \{(x,y) \in \mathbb{R}^2 : x < 0 \text{ and } y < 0\},\\
R_{LV}^{4} &= \{(x,y) \in \mathbb{R}^2 : x > 0 \text{ and } y < 0\}.
\end{aligned} \tag{3-22}$$

Moreover, $\Sigma = \Gamma_1^+ \cup \Gamma_1^- \cup \Gamma_2^+ \cup \Gamma_2^-$, where $\Gamma_1^+ = \{(x,y) \in \mathbb{R}^2 : x = 0, y \geq 0\}$, $\Gamma_1^- = \{(x,y) \in \mathbb{R}^2 : x = 0, y \leq 0\}$, $\Gamma_2^+ = \{(x,y) \in \mathbb{R}^2 : y = 0, x \geq 0\}$ and $\Gamma_2^- = \{(x,y) \in \mathbb{R}^2 : y = 0, x \leq 0\}$. In this case we have two types of crossing limit cycles, namely crossing limit cycles of type 1 which intersect only two branches of Σ in exactly two points in each branch, and crossing limit cycles of type 2 which intersect in a unique point each branch of the set Σ.

Theorem D *Let $\mathcal{F}_{LV}$ be the family of planar discontinuous PWLC formed by four linear centers and with Σ of the type (LV). The following statements holds.*

(i) The maximum number of crossing limit cycles type 1 is one. Moreover this upper bound is reached.

(ii) The maximum number of crossing limit cycles type 2 is at least four. See Figure 3.13

Theorem D is proved in Section 3.3.2.

Proposition 3.8 *The upper bound for the maximum number of crossing limit cycles of type 1 provided in statement (i) of Theorem D is reached. See Figure 3.8.*

Proposition 3.9 *Consider the family of planar discontinuous piecewise linear differential centers $\mathcal{F}_{LV}$. Then the following statement hold.*

(a) There are systems in $\mathcal{F}_{LV}$ with exactly one crossing limit cycle of type 2, see Figure 3.9.

(b) There are systems in $\mathcal{F}_{LV}$ with exactly two crossing limit cycles of type 2, see Figure 3.10.

(c) There are systems in $\mathcal{F}_{LV}$ with exactly three crossing limit cycles of type 2, see Figure 3.11.

3.3.2 Proof of the main results

In this subsection we provide the proofs of Propositions 3.7, 3.8 and 3.9 and Theorems *C* and *D*.

Proof of Proposition 3.7 for the family $\mathcal{F}_{PL}$. We consider the discontinuous PWLC

$$\begin{aligned}
\dot{x} &= -\frac{3}{16} - \frac{x}{2} - \frac{5}{16}y, & \dot{y} &= \frac{1}{16} + x + \frac{y}{2}, & &\text{in } R_{PL}^1,\\
\dot{x} &= -\frac{67}{500} - \frac{x}{5} - \frac{29}{100}y, & \dot{y} &= -\frac{43}{1000} + x + \frac{y}{5}, & &\text{in } R_{PL}^2,\\
\dot{x} &= \frac{7}{60} - \frac{x}{3} - \frac{13}{36}y, & \dot{y} &= \frac{1}{7} + x + \frac{y}{3}, & &\text{in } R_{PL}^3.
\end{aligned} \tag{3-23}$$

These systems have the first integrals

$$
\begin{aligned}
H_1(x,y) &= 16x^2 + y(6+5y) + 2x(1+8y),\\
H_2(x,y) &= 4\left(x+\frac{y}{5}\right)^2 + y^2 + \frac{1}{125}(-43x+134y),\\
H_3(x,y) &= \frac{8}{7}x - \frac{14}{15}y + y^2 + \frac{4}{9}(3x+y)^2,
\end{aligned}
$$

respectively. Then the discontinuous PWLC formed by the linear differential centers (3-

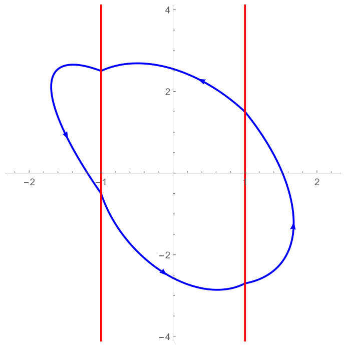

Figure 3.4: *The crossing limit cycle of the discontinuous PWLC (3-23) with three centers separated by the conic (PL).*

23) has one crossing limit cycle that intersects (PL) in four points, because the unique real solution (y_1,y_2,y_3,y_4) with $y_1 > y_2$ and $y_3 > y_4$ of system (3-46) is the point $(y_1,y_2,y_3,y_4) = (3/2,-27/10,5/2,-1/2)$. See the crossing limit cycle of this system in Figure 3.4. This completes the proof of Proposition 3.7 for the family $\mathcal{F}_{PL}$. □

Proof of Proposition 3.7 for the family $\mathcal{F}_P$. We consider the discontinuous PWLC formed by the following linear differential centers

$$
\dot{x} = \frac{831}{128} + x - \frac{17}{16}y, \qquad \dot{y} = \frac{587}{128} + x - y, \qquad \text{in } R_P^1, \tag{3-24}
$$

$$
\dot{x} = \frac{21145}{4176} + \frac{x}{6} - \frac{5}{18}y, \qquad \dot{y} = \frac{127}{174} + x - \frac{y}{6}, \qquad \text{in } R_P^2. \tag{3-25}
$$

These linear differential centers have the first integrals

$$
\begin{aligned}
H_1(x,y) &= 64x^2 + x(587-128y) + y(-831+68y),\\
H_2(x,y) &= \frac{1}{522}(3048x - 21145y) + 4\left(x-\frac{y}{6}\right)^2 + y^2,
\end{aligned}
$$

respectively. The discontinuous PWLC formed by the linear differential centers (3-24) and (3-25) has one crossing limit cycle, because the unique real solution (p_1,p_2,p_3,p_4) of system (3-52) is $p_1 = (6,36),\ p_2 = (-5,25),\ p_3 = (-3/2,9/4),$ and $\ p_4 = (2,4)$. See the

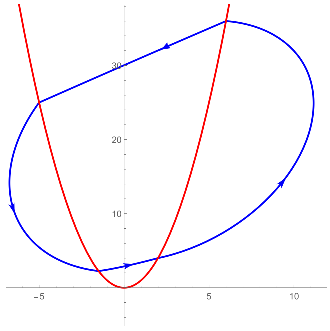

Figure 3.5: *The crossing limit cycle of the discontinuous PWLC formed by (3-24) and (3-25) separated by the conic (P).*

crossing limit cycle of this PWLC in Figure 3.5. This completes the proof of Proposition 3.7 for the family $\mathcal{F}_P$. □

Proof of Proposition 3.7 for the family $\mathcal{F}_E$. Consider the discontinuous PWLC in the

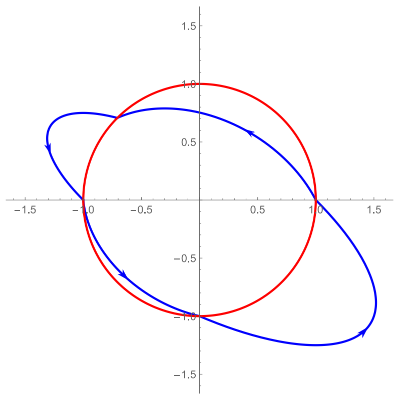

Figure 3.6: *The crossing limit cycle of the discontinuous PWLC formed by the centers (3-26) and (3-27) separated by the conic (E).*

family $\mathcal{F}_E$ formed by the following two linear differential centers

$$\dot{x} = \frac{-107-89\sqrt{2}}{1024} - \frac{5}{16}x - \frac{345}{256}y, \qquad \dot{y} = \frac{-71+89\sqrt{2}}{1024} + x + \frac{5}{16}y, \quad \text{in } R_E^1, \tag{3-26}$$

$$\dot{x} = -\frac{1}{4} - x - 2y, \qquad \dot{y} = \frac{1}{4} + x + y, \quad \text{in } R_E^2. \tag{3-27}$$

These linear differential centers have the first integrals

$$H_1(x,y) = 512x^2 + x(-71+89\sqrt{2}+320y) + y(107+89\sqrt{2}+690y),$$
$$H_2(x,y) = x + 2x^2 + y + 4xy + 4y^2,$$

respectively. Then the discontinuous PWLC formed by the linear differential centers (3-26) and (3-27) has one crossing limit cycle, because the unique real solution (p_1,p_2,p_3,p_4) of system (3-53) is $p_1=(1,0),\ p_2=\left(-\sqrt{2}/2,1/\sqrt{2}\right),\ p_3=(-1,0)$, and $p_4=(0,-1)$. See the crossing limit cycle of this system in Figure 3.6. This completes the proof of Proposition 3.7 for the family $\mathcal{F}_E$. $\square$

Proof of Proposition 3.7 for the family $\mathcal{F}_H$. We consider the discontinuous PWLC in the family $\mathcal{F}_H$ formed by the following linear differential centers

$$
\begin{aligned}
\dot{x} &= \frac{-355+64\sqrt{10}+80\sqrt{21}}{64(6+\sqrt{21})}-\frac{x}{2}-\frac{29}{16}y, && \dot{y}=1+x+\frac{y}{2}, \quad \text{in } R_H^1,\\
\dot{x} &= K_1-\frac{x}{10}-\frac{101}{100}y, && \dot{y}=K_2+x+\frac{y}{10}, \ \text{in } R_H^2,\\
\dot{x} &= \frac{3}{4}(-11+2\sqrt{3})+\frac{337(-3+2\sqrt{3})}{64\sqrt{5}}-\frac{3}{2}x-\frac{45}{16}y, && \dot{y}=-\frac{3}{2}x+\frac{3}{2}y, \quad \text{in } R_H^3.
\end{aligned}
\tag{3-28}
$$

Where

$$
\begin{aligned}
K_1 = {} & \frac{1}{200\sqrt{81-12\sqrt{35}}(-20\sqrt{2}+26\sqrt{3}-14\sqrt{5}+7\sqrt{7}-13\sqrt{15}+2\sqrt{70})}(27300\\
& -62790\sqrt{2}+8750\sqrt{3}-31356\sqrt{5}+15678\sqrt{7}+2500\sqrt{30}-2730\sqrt{35}\\
& -600\sqrt{42}+6279\sqrt{70}-420\sqrt{105}\Big)\\
K_2 = {} & \frac{1}{400(-20\sqrt{2}+26\sqrt{3}-14\sqrt{5}+7\sqrt{7}-13\sqrt{15}+2\sqrt{70})}\Big(3(200-800\sqrt{2}\\
& +5226\sqrt{3}-1846\sqrt{5}+403\sqrt{7}-2613\sqrt{15}+240\sqrt{35}+80\sqrt{70})\Big).
\end{aligned}
$$

These linear differential centers have first integrals

$$
\begin{aligned}
H_1(x,y) = {} & (355-64\sqrt{10}-80\sqrt{21})y+2(6+\sqrt{21})(16x(2+x)+16xy+29y^2),\\
H_2(x,y) = {} & \frac{1}{200\sqrt{27-4\sqrt{35}}(-20\sqrt{2}+26\sqrt{3}-14\sqrt{5}+7\sqrt{7}-13\sqrt{15}+2\sqrt{70})}\Big(y^2\\
& +\left(x+\frac{y}{10}\right)^2+(3\sqrt{27-4\sqrt{35}}(200-800\sqrt{2}+5226\sqrt{3}-1846\sqrt{5}\\
& +403\sqrt{7}-2613\sqrt{15}+240\sqrt{35}+80\sqrt{70})x-2(8750+9100\sqrt{3}\\
& -20930\sqrt{6}+2500\sqrt{10}-600\sqrt{14}-10452\sqrt{15}+5226\sqrt{21}-420\sqrt{35}\\
& -910\sqrt{105}+2093\sqrt{210})y)\Big),\\
H_3(x,y) = {} & 4x^2+12x(-1+y)+\frac{1}{40}y(2640-480\sqrt{3}+1011\sqrt{5}-674\sqrt{15}+450y),
\end{aligned}
$$

respectively. The unique real solution (p_1,p_2,p_3,p_4) that satisfies (3-58) in this case is $p_1=(\sqrt{10},-3),\ p_2=(-5/2,\sqrt{21/4}),\ p_3=(-4,\sqrt{15})$ and $p_4=(-7/2,-\sqrt{45/4})$. See the crossing limit cycle of this system in Figure 3.7. This completes the proof of

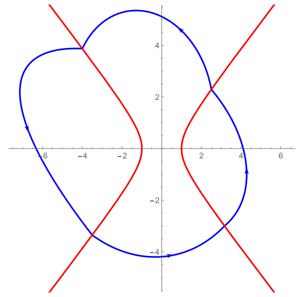

Figure 3.7: *The crossing limit cycle of the discontinuous PWLC (3-28) with discontinuity curve the conic (H).*

Proposition 3.7 for the family $\mathcal{F}_H$. □

Proof of Proposition 3.8 We consider the following discontinuous PWLC

$$
\begin{aligned}
\dot{x} &= \frac{23177}{9000} - \frac{11}{10}x - \frac{557}{450}y, & \dot{y} &= -\frac{1837}{1125} + x + \frac{11}{10}y, && \text{in } R^1_{LV},\\
\dot{x} &= \frac{477}{64} - \frac{x}{2} - \frac{53}{16}y, & \dot{y} &= 1 + x + \frac{y}{2}, && \text{in } R^2_{LV},\\
\dot{x} &= -y - \beta, & \dot{y} &= x + \alpha, && \text{in } R^3_{LV},\\
\dot{x} &= 2 - \frac{x}{2} - \frac{17}{4}y, & \dot{y} &= -2 + x + \frac{y}{2}, && \text{in } R^4_{LV}.
\end{aligned}
\tag{3-29}
$$

In the region R^3_{LV} we can consider any linear differential center, because the crossing limit cycle will be formed by parts of the orbits of the centers of the regions R^1_{LV}, R^2_{LV} and R^4_{LV}. The centers in (3-29) have the first integrals

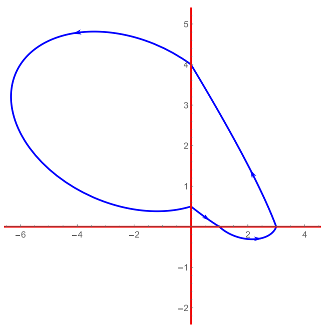

Figure 3.8: *The crossing limit cycle of type* 1 *of the discontinuous PWLC (3-29) separated by the conic (LV).*

$$
\begin{aligned}
H_1(x,y) &= 4500x^2 + 44x(-334 + 225y) + y(-23177 + 5570y),\\
H_2(x,y) &= 4x^2 + 4x(2 + y) + \frac{53}{8}y(-9 + 2y),\\
H_3(x,y) &= (x + \alpha)^2 + (y + \beta)^2,\\
H_4(x,y) &= 4(-4 + x)x + 4(-4 + x)y + 17y^2,
\end{aligned}
$$

in R^i_{LV}, $i=1,2,3,4$, respectively. Then for the discontinuous PWLC (3-29) system (3-62) becomes

$$\begin{aligned} -14696x_2+4500x_2^2+(23177-5570y_2)y_2 &= 0,\\ (y_1-y_2)(-9+2y_1+2y_2) &= 0,\\ 14696x_1-4500x_1^2+y_1(-23177+5570y_1) &= 0,\\ (x_1-x_2)(-4+x_1+x_2) &= 0. \end{aligned} \tag{3-30}$$

Taking into account that the solutions (x_1,x_2,y_1,y_2) must satisfy $x_1<x_2$ and $y_1<y_2$, we have that the unique solution of system (3-30) is the point $(x_1,x_2,y_1,y_2)=(1,3,1/2,4)$. See the crossing limit cycle of type 1 of the PWLC (3-29) in Figure 3.8. This completes the proof of Proposition 3.8. □

Proof of statement (a) ***of Proposition*** 3.9 In the region R^1_{LV} we consider the linear differential center

$$\dot{x}=-\frac{13}{4}-\frac{x}{2}-\frac{y}{2},\qquad \dot{y}=1+x+\frac{y}{2}, \tag{3-31}$$

this system has the first integral $H_1(x,y)=2(2x^2+2x(2+y)+y(13+y))$. In the region R^2_{LV} we have the linear differential center

$$\dot{x}=-\frac{851}{3600}-\frac{x}{3}-\frac{181}{900}y,\qquad \dot{y}=\frac{3}{2}+x+\frac{y}{3}, \tag{3-32}$$

which has the first integral $H_2(x,y)=4x^2+4x(9+2y)/3+y(851+362y)/450$. In the region R^3_{LV} we have the linear differential center

$$\dot{x}=-\frac{43}{32}+\frac{x}{4}-\frac{5}{16}y,\qquad \dot{y}=-\frac{1}{2}+x-\frac{y}{4}, \tag{3-33}$$

which has the first integral $H_3(x,y)=4x^2-3x(2+y)+y(-43+5y)/4$. And in the region R^4_{LV} we have the linear differential center

$$\dot{x}=\frac{137}{72}+\frac{x}{3}-\frac{25}{144}y,\qquad \dot{y}=\frac{3}{2}+x-\frac{y}{3}, \tag{3-34}$$

which has the first integral $H_4(x,y)=4x(3+x)-(137+24x)y/9+25y^2/36$.

In order to have a crossing limit cycle of type 2, which intersects the discontinuity conic (LV) in four different points $p_1=(x_1,0)$, $q_1=(0,y_1)$, $p_2=(x_2,0)$ and $q_2=(0,y_2)$, with $x_1,y_1>0$ and $x_2,y_2<0$, these points must satisfy the *closing equations*

$$\begin{aligned} e_1&: H_1(x_1,0)-H_1(0,y_1)=0,\\ e_2&: H_2(0,y_1)-H_2(x_2,0)=0,\\ e_3&: H_3(x_2,0)-H_3(0,y_2)=0,\\ e_4&: H_4(0,y_2)-H_4(x_1,0)=0. \end{aligned} \tag{3-35}$$

Considering the four above linear differential centers (3-31), (3-32), (3-33) and (3-34) and their respective first integrals $H_i(x,y)$, $i=1,2,3,4$, we have the following equivalent system

$$\begin{aligned} 4x_1(2+x_1)-2y_1(13+y_1) &= 0,\\ -4x_2(3+x_2)+\frac{1}{450}y_1(851+361y_1) &= 0,\\ 4(x_2-1)x_2+\frac{1}{4}(43-5y_2)y_2 &= 0,\\ -4x_1(x_1+3)+\frac{1}{36}y_2(-548+25y_2) &= 0, \end{aligned} \tag{3-36}$$

the unique real solution (p_1,q_1,p_2,q_2) of (3-36) is $p_1=(3,0)$, $q_1=(0,2)$, $p_2=(-7/2,0)$

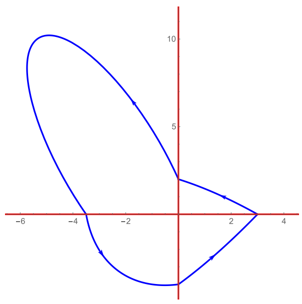

Figure 3.9: *The crossing limit cycle of type 2 of the discontinuous PWLC formed by the linear centers (3-31), (3-32), (3-33) and (3-34) separated by (LV).*

and $q_2=(0,-4)$, therefore the PWLC formed by the linear differential centers (3-31), (3-32), (3-33) and (3-34) has exactly one crossing limit cycle of type 2. See the crossing limit cycle of this system in Figure 3.9. This completes the proof of statement (a) of Proposition 3.9. □

Proof of statement (b) ***of Proposition*** 3.9 In the region R^1_{LV} we consider the linear differential center

$$\dot{x}=-\frac{25}{8}+\frac{x}{2}+\frac{y}{2},\quad \dot{y}=\frac{11}{2}-x-\frac{y}{2}, \tag{3-37}$$

which has the first integral $H_1(x,y)=4x^2+4x(-11+y)+y(-25+2y)$. In the region R^2_{LV} we consider the linear differential center

$$\dot{x}=-\frac{251}{400}-x-\frac{109}{100}y,\quad \dot{y}=-\frac{293}{200}+x+y, \tag{3-38}$$

this system has the first integral $H_2(x,y)=200x^2+y(251+218y)+x(-586+400y)$. In the region R^3_{LV} we have the linear differential center

$$\dot{x}=\frac{5}{96}+\frac{x}{4}-\frac{5}{16}y,\quad \dot{y}=\frac{23}{24}+x-\frac{y}{4}, \tag{3-39}$$

this system has the first integral $H_3(x,y) = 4x^2 + x(23/3 - 2y) + 5y(-1+3y)/12$. In the region R_{LV}^4 we have the linear differential center

$$\dot{x} = -\frac{73}{800} + \frac{x}{10} - \frac{29}{400}y, \quad \dot{y} = -\frac{31}{40} + x - \frac{y}{10}, \tag{3-40}$$

this system has the first integral $H_4(x,y) = 400x^2 - 20x(31+4y) + y(73+29y)$. This discontinuous PWLC formed by the linear differential centers (3-37), (3-38),(3-39) and (3-40) has two crossing limit cycles of type 2, because the unique real solutions $(p_1^i, q_1^i, p_2^i, q_2^i)$, with $i = 1,2$ of system (3-35) are $p_1^1 = (3/2, 0)$, $q_1^1 = (0,3)$, $p_2^1 = (-5/2, 0)$ and $q_2^1 = (0,-2)$ and $p_1^2 = (2,0)$, $q_1^2 = (0, 9/2)$, $p_2^2 = (-4,0)$ and $q_2^2 = (0,-5)$. See these two crossing limit cycles in Figure 3.10. This completes the proof of statement (b) of Proposition 3.9. □

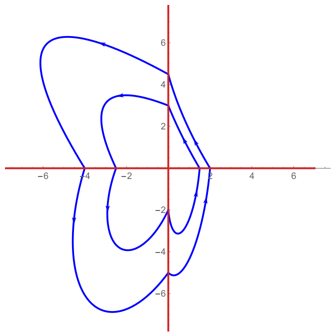

Figure 3.10: *The two crossing limit cycle of type 2 of the discontinuous PWLC formed by the linear centers (3-37), (3-38), (3-39) and (3-40) separated by (LV).*

Proof of statement (c) ***of Proposition*** 3.9 In the region R_{LV}^1 we consider the linear differential center

$$\dot{x} = \frac{813}{803} - \frac{x}{2} - \frac{300}{803}y, \quad \dot{y} = -\frac{1207}{730} + x + \frac{y}{2}, \tag{3-41}$$

which has the first integral $H_1(x,y) = 4x^2 + x(-4828/365 + 4y) + 24y(-271+50y)/803$. In the region R_{LV}^2 we have the linear differential center

$$\dot{x} = \frac{210061}{55055} + \frac{11}{10}x - \frac{15760}{11011}y, \quad \dot{y} = \frac{63667}{20020} + x - \frac{11}{10}y, \tag{3-42}$$

this system has the first integral $H_2(x,y) = 110110x^2 + x(700337 - 242242y) + 4y(-210061 + 39400y)$. In the region R_{LV}^3 we have the linear differential center

$$\dot{x} = -\frac{79831}{38904} - \frac{7}{10}x - \frac{3875}{4863}y, \quad \dot{y} = \frac{421379}{194520} + x + \frac{7}{10}y, \tag{3-43}$$

this system has the first integral $H_3(x,y) = 97260x^2 + 5y(79831 + 15500y) + 7x(60197 + 19452y)$. In the region R^4_{LV} we have the linear differential center

$$\dot{x} = -\frac{15513}{28057} + \frac{2}{5}x - \frac{5700}{28057}y, \quad \dot{y} = -\frac{330343}{280570} + x - \frac{2}{5}y, \tag{3-44}$$

this system has the first integral $H_4(x,y) = 140285x^2 + 30y(5171 + 950y) - x(330343 + 112228y)$. This discontinuous PWLC formed by the linear differential centers (3-41), (3-42), (3-43) and (3-44) has three crossing limit cycles of type 2, because the unique real solutions $(p^i_1, q^i_1, p^i_2, q^i_2)$, with $i = 1,2,3$ of system (3-35) are $p^1_1 = (9/5, 0)$, $q^1_1 = (0,3)$, $p^1_2 = (-7/2, 0)$ and $q^1_2 = (0, -43/10)$; $p^2_1 = (2,0)$, $q^2_1 = (0, 33/10)$, $p^2_2 = (-39/10, 0)$ and $q^2_2 = (0, -47/10)$; and $p^3_1 = (17/10, 0)$, $q^3_1 = (0, 289/100)$, $p^3_2 = (-33/10, 0)$ and $q^3_2 = (0, -411/100)$. See these three crossing limit cycles of type 2 in Figure 3.11. This completes the proof of statement (c) of Proposition 3.9. □

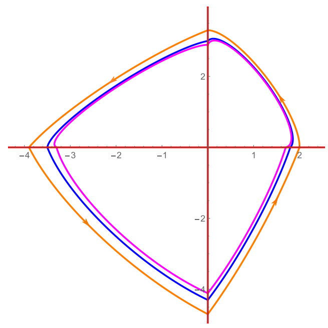

Figure 3.11: *The three crossing limit cycle of type 2 of the discontinuous PWLC formed by the centers (3-41), (3-42),(3-43) and (3-44) separated by (LV).*

In what follows we prove Theorem C.

Proof of Theorem C for the family $\mathcal{F}_{PL}$. When the discontinuity curve Σ is of the type (PL), we have following three regions in the plane:

$$\begin{aligned} R^1_{PL} &= \{(x,y) \in \mathbb{R}^2 : x < -1\}, \\ R^2_{PL} &= \{(x,y) \in \mathbb{R}^2 : -1 < x < 1\}, \\ R^3_{PL} &= \{(x,y) \in \mathbb{R}^2 : x > 1\}. \end{aligned}$$

We consider a planar discontinuous PWLC separated by two parallel straight lines and formed by three arbitrary linear centers. By Lemma 3.1, we have that these linear centers can be as follows

$$
\begin{aligned}
\dot{x} &= -bx - \frac{4b^2+\omega^2}{4a}y + d, & \dot{y} &= ax + by + c, \quad \text{in} \quad R^1_{PL},\\
\dot{x} &= -Bx - \frac{4B^2+\Omega^2}{4A}y + D, & \dot{y} &= Ax + By + C, \quad \text{in} \quad R^2_{PL},\\
\dot{x} &= -\beta x - \frac{4\beta^2+\lambda^2}{4\alpha}y + \delta, & \dot{y} &= \alpha x + \beta y + \gamma, \quad \text{in} \quad R^3_{PL}.
\end{aligned} \tag{3-45}
$$

These linear centers have the first integrals

$$
\begin{aligned}
H_1(x,y) &= 4(ax+by)^2 + 8a(cx-dy) + y^2\omega^2,\\
H_2(x,y) &= 4(Ax+By)^2 + 8A(Cx-Dy) + y^2\Omega^2,\\
H_3(x,y) &= 4(\alpha x+\beta y)^2 + 8\alpha(\gamma x-\delta y) + y^2\lambda^2,
\end{aligned}
$$

respectively.

We are going to analyze if the discontinuous PWLC (3-45) has crossing periodic solutions. Since the orbits in each region R^i_{PL}, for $i = 1,2,3$, are ellipses or pieces of one ellipse, we have that if there is a crossing limit cycle this must intersect each straight line $x = \pm 1$ in exactly two points, namely $(1,y_1),(1,y_2)$ and $(-1,y_3),(-1,y_4)$, with $y_1 > y_2$ and $y_3 > y_4$. Therefore we must study the solutions of the *closing equations* (3-4), this is,

$$
\begin{aligned}
H_3(\ 1,y_2) &= H_3(\ 1,y_1),\\
H_2(\ 1,y_1) &= H_2(-1,y_3),\\
H_1(-1,y_3) &= H_1(-1,y_4),\\
H_2(-1,y_4) &= H_2(\ 1,y_2),
\end{aligned}
$$

or equivalently, we have the system

$$
\begin{aligned}
-(y_1-y_2)(8\beta-8\delta+(4\beta^2+\lambda^2)(y_1+y_2)) &= 0,\\
16C-8D(y_1-y_3)+8B(y_1+y_3)+(4B^2+\Omega^2)(y_1^2-y_3^2) &= 0,\\
(y_3-y_4)(-8b-8d+(4b^2+\omega^2)(y_3+y_4)) &= 0,\\
-16C+8D(y_2-y_4)-8B(y_2+y_4)-(4B^2+\Omega^2)(y_2^2-y_4^2) &= 0.
\end{aligned} \tag{3-46}
$$

By hypothesis $y_1 > y_2$ and $y_3 > y_4$ and therefore system (3-46) is equivalently to the system

$$
\begin{aligned}
\gamma_3-\delta_3+l_3(y_1+y_2) &= 0,\\
\eta-\delta_2(y_1-y_3)+\gamma_2(y_1+y_3)+l_2(y_1^2-y_3^2) &= 0,\\
-\gamma_1-\delta_1+l_1(y_3+y_4) &= 0,\\
-\eta+\delta_2(y_2-y_4)-\gamma_2(y_2+y_4)-l_2(y_2^2-y_4^2) &= 0,
\end{aligned} \tag{3-47}
$$

where $\gamma_1 = 8b, \gamma_2 = 8B, \gamma_3 = 8\beta, \delta_1 = 8d, \delta_2 = 8D, \delta_3 = 8\delta, l_1 = 4b^2+\omega^2, l_2 = 4B^2+$

$\Omega^2, l_3 = 4\beta^2 + \lambda^2$ and $\eta = 16C$. As $l_1 \neq 0$ and $l_3 \neq 0$, we can isolated y_1 and y_4 of the first and the third equations of system (3-47), respectively. Then, we obtain

$$y_1 = -\frac{l_3 y_2 + \gamma_3 - \delta_3}{l_3}, \qquad y_4 = \frac{-l_1 y_3 + \gamma_1 + \delta_1}{l_1}.$$

Now replacing these expressions of y_1 and y_4 in the second and fourth equations of (3-47), we have the system of two equations

$$E_1 = \frac{l_2(l_3(y_2-y_3)+\psi_3)(l_3(y_2+y_3)+\psi_3)+l_3(l_3(\eta+(y_3-y_2)\gamma_2+(y_2+y_3)\delta_2)-\psi_2\psi_3)}{l_3^2},$$
$$E_2 = \frac{l_2\psi_1^2 - l_1\psi_1(2l_2y_3+\gamma_2+\delta_2) - l_1^2(\eta+(y_2-y_3)(l_2(y_2+y_3)+\gamma_2)-(y_2+y_3)\delta_2)}{l_1^2}.$$

Doing the Groebner basis of the two polynomials E_1 and E_2 with respect to the variables y_2 and y_3, we obtain the equations

$$m_0 + m_1 y_3 + m_2 y_3^2 = 0, \quad k_0 + k_1 y_3 + k_2 y_2 = 0, \tag{3-48}$$

with

$$\begin{aligned}
m_0 =& \frac{1}{l_1^4 l_3^2}\big(2l_1^3 l_3^3\psi_2^2(l_3\psi_1(\gamma_2+\delta_2)+l_1(2l_3\eta-\psi_2\psi_3)) - l_1^2 l_2 l_3^2(l_3^2\psi_1^2(\gamma_2^2-6\gamma_2\delta_2+\delta_2^2)\\
&+4l_1l_3\psi_2(2l_1\eta+\psi_1(\gamma_2+\delta_2))\psi_3 - 5l_1^2\psi_2^2\psi_3^2) + 2l_1l_2^2l_3(-l_3^3\psi_1^3(\gamma_2+\delta_2)\\
&+2l_1l_3^2\psi_1^2\psi_2\psi_3 - 2l_1^3\psi_2\psi_3^3) + l_1^2l_3(2l_1\eta+\psi_1(\gamma_2+\delta_2))\psi_3^2\\
&+l_2^3(l_3\psi_1+l_1\psi_3)^2(l_3\psi_1-l_1\psi_3)^2\big),\\
m_1 =& \frac{4l_2\psi_1(-2l_1l_3\psi_1+l_1\psi_3))(2l_1l_3\delta_2-l_2(l_3\psi_1-l_1\psi_3))}{l_1^3},\\
m_2 =& \frac{4l_2(l_2(l_3\psi_1+l_1\psi_3-2l_1l_3\gamma_2))(l_2(l_3\psi_1-l_1\psi_3-2l_1l_2\delta_2))}{l_1^2},\\
k_0 =& \frac{(l_1l_3(l_3\psi_1(\gamma_2+\delta_2)\psi_1^2+l_1^2\psi_3^2))}{l_1^2l_3},\\
k_1 =& \frac{(2l_3(l_2\psi_1-l_1(\gamma_2+\delta_2))}{l_1},\\
k_2 =& 2(l_3\psi_2-l_2\psi_3),
\end{aligned}$$

where $\psi_1 = \gamma_1+\delta_1, \psi_2 = \gamma_2-\delta_2$ and $\psi_3 = \gamma_3-\delta_3$.

Then by Bézout's Theorem 1.11 in this case, we have that system (3-48) has at most two solutions. Moreover, from these two solutions (y_2^1, y_3^1) and (y_2^2, y_3^2) of (3-48), we will have two solutions of (3-47) which are of the form $(y_1^1, y_2^1, y_3^1, y_4^1)$ and $(y_1^2, y_2^2, y_3^2, y_4^2)$, but analyzing system (3-47) we have that if $(y_1^1, y_2^1, y_3^1, y_4^1)$ is a solution, then $(y_2^1, y_1^1, y_4^1, y_3^1)$

is another solution. Finally due to the fact that $y_1 > y_2$ and $y_3 > y_4$, at most one of these two solutions will be satisfactory. Therefore we have proved that the planar discontinuous PWLC of the family $\mathcal{F}_{PL}$, can have at most one crossing limit cycle.

In Proposition 3.7 we verify that this upper bound is reached, this is, we verify that there is a discontinuous PWLC that belongs to the family $\mathcal{F}_{PL}$ and it has exactly one crossing limit cycle.

This completes the proof of Theorem C for the family $\mathcal{F}_{PL}$. □

Proof of Theorem C for the family $\mathcal{F}_P$. If the discontinuity curve Σ is of the type (P), we have following two regions in the plane:

$$\begin{aligned} R_P^1 = & \quad \{(x,y) \in \mathbb{R}^2 : x^2 < y\}, \\ R_P^2 = & \quad \{(x,y) \in \mathbb{R}^2 : x^2 > y\}. \end{aligned}$$

We consider a planar discontinuous PWLC formed by two linear arbitrary centers. By Lemma 3.1 these PWLC can be as follows

$$\begin{aligned} \dot{x} &= -bx - \frac{4b^2+\omega^2}{4a}y + d, & \dot{y} &= ax + by + c, \text{ in } R_P^1, \\ \dot{x} &= -\beta x - \frac{4\beta^2+\omega^2}{4\alpha}y + \delta, & \dot{y} &= \alpha x + \beta y + \gamma, \text{ in } R_P^2. \end{aligned} \tag{3-49}$$

These linear differential centers have the first integrals

$$\begin{aligned} H_1(x,y) &= 4(ax+by)^2 + 8a(cx-dy) + y^2\omega^2, \\ H_2(x,y) &= 4(\alpha x+\beta y)^2 + 8\alpha(\gamma x-\delta y) + y^2\Omega^2, \end{aligned} \tag{3-50}$$

respectively. After two rescaling of time as in the proof Theorem B we can assume without loss of generality that $a = \alpha = 1$.

In order that the PWLC (3-49) has crossing limit cycles with four point on (P). We must study the solutions of the system:

$$\begin{aligned} e_1 &: H_1(x_1,x_1^2) - H_1(x_2,x_2^2) = 0, \\ e_2 &: H_2(x_2,x_2^2) - H_2(x_3,x_3^2) = 0, \\ e_3 &: H_1(x_3,x_3^2) - H_1(x_4,x_4^2) = 0, \\ e_4 &: H_2(x_4,x_4^2) - H_2(x_1,x_1^2) = 0, \end{aligned} \tag{3-51}$$

or equivalently

$$
\begin{aligned}
e_1: 4x_1^2(1+bx_1)^2-4x_2^2(1+bx_2)^2+8x_1(c-dx_1)+8x_2(dx_2-c)+(x_1^4-x_2^4)\omega^2 &= 0,\\
e_2: 4x_2^2(1+\beta x_2)^2+8x_2(\gamma-\delta x_2)^2-4x_3^2(1+\beta x_3)^2+8x_3(\delta x_3-\gamma)+(x_2^4-x_3^4)\Omega &= 0,\\
e_3: 4x_3^2(1+bx_3)^2-4x_4^2(1+bx_4)^2+8x_3(c-dx_3)+8x_4(dx_4-c)+(x_3^4-x_4^4)\omega^2 &= 0,\\
e_4: 4x_4^2(1+\beta x_4)^2+8x_1(\delta x_1-\gamma)-4x_1^2(1+\beta x_1)^2+8x_4(\gamma-\delta x_4)+(x_4^4-x_1^4)\Omega^2 &= 0.
\end{aligned}
\tag{3-52}
$$

We assume that the discontinuous PWLC (3-49) has two crossing periodic solutions. For this we must have that system of equations (3-52) has two real solutions, namely (p_1,p_2,p_3,p_4) and (q_1,q_2,q_3,q_4), where $p_i=(k_i,k_i^2)$ and $q_i=(L_i,L_i^2)$, with $i=1,2,3,4$. These points are the points where the two crossing periodic solution intersect discontinuity curve (P).

If the point (p_1,p_2,p_3,p_4) satisfies system (3-52), from equation e_1 of (3-52) we obtain the following expression

$$
d=\frac{8c+4(k_1+k_2)(1+b(k_1+k_2))+4b(k_1^2+k_2^2)+(k_1+k_2)(k_1^2+k_2^2)l_1}{8(k_1+k_2)},
$$

by the equation e_2 of (3-52) we get the expression

$$
\delta=\frac{8\gamma+4(k_2+k_3)(1+\beta(k_2+k_3))+4\beta(k_2^2+k_3^2)+(k_2+k_3)(k_2^2+k_3^2)l_2}{8(k_2+k_3)},
$$

from equation e_3 of (3-52) we obtain the expression

$$
\begin{aligned}
c=&\frac{k_1+k_2}{2(k_1+k_2-k_3-k_4)(k_3-k_4)}\left((k_4^2-k_3^2)\left(\frac{(k_1^2+k_2^2)l_1}{4}+1+(1+b(k_1+k_2))\right.\right.\\
&\left.\left.+b\frac{k_1^2+k_2^2}{k_1+k_2}\right)+2b(k_4^3-k_3^3)+(k_4^4-k_3^4)l_1\right)
\end{aligned}
$$

and from equation e_4 of (3-52) we obtain the expression

$$
\begin{aligned}
\gamma=&\frac{1}{8(k_1-k_2-k_3+k_4)}\left(8\beta(k_1+k_4)k_4-l_2(k_2^2+k_3^2-k_4^2)(k_2+k_3)(k_1+k_4)-2(k_2^2+k_2k_3\right.\\
&\left.+k_3^2)+(k_2+k_3)(k_1^3l_2+k_1^2(8\beta+k_4l_2))\right),
\end{aligned}
$$

here we consider $l_1=4b^2+\omega^2$ and $l_2=4\beta^2+\Omega^2$.

We assume that the point (q_1,q_2,q_3,q_4) satisfies system (3-52), then we can obtain the remaining parameters of discontinuous PWLC (3-49).

From equation e_1 of (3-52) we obtain $\omega^2 = S/T$, where

$$S = \frac{-4b(L_1-L_2)}{k_1+k_2-k_3-k_4}\big((bk_1+(bk_2+2))(k_3+k_4-L_1-L_2)k_1^2+k_1\left(-bk_3^3+k_2(bk_2+2)\right.$$
$$(k_3+k_4-L_1-L_2)-(k_3^2+k_3k_4+k_4^2)(bk_4+2)+bL_2^3+(L_1^2+L_1L_2+L_2^2)(bL_1+2)\big)$$
$$+k_2(bk_2+2)(k_3+k_4-L_1-L_2)+k_2\left((bL_1+2)(L_1^2+L_1L_2+L_2^2)-(k_3^2+k_3k_4+k_4^2)\right.$$
$$\left.(bk_4+2)-bk_3^3\right)+bk_2L_2^3+(L_2+L_1)\left(bk_3^3+(k_3+k_4)(k_4-L_1)(b(k_4+L_1)+2)+k_3^2\right.$$
$$(bk_4+2)\,)-L_2^2(k_3+k_4)((bL_1+2)-bL_2)\,),$$

and

$$T = \frac{L_1-L_2}{k_1+k_2-k_3-k_4}\left((k_1^3+k_1^2k_2+k_1k_2^2+k_2^3)(k_3+k_4-L_1-L_2)+(k_1+k_2)\left((L_1+L_2)\right.\right.$$
$$\left.(L_1^2+L_2^2)-(k_3+k_4)(k_3^2+k_4^2)\right)+(k_3+k_4)(L_1+L_2)\left(k_3^2+k_4^2-L_1^2-L_2^2\right)\big),$$

by the equation e_2 of (3-52) we obtain $\Omega^2 = V/W$, where

$$V = \frac{-4\beta(L_2-L_3)}{k_1-k_2-k_3+k_4}\big(k_1(\beta k_1+(\beta k_4+2))(k_2+k_3-L_2-L_3)-2k_1\left(k_2^2+k_2(k_3-k_4)\right.$$
$$\left.+(k_3-L_2)(k_3-k_4+L_2)+L_3(k_4-L_2)-L_3^2\right)+\beta k_1\left(-k_2^3-k_2^2k_3+k_2\left(k_4^2-k_3^2\right)\right.$$
$$\left.-k_3^3+k_3k_4^2+(L_2+L_3)\left(-k_4^2+L_2^2+L_3^2\right)\right)+\beta k_2^3(-k_4+L_2+L_3)-k_2^2(\beta k_3+2)$$
$$(k_4-L_2-L_3)-\beta k_2\left(k_3^2(k_4-L_2-L_3)-k_4^3+(L_2+L_3)(L_2^2+L_3^2)\right)-2k_2\left(L_2^2-k_4^2\right.$$
$$\left.+k_3(k_4-L_2-L_3)+L_2L_3+L_3^2\right)-(k_3-k_4)\left((\beta k_3^2+k_3(\beta k_4+2))(k_4-L_2-L_3)\right.$$
$$\left.+\beta(L_2+L_3)\left(-k_4^2+L_2^2+L_3^2\right)+2\left(-k_4(L_2+L_3)+L_2^2+L_2L_3+L_3^2\right)\right)\big),$$

and

$$W = \frac{L_2-L_3}{k_1-k_2-k_3+k_4}\big(k_1^2(k_2+k_3-L_2-L_3)(k_1+k_4)+k_1\left(k_2(k_4^2-k_3^2)-k_2^3-k_2^2k_3-k_3^3\right.$$
$$\left.+k_3k_4^2+(L_2+L_3)\left(-k_4^2+L_2^2+L_3^2\right)\right)+L_2\left((k_2+k_3)\left(k_2^2+k_3^2\right)-k_4^3\right)-k_4(k_2+k_3)$$
$$\left(k_2^2+k_3^2-k_4^2\right)+L_3\left(k_2^3+k_2^2k_3+k_2k_3^2-L_2^2(k_2+k_3-k_4)+k_3^3-k_4^3\right)-(k_2+k_3-k_4)$$
$$(L_2^3+L_2L_3^2+L_3^3)\,),$$

from equations e_3 and e_4 we get that $b=\beta=0$. This implies that the linear differential systems in R_P^1 and in R_P^2 are of the form

$$\dot{x} = \frac{1}{2}, \quad \dot{y} = x,$$

which is a contradiction because by hypothesis each of the linear differential systems considered is a center. Therefore we have proved that the maximum number of crossing limit cycles of the discontinuous PWLC in $\mathcal{F}_P$ is one.

In Proposition 3.7 we verify that this upper bound is reached. That is, that there are PWLC in the family $\mathcal{F}_P$ having one crossing limit cycle.

This completes the proof of Theorem C for the family $\mathcal{F}_P$. $\square$

Proof of Theorem C for the family $\mathcal{F}_E$. When the discontinuity curve Σ is of the type (E), we have following two regions in the plane:

$$\begin{aligned} R_E^1 = & \quad \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}, \\ R_E^2 = & \quad \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > 1\}. \end{aligned}$$

By Lemma 3.1 a PWLC of family $\mathcal{F}_E$ can be consider as (3-49) where the first integrals are given in (3-50).

Now we are going to study the conditions in order that a PWLC in the family $\mathcal{F}_E$ has crossing limit cycles intersecting the discontinuity curve (E) in exactly four points. Taking into account the first integrals (3-50) a PWLC in $\mathcal{F}_E$ has crossing limit cycles if there are points (x_i, y_i) for $i = 1,2,3,4$ satisfying the equations

$$\begin{aligned} e_1 : H_1(x_1,y_1) - H_1(x_2,y_2) = & \quad 0, \\ e_2 : H_2(x_2,y_2) - H_2(x_3,y_3) = & \quad 0, \\ e_3 : H_1(x_3,y_3) - H_1(x_4,y_4) = & \quad 0, \\ e_4 : H_2(x_4,y_4) - H_2(x_1,y_1) = & \quad 0, \\ E_1 : x_1^2 + y_1^2 - 1 = & \quad 0, \\ E_2 : x_2^2 + y_2^2 - 1 = & \quad 0, \\ E_3 : x_3^2 + y_3^2 - 1 = & \quad 0, \\ E_4 : x_4^2 + y_4^2 - 1 = & \quad 0, \end{aligned}$$

considering $l_1 = 4b^2 + \omega^2$ and $l_2 = 4\beta^2 + \Omega^2$, we have the equivalent system

$$\begin{aligned} e_1 : \; & 4(x_1^2 - x_2^2) + 8(bx_1y_1 - bx_2y_2 + c(x_1 - x_2) - dy_1 + dy_2) + l_1(y_1^2 - y_2^2) & = 0, \\ e_2 : \; & 4(x_2^2 - x_3^2) + 8x_2(\gamma + \beta y_2) - 8x_3(\gamma + \beta y_3) + (y_2 - y_3)(l_2(y_2 + y_3) - 8\delta) & = 0, \\ e_3 : \; & 4(x_3^2 - x_4^2) + 8(bx_3y_3 - bx_4y_4 + c(x_3 - x_4) - dy_3 + dy_4) + l_1(y_3^2 - y_4^2) & = 0, \\ e_4 : \; & 8\delta(y_1 - y_4) + 8x_4(\gamma + \beta y_4) - 4(x_1^2 - x_4^2) - 8x_1(\gamma + \beta y_1) + l_2(y_4^2 - y_1^2) & = 0, \\ & E_1 = 0, \; E_2 = 0, \; E_3 = 0, \; E_4 & = 0. \end{aligned} \tag{3-53}$$

Where we consider without generality $a = \alpha = 1$ as in the proof Theorem B.

We assume that this PWLC has at least two crossing periodic solutions. For this we have that system (3-53) has two pairs of solutions, (p_1, p_2, p_3, p_4) and (q_1, q_2, q_3, q_4) with $p_i \neq p_j$, and $q_i \neq q_j$, for $i \neq j$ and $i, j = 1,2,3,4$. Since these solution points are on the circle (E), then we can consider them in the following way

$$p_i = (k_i, \lambda_i), \text{ with } k_i = \cos(s_i),\ \lambda_i = \sin(s_i) \text{ and } q_i = (m_i, n_i), \text{ with } m_i = \cos(t_i),$$
$$n_i = \sin(t_i), \quad \text{with } s_i, t_i \in [0, 2\pi), \text{ for } i = 1,2,3,4. \tag{3-54}$$

Substituting the first solution (p_1, p_2, p_3, p_4) with p_i as in (3-54) in (3-53) we can determine the parameters d, δ, c, γ of the PWLC (3-49), and we get

$$d = \frac{(8c(k_1-k_2)+4(k_1-k_2+b\lambda_1-b\lambda_2)(k_1+k_2+b(\lambda_1+\lambda_2))+(\lambda_1^2-\lambda_2^2)\omega^2)}{8(\lambda_1-\lambda_2)},$$
$$\delta = \frac{4k_2^2-4k_3^2+8k_2(\lambda_2\beta+\gamma)-8k_3(\lambda_3\beta+\gamma)+(\lambda_2^2-\lambda_3^2)l_2}{8(\lambda_2-\lambda_3)},$$
$$c = \frac{1}{8((k_3-k_4)(\lambda_1-\lambda_2)-(k_1-k_2)(\lambda_3-\lambda_4))}\left((4(\lambda_1-\lambda_2)(k_4^2-k_3^2-2b\lambda_3k_3+2bk_4\lambda_4)\right.$$
$$\left.+(\lambda_3-\lambda_4)(4(k_1^2-k_2^2)+8b(k_1\lambda_1-k_2\lambda_2)+(\lambda_1-\lambda_2)(\lambda_1+\lambda_2-\lambda_3-l_4)\,l_1)\right),$$
$$\gamma = \frac{1}{8(-(k_1-k_4)(\lambda_2-\lambda_3)+(k_2-k_3)(\lambda_1-\lambda_4))}\left(4(\lambda_1-\lambda_4)(k_3^2-k_2^2+2k_3\lambda_3\beta-2k_2\lambda_2)\right.$$
$$\left.+(\lambda_2-\lambda_3)(4k_1^2-4k_4^2+8k_1\lambda_1\beta-8k_4\lambda_4\beta+(\lambda_1-\lambda_4)(\lambda_1-\lambda_2-\lambda_3+\lambda_4)\,l_2)\right).$$

Analogously, substituting the second solution (q_1, q_2, q_3, q_4) with q_i as in (3-54) in (3-53) we get remaining parameters ω, Ω, b, β. Substituting k_i, λ_i, m_i, n_i as was done in (3-54) we obtain that $b = \beta = 0$. Therefore we get that the PWLC is formed by linear differential center $\dot{x} = -y, \quad \dot{y} = x$, in the regions R_E^1 and R_E^2. This is a contradiction because with this linear differential center is not possible to generate crossing limit cycles. Then we proved that the maximum number of crossing limits cycles for PWLC in $\mathcal{F}_E$ is one.

In Proposition 3.7 we prove that the maximum number provided in Theorem C is reached, that is, there are PWLC in $\mathcal{F}_E$ such that have one crossing limit cycle with four points on (E).

This completes the proof of Theorem *C* for the family $\mathcal{F}_E$. □

Proof of Theorem C for the family $\mathcal{F}_H$. If the discontinuity curve Σ is of the type (H) we have following three regions in the plane given in (3-5):

$$\begin{aligned} R_H^1 &= \{(x,y)\in\mathbb{R}^2 : x^2-y^2>1, x>0\},\\ R_H^2 &= \{(x,y)\in\mathbb{R}^2 : x^2-y^2<1\},\\ R_H^3 &= \{(x,y)\in\mathbb{R}^2 : x^2-y^2>1, x<0\}. \end{aligned}$$

We consider a planar discontinuous PWLC formed by three linear arbitrary centers. By Lemma 3.1 these linear differential centers can be as follows

$$\begin{aligned}
\dot{x} &= -b_1x - \frac{4b_1^2+\omega_1^2}{4a_1}y + d_1, & \dot{y} &= a_1x + b_1y + c_1, \ \text{ in } \ R_H^1,\\
\dot{x} &= -b_2x - \frac{4b_2^2+\omega_2^2}{4a_2}y + d_2, & \dot{y} &= a_2x + b_2y + c_2, \ \text{ in } \ R_H^2,\\
\dot{x} &= -b_3x - \frac{4b_3^2+\omega_3^2}{4a_3}y + d_3, & \dot{y} &= a_3x + b_3y + c_3, \ \text{ in } \ R_H^3.
\end{aligned} \tag{3-55}$$

These linear differential centers have the first integrals

$$\begin{aligned}
H_1(x,y) &= 4(a_1x+b_1y)^2 + 8a_1(c_1x-d_1y) + y^2\omega_1^2,\\
H_2(x,y) &= 4(a_2x+b_2y)^2 + 8a_2(c_2x-d_2y) + y^2\omega_2^2,\\
H_3(x,y) &= 4(a_3x+b_3y)^2 + 8a_3(c_3x-d_3y) + y^2\omega_3^2,
\end{aligned} \tag{3-56}$$

respectively.

In order to have a crossing limit cycle, which intersects the discontinuity curve (H) in four different points $p_i = (x_i, y_i)$, $i = 1,2,3,4$, these points must satisfy the following equations

$$\begin{aligned}
e_1 : H_1(x_1,y_1) - H_1(x_2,y_2) = &\ 0,\\
e_2 : H_2(x_2,y_2) - H_2(x_3,y_3) = &\ 0,\\
e_3 : H_3(x_3,y_3) - H_3(x_4,y_4) = &\ 0,\\
e_4 : H_2(x_4,y_4) - H_2(x_1,y_1) = &\ 0,\\
E_1 : x_1^2 - y_1^2 - 1 = &\ 0,\\
E_2 : x_2^2 - y_2^2 - 1 = &\ 0,\\
E_3 : x_3^2 - y_3^2 - 1 = &\ 0,\\
E_4 : x_4^2 - y_4^2 - 1 = &\ 0,
\end{aligned} \tag{3-57}$$

equivalently, we have

$$\begin{aligned}
e_1 : 4(x_1^2-x_2^2) + 8(c_1(x_1-x_2) - d_1y_1 + b_1x_1y_1 + d_1y_2 - b_1x_2y_2) + (y_1^2-y_2^2)l_1 &= 0,\\
e_2 : 4(x_2^2-x_3^2) + 8(c_2(x_2-x_3) - d_2y_2 + b_2x_2y_2 + d_2y_3 - b_2x_3y_3) + (y_2^2-y_3^2)l_2 &= 0,\\
e_3 : 4(x_3^2-x_4^2) + 8(c_3(x_3-x_4) - d_3y_3 + b_3x_3y_3 + d_3y_4 - b_3x_4y_4) + (y_3^2-y_4^2)l_3 &= 0,\\
e_4 : 4(x_4^2-x_1^2) + 8(c_2(x_4-x_1) - d_2y_4 - b_2x_1y_1 + d_2y_1 + b_4x_4y_4) + (y_4^2-y_1^2)l_2 &= 0,\\
E_1 = 0,\ E_2 = 0,\ E_3 = 0,\ E_4 &= 0,
\end{aligned} \tag{3-58}$$

where $l_i = 4b_i^2 + \omega_i^2$, for $i = 1,2,3$.

Here we are taking without generality $a_1 = a_2 = a_3 = 1$ as in the proofs of the previous theorems.

We assume that the discontinuous PWLC formed by the three linear differential centers in (3-55) has at least two crossing periodic solutions. For this we must impose that the system of equations (3-58) has two real solution, namely (p_1, p_2, p_3, p_4) and (q_1, q_2, q_3, q_4). Since these solutions provide crossing periodic solutions and these points are the points where the crossing periodic solutions intersect the hyperbola (H) we can consider

$$\begin{aligned} & p_i = (k_i, \lambda_i) = (\cosh(r_i), \sinh(r_i)) \quad \text{and} \quad q_i = (m_i, n_i) = (\cosh(s_i), \sinh(s_i)), \\ & \text{with } \ r_i, s_i \in \mathbb{R} \quad \text{for } \ i = 1,2,3,4. \end{aligned} \tag{3-59}$$

Now we assume that the point (p_1, p_2, p_3, p_4) with $p_i = (k_i, \lambda_i)$, $i = 1,2,3,4$ satisfies system (3-58), and then we obtain the following expressions

$$\begin{aligned} d_i = & \frac{1}{8(\lambda_i - \lambda_{i+1})} \left(c_i(k_i - k_{i+1}) + 4(k_i - k_{i+1} + b_i(\lambda_i - \lambda_{i+1}))(k_i + k_{i+1} + b_i(\lambda_i + \lambda_{i+1})) \right. \\ & \left. + (\lambda_i^2 - \lambda_{i+1}^2)\omega_i^2 \right), \end{aligned}$$

for $i = 1,2,3$, and

$$\begin{aligned} c_2 = & \frac{1}{8((k_2 - k_3)(\lambda_1 - \lambda_4) - (k_1 - k_4)(\lambda_2 - \lambda_3))} \left(4(k_3^2 + 2b_2k_3\lambda_3 - k_2^2 - 2b_2k_2\lambda_2) \right. \\ & (\lambda_4 - \lambda_1) + (\lambda_2 - \lambda_3)(4(k_1^2 - k_4^2) + 8b_2(k_1\lambda_1 - k_4\lambda_4) + (\lambda_1 - \lambda_4) \\ & \left. (\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4)\, l_2) \right). \end{aligned}$$

We assume that the point (q_1, q_2, q_3, q_4) with q_i as in (3-59) satisfies system (3-58). By the first equation in (3-58) we get that

$$\begin{aligned} c_1 = & \frac{1}{8((\lambda_1 - \lambda_2)(m_1 - m_2) - (k_1 - k_2)(n_1 - n_2))} \left((\lambda_1 - \lambda_2)(-4(m_1^2 - m_2^2) - 8b_1m_1n_1 \right. \\ & + \frac{4(k_1 - k_2 + b_1(\lambda_1 - \lambda_2))(k_1 + k_2 + b_1(\lambda_1 + \lambda_2))(n_1 - n_2)}{\lambda_1 - \lambda_2} + 8b_1m_2n_2 + (\lambda_1 + \lambda_2) \\ & \left. \omega_1^2(n_1 - n_2) - (n_1^2 - n_2^2)l_1) \right). \end{aligned}$$

By the second equation of (3-58) we obtain that $\omega_2^2 = K/S$ where

$$\begin{aligned} K = & \frac{4}{(k_1 - k_4)(\lambda_2 - \lambda_3) - (k_2 - k_3)(\lambda_1 - \lambda_4)} \left((k_3^2 - k_2^2)((\lambda_1 - \lambda_4)\psi_2 - (k_1 - k_4)\psi_1) + (\lambda_2 \right. \\ & - \lambda_3)((k_1^2 + b_2^2(\lambda_1 - \lambda_4)(\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4) - k_4^2)\psi_2 + k_4 \left(\psi_2 - b_2^2(\lambda_2 + \lambda_3 - n_2 - n_3) \right. \\ & \psi_1 + 2b_2(-\lambda_4\psi_2 + m_2n_2 - m_3n_3)) + k_1(-\psi_2 + b_2^2(\lambda_2 + \lambda_3 - n_2 - n_3)\psi_1 + 2b_2(\lambda_1\psi_2 \\ & - m_2n_2 + m_3n_3))) + k_2((\lambda_1 - \lambda_4)(\psi_2 - b_2^2(\lambda_1 + \lambda_4 - n_2 - n_3)\psi_1) - (k_1^2 - k_4^2)\psi_1 \\ & - 2b_2(\lambda_2(-\lambda_4\psi_2 - (k_1 - k_4)\psi_1) + \lambda_4((m_2 - k_4)n_2 + (k_4 - m_3)n_3)\lambda_1(\lambda_2\psi_2 \end{aligned}$$

$$
\begin{aligned}
&+(k_1-m_2)n_2+(m_3-k_1)n_3)))+k_3((\lambda_4-\lambda_1)\psi_2+(k_1^2-k_4^2)\psi_1+b_2^2(\lambda_1-\lambda_4)\\
&(\lambda_1+\lambda_4-n_2-n_3)\psi_1+2b_2(\lambda_3(-\lambda_4\psi_2-(k_1-k_4)\psi_1)+\lambda_4(m_2n_2-k_4\psi_1-m_3n_3)+\\
&\lambda_1(\lambda_3\psi_2+k_1\psi_1-m_2n_2+m_3n_3)),\\
S=&\frac{1}{(k_1-k_4)(\lambda_2-\lambda_3)-(k_2-k_3)(\lambda_1-\lambda_4)}(\lambda_3^2\lambda_4\psi_2+\lambda_3\lambda_4^2\psi_2+(k_1-k_4)(\lambda_3^2n_2-\lambda_3n_2^2)\\
&+(k_3-k_2)(\lambda_4^2n_2-\lambda_4n_2^2)+\lambda_1^2((k_2-k_3)\psi_1-(\lambda_2-\lambda_3)\psi_2)+\lambda_2^2(\lambda_4\psi_2+(k_1-k_4)\psi_1)\\
&+((k_4-k_1)\lambda_3^2+(k_2-k_3)\lambda_4^2)n_3+(k_1\lambda_3-k_4\lambda_3-k_2\lambda_4+k_3\lambda_4)n_3^2+\lambda_1((\lambda_2^2-\lambda_3^2)\psi_2\\
&-(k_2-k_3)\psi_1(n_2+n_3))+\lambda_2(\lambda_4^2\psi_2+(k_1-k_4)\psi_1(n_2+n_3)))),
\end{aligned}
$$

by the third equation we get that

$$
\begin{aligned}
c_3=&\frac{-1}{8(\lambda_4\psi_4-\lambda_3\psi_2+(k_3-k_4)\psi_1))}(((\lambda_3-\lambda_4)(4(m_4^2-m_3^2)+8b_3(m_4n_4-m_3n_3)\\
&+\frac{4(k_3-k_4+b_3(\lambda_3-\lambda_4))(k_3+k_4+b_3(\lambda_3+\lambda_4))\psi_1}{\lambda_3-\lambda_4}+(\lambda_3+\lambda_4)\omega_3^2\psi_1-\\
&(n_3^2-n_4^2)l_3)),
\end{aligned}
$$

where $\psi_1=n_2-n_3, \psi_2=m_2-m_3, \psi_3=n_3-n_4$ and $\psi_4=m_3-m_4$.

Finally by the fourth equation in (3-58) we have that $b_2=0$. With these expressions for d_2, c_2, ω_2 and b_2 we obtain that the linear differential system in the region R_H^2 is $\dot{x}=y,\ \dot{y}=x$, which is a linear differential system type saddle. This is a contradiction because we are working with centers in each region R_H^i for $i=1,2,3$. Therefore we have proved that the maximum number of crossing limit cycles for systems in $\mathcal{F}_H$ is one.

In Proposition 3.7 we prove that there are PWLC in $\mathcal{F}_H$ such that have one crossing limit cycle.

This completes the proof of Theorem *C* for the family $\mathcal{F}_H$. □

In what follows we prove Theorem D.

Proof of statement (i) ***of Theorem*** ***D*****.** We consider a planar discontinuous PWLC with four zones separated by (LV) and formed by four arbitrary linear centers in each region R_{LV}^i. By Lemma 3.1 this PWLC can be as follows

$$
\begin{aligned}
&\dot{x}=-b_1x-\frac{4b_1^2+\omega_1^2}{4a_1}y+d_1, && \dot{y}=a_1x+b_1y+c_1,\ \text{in}\ R_{LV}^1,\\
&\dot{x}=-b_2x-\frac{4b_2^2+\omega_2^2}{4a_2}y+d_2, && \dot{y}=a_2x+b_2y+c_2,\ \text{in}\ R_{LV}^2,\\
&\dot{x}=-b_3x-\frac{4b_3^2+\omega_3^2}{4a_3}y+d_3, && \dot{y}=a_3x+b_3y+c_3,\ \text{in}\ R_{LV}^3,\\
&\dot{x}=-b_4x-\frac{4b_4^2+\omega_4^2}{4a_4}y+d_4, && \dot{y}=a_4x+b_4y+c_4,\ \text{in}\ R_{LV}^4,
\end{aligned}
\tag{3-60}
$$

with $a_i \neq 0$ and $\omega_i > 0$ for $i = 1,2,3,4$. The regions R^i_{LV} for $i = 1,2,3,4$ are defined just before the statement of Theorem D in (3-22). These linear differential centers have the first integrals H_1, H_2, H_3 and H_4 respectively, where

$$H_i(x,y) = 4(a_i x + b_i y)^2 + 8a_i(c_i x - d_i y) + y^2\omega_i^2, \text{ for } i = 1,2,3,4. \tag{3-61}$$

If the discontinuous PWLC (3-60) has two crossing limit cycles of type 1, these two crossing limit cycles should be some of Figure 3.12. We observe that the cases of Figures

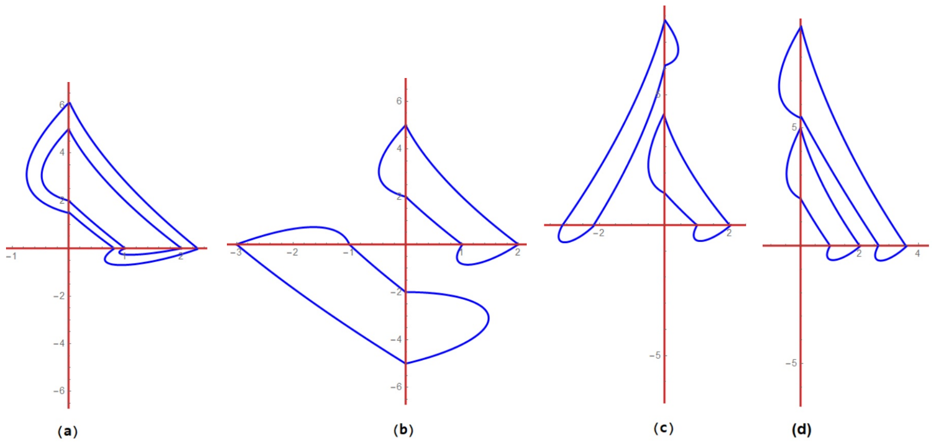

Figure 3.12: *Possible cases of two crossing limit cycles of type* 1 *of discontinuous PWLC (3-60).*

3.12(b), 3.12(c) and 3.12(d) are not possible because in these cases the pieces of the ellipses of linear differential centers in the regions R^4_{LV}, R^1_{LV} and R^2_{LV}, respectively would not be nested which contradicts that the linear differential systems in each of these regions are linear centers. Therefore if the discontinuous PWLC (3-60) has two crossing limit cycles of type 1 these could be as in Figure 3.12(a).

Now we going to study the conditions in order that the PWLC (3-60) has crossing limit cycles of type 1 and we will show that the maximum number of crossing limits cycles of type 1 is one. Without loss of generality we assume that the crossing limit cycles intersect the branches Γ_1^+ and Γ_2^+ in the points $(0,y_1),(0,y_2)$ and $(x_1,0),(x_2,0)$, respectively, where $0 < y_1 < y_2$ and $0 < x_1 < x_2$. Then taking into account the first integrals (3-61) for each linear center, these points must satisfy the following equations

$$\begin{aligned} H_1(x_2,0) &= H_1(0,y_2), \\ H_2(0,y_2) &= H_2(0,y_1), \\ H_1(0,y_1) &= H_1(x_1,0), \\ H_4(x_1,0) &= H_4(x_2,0), \end{aligned}$$

equivalently we have

$$
\begin{aligned}
4a_1^2x_2^2+8a_1(c_1x_2+d_1y_2)-y_2^2l_1 &= 0,\\
-(y_1-y_2)(-8a_2d_2+(y_1+y_2)l_2) &= 0,\\
-4a_1^2x_1^2-8a_1(c_1x_1+d_1y_1)+y_1^2l_1 &= 0,\\
4a_4(x_1-x_2)(2c_4+a_4(x_1+x_2)) &= 0,
\end{aligned}
\tag{3-62}
$$

where $l_1=4b_1^2+\omega_1^2, l_2=4b_2^2+\omega_2^2$ and $\eta=(a_4c_1-a_1c_4)$.

Moreover, by hypothesis $x_1<x_2$ and $y_1<y_2$, then from the second and the fourth equations of (3-62), we have

$$
y_1=\frac{8a_2d_2-l_2y_2}{l_2},\quad x_2=-\frac{2c_4+a_4x_1}{a_4}.
$$

Substituting these expressions of y_1 and x_2 in the first and third equations of (3-62) we obtain the two equations

$$
\begin{aligned}
E_1 &= \frac{4a_1^2(2c_4+a_4x_1)^2-8a_1a_4(2c_1c_4+a_4c_1x_1-a_4d_1y_2)-a_4^2y_2^2l_1}{a_4^2},\\
E_2 &= 4a_1^2x_1^2-\frac{l_1(y_2l_2-8a_2d_2)^2}{l_2^2}-8a_1\left(d_1y_2-\frac{8a_2d_1d_2}{l_2}-c_1x_1\right).
\end{aligned}
$$

Doing the Groebner basis of the two polynomials E_1 and E_2 with respect to the variables x_1 and y_2 we get the two equations

$$
\alpha_0+\alpha_1y_2+\alpha_2y_2^2=0,\qquad \beta_0+\beta_1x_1+\beta_2y_2=0,
\tag{3-63}
$$

where

$$
\begin{aligned}
\alpha_0 =& 4\Bigg(\frac{a_1c_4\eta^2(-2a_4c_1+a_1c_4)}{a_4^2}+\frac{16a_2^3a_4^2d_2^3l_1(a_2d_2l_1-2a_1d_1l_1l_2)}{l_2^4}+\big(8a_2d_2(a_1\eta^2d_1l_2\\
&+a_2d_2(2a_1^2a_4^2d_1^2-\eta^2l_1))\big)\frac{1}{l_2^2}\Bigg),\\
\alpha_1 =& \frac{8a_2d_2}{l_2^3}\left(-32a_2^2a_4^2b_1^2d_2^2(2b_1^2+\omega_1^2)-4a_2^2a_4^2d_2^2\omega_1^4+8a_1a_2a_4^2d_1d_2l_1l_2+a_4^2c_1^2l_2^2l_1^2\right.\\
&\left.-2a_1a_4c_1c_4l_1l_2^2+a_1^2(-4a_4^2d_1^2+c_4^2l_1l_2)\right),\\
\alpha_2 =& 2a_1a_4c_1c_4l_1-a_1^2c_4^2l_1+a_4^2\left(-c_1^2l_1+4a_1^2d_1^2+\frac{4a_2^2d_2^2l_1^2}{l_2^2}-\frac{8a_1a_2d_1d_2l_1}{l_2}\right),\\
\beta_0 =& -\frac{a_1c_4\eta}{a_4}+\frac{4a_2^2a_4d_2^2l_1}{l_2^2}-\frac{4a_1a_2a_4d_1d_2}{l_2},\\
\beta_1 =& -a_1\eta,\qquad \beta_2=a_1a_4d_1-\frac{a_2a_4d_2l_1}{l_2}.
\end{aligned}
$$

The Bézout's Theorem 1.11 applied to system (3-63) says that this system has at most two isolated solutions. Therefore system (3-62) has two solutions which are of the form $(x_1^1, x_2^1, y_1^1, y_2^1)$ and $(x_1^2, x_2^2, y_1^2, y_2^2)$, but it is possible to prove that if (x_1, x_2, y_1, y_2) is a solution of system (3-62), then (x_2, x_1, y_2, y_1) is also a solution of this system. Since we must have that $x_1 < x_2$ and $y_1 < y_2$, then system (3-62) has a unique solution, and therefore the discontinuous PWLC (3-60) that belongs to the family $\mathcal{F}_{LV}$ can have at most one crossing limit cycle of type 1 intersecting Γ_1^+ and Γ_2^+.

In Proposition 3.8 we verify that this upper bound is reached. That is, that there are PWLC in the family $\mathcal{F}_{LV}$ having one crossing limit cycle of type 1. This completes the proof of statement (i) of Theorem *D*. □

Proof of statement (ii) ***of Theorem D.*** We consider the following discontinuous PWLC

$$\begin{aligned}
&\dot{x} = \frac{19763}{19980} - \frac{13}{25}x - \frac{427}{999}y, && \dot{y} = -\frac{9751}{6660} + x + \frac{13}{25}y, && \text{in } R_{LV}^1,\\
&\dot{x} = \frac{78049}{19680} + x - \frac{1397}{984}y, && \dot{y} = \frac{682}{205} + x - y, && \text{in } R_{LV}^2,\\
&\dot{x} = \frac{108179}{38400} - \frac{4}{5}x - \frac{1367}{1920}y, && \dot{y} = \frac{2137}{640} + x + \frac{4}{5}y, && \text{in } R_{LV}^3,\\
&\dot{x} = -\frac{5539}{8220} + \frac{2}{5}x - \frac{91}{411}y, && \dot{y} = -\frac{3743}{2740} + x - \frac{2}{5}y, && \text{in } R_{LV}^4.
\end{aligned} \tag{3-64}$$

The linear differential centers in (3-64) have the first integrals

$$\begin{aligned}
H_1(x,y) &= 15x(-9751 + 3330x) + (-98815 + 51948x)y + 21350y^2,\\
H_2(x,y) &= 4x^2 + x\left(\frac{5456}{205} - 8y\right) + \frac{y(-78049 + 13970y)}{2460},\\
H_3(x,y) &= 4x^2 + \frac{x}{80}(2137 + 512y) + \frac{y(108179 + 13670y)}{4800},\\
H_4(x,y) &= 4110x^2 + y(5539 + 910y) - 3x(3743 + 1096y),
\end{aligned}$$

in R_{LV}^i, $i = 1,2,3,4$, respectively. This discontinuous PWLC has four crossing limit cycles of type 2, because the unique real solutions $(p_1^i, q_1^i, p_2^i, q_2^i)$, with $i = 1,2,3,4$ of system (3-35) are $p_1^1 = (3/2, 0)$, $q_1^1 = (0, 29/10)$, $p_2^1 = (-17/5, 0)$ and $q_2^1 = (0, -4)$; $p_1^2 = (1.529206.., 0)$, $q_1^2 = (0, 2.905859..)$, $p_2^2 = (-3.411537.., 0)$ and $q_2^2 = (0, -4.020269..)$; $p_1^3 = (17/10, 0)$, $q_1^3 = (0, 3)$, $p_2^3 = (-71/20, 0)$ and $q_2^3 = (0, -21/5)$ and $p_1^4 = (19/10, 0)$, $q_1^4 = (0, 16/5)$, $p_2^4 = (-19/5, 0)$ and $q_2^4 = (0, -9/2)$. See these four crossing limit cycles of type 2 in Figure 3.13. This completes the proof of statement (ii) of Theorem *D*. □

In Theorem *C* and Proposition 3.7 we get to provided the maximum number of crossing limit cycles which is reached for PWLC when the discontinuity curve is a conic of the type either $(PL), (P), (E)$ or (H). We observed that when the discontinuity curve

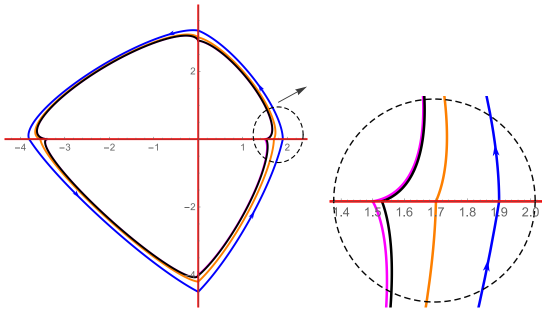

Figure 3.13: *The four crossing limit cycles of type* 2 *of the discontinuous PWLC* (3-64).

is of the conic (LV) we obtain two types of crossing limits with four point on (LV), and in Theorem *D* and Proposition 3.8 we get proved that the maximum number of crossing limit of type 1 is one and this upper bound is reached. With regard to the crossing limit cycles of type 2 we get to provided a lower bound for the maximum number, illustrated in Proposition 3.9 which is equal to four.

Now we study the case where the PWLC have crossing limit cycles that intersect the discontinuity curve in two and four points simultaneously.

3.4 Crossing limit cycles with four and with two points on the discontinuity curve Σ simultaneously

In this section we study the maximum number of crossing limit cycles of planar discontinuous PWLC that intersect the discontinuity curve Σ in two and in four points simultaneously.

3.4.1 Statement of the main results

We do not consider planar discontinuous PWLC with discontinuity curve a conic of type (DL), (PL) and (LV) because as in the proof of Theorem 3.3 they do not have crossing limit cycles that intersect the discontinuity curve in two points. Then we study the maximum number of crossing limit cycles with two and with four points in Σ simultaneously by the families $\mathcal{F}_P$, $\mathcal{F}_E$ and $\mathcal{F}_H$.

Theorem E *The following statements hold.*

(a) The planar discontinuous PWLC that belong to the family $\mathcal{F}_P$, can have simultaneous one crossing limit cycle that intersects (P) in two points and one crossing limit cycle that intersects (P) in four points.

(b) *The planar discontinuous PWLC that belong to the family $\mathcal{F}_E$, can have simultaneous one crossing limit cycle that intersects (E) in two points and one crossing limit cycle that intersects (E) in four points.*

(c) *The planar discontinuous PWLC that belong to the family $\mathcal{F}_H$, can have simultaneous one crossing limit cycle that intersects (H) in two points and one crossing limit cycle that intersects (H) in four points.*

Theorem E is proved in Section 3.4.2.

Proposition 3.10 *The upper bounds for the maximum number of crossing limit cycles provided in Theorem E are reached. See Figures 3.14, 3.15 and 3.16.*

3.4.2 Proof of the main results

In this subsection we prove the Proposition 3.10 and Theorem *E*.

Proof of Proposition 3.10 for the family $\mathcal{F}_P$. We verify that the upper bound provided in statement (a) of Theorem E is reached, that is there are systems in $\mathcal{F}_P$ with one crossing limit cycle with four points on (P) and one crossing limit cycle with two points on (P) simultaneously. We consider the discontinuous PWLC formed by the linear centers

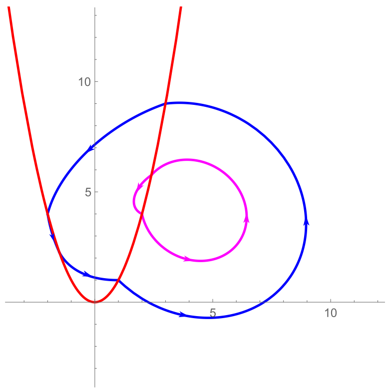

Figure 3.14: *The two crossing limit cycles of the discontinuous PWLC formed by the centers (3-65) and (3-66).*

$$\dot{x} = \frac{1225}{229} + \frac{x}{2} - \frac{310}{229}y, \qquad \dot{y} = -\frac{103}{229} + x - \frac{y}{2}, \quad \text{in } R_P^1, \tag{3-65}$$

$$\dot{x} = \frac{6411}{1424} - \frac{x}{8} - \frac{85}{89}y, \qquad \dot{y} = -\frac{3359}{712} + x + \frac{y}{8}, \quad \text{in } R_P^2. \tag{3-66}$$

These linear differential centers have the first integrals

$$H_1(x,y) = 229x^2 + 10y(-245 + 31y) - x(206 + 229y),$$
$$H_2(x,y) = 4x^2 + x\left(-\frac{3359}{89} + y\right) + \frac{y}{178}(-6411 + 680y),$$

respectively. The unique real solution of systems (3-52) and (3-72) is $(x_1,x_2,x_3,x_4,x_5,x_6) = (3, -2,-3/2, 1,2,12/5)$, therefore we have one crossing limit cycle that intersects (P) in the points $(3,9)$, $(-2,4)$, $(-3/2,9/2)$ and $(1,1)$, and one crossing limit cycle that intersects (P) in the points $(2,4)$ and $(12/5,144/25)$. See these crossing limit cycles in Figure 3.14.

This completes the proof of Proposition 3.10 for the family $\mathcal{F}_P$. □

Proof of Proposition 3.10 for the family $\mathcal{F}_E$. We verify that the upper bound provided in statement (b) of Theorem E is reached, that is there are systems in $\mathcal{F}_E$ with one crossing limit cycle with four points on (E) and one crossing limit cycle with two points on (E) simultaneously. We consider the discontinuous PWLC in $\mathcal{F}_E$ formed by the linear centers

$$
\begin{aligned}
\dot{x} &= -\frac{(-6+3\sqrt{2}+\sqrt{6}+(6-4\sqrt{2}-6\sqrt{3})x+8(-1+\sqrt{2}+2\sqrt{3})y}{4(-3+2\sqrt{2}+3\sqrt{3})},\\
\dot{y} &= -\frac{-4+3\sqrt{2}+2\sqrt{3}+\sqrt{6}}{2(-6+4\sqrt{2}+6\sqrt{3})}+x-\frac{y}{2}, \quad \text{in } R_E^1,
\end{aligned} \tag{3-67}
$$

$$
\begin{aligned}
\dot{x} &= -\frac{18-93\sqrt{2}+4\sqrt{3}+33\sqrt{6}-230(1+\sqrt{3})x+4(335-2\sqrt{2}+261\sqrt{3}+20\sqrt{6})y}{920(1+\sqrt{3})},\\
\dot{y} &= x+\frac{1}{920}\left(9+34\sqrt{2}-67\sqrt{3}-41\sqrt{6}-230y\right), \quad \text{in } R_E^2.
\end{aligned} \tag{3-68}
$$

The unique real solution of systems (3-53) and (3-73) in this case is $(p_1,p_2,p_3,p_4,p_5,p_6)$

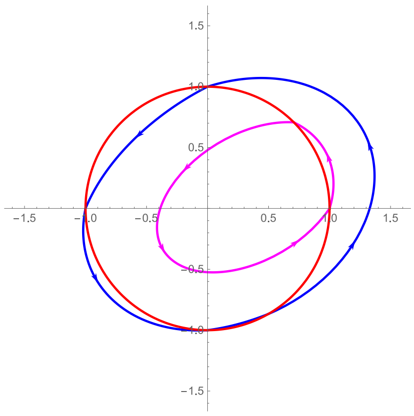

Figure 3.15: *The two limit cycles of the discontinuous PWLC formed by the centers (3-67) and (3-68).*

with $p_1 = (\cos(\pi/2),\sin(\pi/2))$, $p_2 = (\cos(\pi),\sin(\pi))$, $p_3 = (\cos(3\pi/2),\sin(3\pi/2))$, $p_4 = (\cos(-\pi/3), \sin(-\pi/3))$, $p_5 = (\cos(\pi/4),\sin(\pi/4))$ and $p_6 = (\cos(0),\sin(0))$. See these crossing limit cycles in Figure 3.15.

This completes the proof of Proposition 3.10 for the family $\mathcal{F}_E$. □

Proof of Proposition 3.10 for the family $\mathcal{F}_H$. We verify that the upper bound provided in statement (c) of Theorem E is reached. We consider the PWLC formed by the linear centers

$$\dot{x} = -\frac{-1215-576\sqrt{2}+256\sqrt{7}+112\sqrt{13}-384\sqrt{15}}{192(2\sqrt{7}+\sqrt{13}-\sqrt{23+4\sqrt{30}})}+\frac{x}{2}+\frac{29}{16}y,$$
$$\dot{y} = K_1 - x - \frac{y}{2}, \quad \text{in } R_H^1, \tag{3-69}$$
$$\dot{x} = \frac{1125+432\sqrt{14}+189\sqrt{26}+207\sqrt{30}+160\sqrt{105}+70\sqrt{195}}{6(87+108\sqrt{14}+54\sqrt{26}+16\sqrt{30}+40\sqrt{105}+20\sqrt{195})}-\frac{x}{2}-K_2 y,$$
$$\dot{y} = -\frac{855\sqrt{2}+3516\sqrt{7}+1797\sqrt{13}+315\sqrt{15}+644\sqrt{210}+329\sqrt{390}}{6(87+108\sqrt{14}+54\sqrt{26}+16\sqrt{30}+40\sqrt{105}+20\sqrt{195})}+x+\frac{y}{2}, \quad \text{in } R_H^2, \tag{3-70}$$

$$\dot{x} = -\frac{9}{2}+\frac{73}{8\sqrt{2}}-\frac{3}{2}x-\frac{45}{16}y, \qquad \dot{y} = -\frac{3}{2}+x+\frac{3}{2}y, \text{ in } R_H^3. \tag{3-71}$$

Where

$$K_1 = \frac{1}{48\left(-1+2\sqrt{\frac{7}{s_1}}+\sqrt{\frac{13}{s_1}}\right)}\left(-\frac{675}{4}-64\sqrt{7}-28\sqrt{13}+288\sqrt{\frac{2}{7}(23-4\sqrt{30})}\right.$$
$$\left.+945\sqrt{\frac{7}{s_1}}+\frac{945}{2}\sqrt{\frac{13}{s_1}}+144\sqrt{\frac{26}{s_1}}+192\sqrt{\frac{105}{s_1}}+96\sqrt{\frac{195}{s_1}}\right),$$
$$K_2 = \frac{1}{42+48\sqrt{14}+24\sqrt{26}+9\sqrt{30}+24\sqrt{105}+12\sqrt{195}}\left(42-54\sqrt{2}-336\sqrt{7}\right.$$
$$-174\sqrt{13}+48\sqrt{14}-24\sqrt{15}+24\sqrt{26}+9\sqrt{30}+24\sqrt{105}+12\sqrt{195}$$
$$\left.-68\sqrt{210}-35\sqrt{390}\right),$$

with $s_1 = 23+4\sqrt{30}$. The unique real solution of systems (3-57) and (3-74) in this case is $(p_1,p_2,p_3,p_4,p_5,p_6)$ with $p_1 = (3,-\sqrt{8})$, $p_2 = (4,\sqrt{15})$, $p_3 = (-3,\sqrt{8})$, $p_4 = (-1,0)$, $p_5 = (7/6,-\sqrt{13}/6)$ and $p_6 = (4/3,\sqrt{7}/3)$. See these crossing limit cycles in Figure 3.16.

This completes the proof of Proposition 3.10 for the family $\mathcal{F}_H$. □

In what follows we prove Theorem E.

Proof of statement (a) of Theorem E In this case we use the notations given in the proof of Theorem C for the family $\mathcal{F}_P$, then we consider the planar discontinuous PWLC (3-49) and the first integrals (3-50). In order that the discontinuous PWLC (3-49) has crossing limit cycles with four points, namely $(x_1,x_1^2),(x_2,x_2^2),(x_3,x_3^2),(x_4,x_4^2)$ and one crossing limit cycle with two points, namely $(x_5,x_5^2),(x_6,x_6^2)$ on (P), we must study the solutions

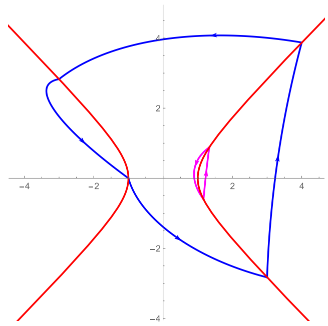

Figure 3.16: *The two limit cycles of the discontinuous PWLC formed by the centers (3-69), (3-70) and (3-71).*

$(x_1,x_2,x_3,x_4,x_5,x_6)$ of system (3-51) and the equations

$$e_5 : H_1(x_5,x_5^2)-H_1(x_6,x_6^2)=0,$$
$$e_6 : H_2(x_6,x_6^2)-H_2(x_5,x_5^2)=0,$$

or equivalently systems (3-52) and

$$\begin{aligned} e_5 &: 4x_5^2(1+bx_5)^2+\ 8x_5(c-dx_5)-4x_6^2(1+bx_6)^2+8x_6(dx_6-c)+\ (x_5^4-x_6^4)\omega^2=0,\\ e_6 &: 4x_6^2(1+x_6\beta)^2-4x_5^2(1+x_5\beta)^2+8x_5(x_5\delta-\gamma)+8x_6(\gamma-x_6\delta)+(x_6^4-x_5^4)\Omega^2=0. \end{aligned} \tag{3-72}$$

We assume that systems (3-52) and (3-72) have two real solutions where each real solution provides one crossing limit cycle with four points on (P) and one crossing limit cycle whit two points on (P), but by Theorem C we have that discontinuous PWLC (3-49) has at most 1 crossing limit cycle with four points on (P), therefore if we have two real solutions of systems (3-52) and (3-72) they are of the form $(x_1,x_2,x_3,x_4,x_5,x_6)=(k_1,k_2,k_3,k_4,k_5,k_6)$ and $(x_1,x_2,x_3,x_4,x_5,x_6)=(k_1,k_2,k_3,k_4,\lambda_5,\lambda_6)$, with $k_i,\lambda_5,\lambda_6\in\mathbb{R}$ for $i=1,2,3,4,5,6$.

If the point $(k_1,k_2,k_3,k_4,k_5,k_6)$ satisfies systems (3-52) and (3-72), by the equations e_1,e_2,e_3 and e_4 of (3-52) we obtain expressions for the parameters d,δ,c and γ as in the proof of Theorem C, by the equation e_5 of system (3-72) we obtain an expression for $\omega^2=S/T$ with S and T as in the proof of Theorem C changing L_1 and L_2 by k_5 and k_6, respectively. By equation e_6 of system (3-72) we obtain $\Omega^2=V/W$ where the expression for V and W are the same expressions that in the proof of Theorem C changing L_3 by k_5. We assume that the point $(k_1,k_2,k_3,k_4,\lambda_5,\lambda_6)$ satisfies systems (3-52) and (3-72), then we have $e_1=e_2=e_3=e_4=0$ and by the equations e_5 and e_6 of system (3-72) we obtain $b=\beta=0$. As in the proof of Theorem C we can conclude that the two linear centers in (3-49) became $\dot{x}=1/2,\ \ \dot{y}=x$, which is a contradiction. So systems (3-52) and (3-72) have at most one solution and therefore planar discontinuous PWLC in $\mathcal{F}_P$ have at most one crossing limit cycle with four point on (P) and one crossing limit cycle with two points on (P) simultaneously.

In Proposition 3.10 we prove that this upper bound is reached, this is there are systems in $\mathcal{F}_P$ with one crossing limit cycle with four points on (P) and one crossing limit cycle with two points on (P) simultaneously.

This completes the proof of statement (a) of Theorem *E*. □

Proof of statement (b) of Theorem E In this case we consider the notations of the proof of Theorem C for the family $\mathcal{F}_E$ and therefore we consider the planar discontinuous PWLC (3-49) and the first integrals (3-50). In order that the discontinuous PWLC (3-49) has crossing limit cycles with four points on (E), namely $(x_1,y_1),(x_2,y_2),(x_3,y_3),(x_4,y_4)$ and one crossing limit cycle with two points on (E), namely (x_5,y_5), (x_6,y_6), we must study the solutions $(p_1,p_2,p_3,p_4,p_5,p_6)$ of systems (3-53) and (3-73)

$$\begin{aligned} e_5:\ 4(x_5^2-x_6^2)+8(c(x_5-x_6)-dy_5+bx_5y_5+dy_6-bx_6y_6)+(y_5^2-y_6^2)l_1 &= 0,\\ e_6:4(x_6^2-x_5^2)+8(\beta x_6y_6-\beta x_5y_5+y_5\delta-x_5\gamma+x_6\gamma-y_6\delta)+(y_6^2-y_5^2)l_2 &= 0,\\ E_5:x_5^2+y_5^2-1 &= 0,\\ E_6:x_6^2+y_6^2-1 &= 0. \end{aligned} \tag{3-73}$$

We assume that systems (3-53) and (3-73) have two real solutions where each real solution provides one crossing limit cycle with four points on (E) and one crossing limit cycle with two points on (E), like in Theorem C we proved that discontinuous PWLC (3-49) has at most 1 crossing limit cycle with four points on (E), then we have that if there are two real solutions of systems (3-53) and (3-73) they are of the form $(p_1,p_2,p_3,p_4,p_5,p_6)$ and $(p_1,p_2,p_3,p_4,q_5,q_6)$, with p_i and q_j as (3-54) for $i=1,2,3,4,5,6$ and $j=5,6$.

Substituting the first solution $(p_1,p_2,p_3,p_4,p_5,p_6)$ in systems (3-53) and (3-73) we obtain from the equations e_1,e_2,e_3 and e_4 of (3-53) the same expressions than in the proof of Theorem C for d,δ,c,γ, and by the equations e_5 and e_6 of system (3-73) we obtain the same expressions than in the proof of Theorem C for ω and Ω changing (m_1,n_1) by (k_5,λ_5) and (m_2,n_2) by (k_6,λ_6), respectively. We assume that the point $(p_1,p_2,p_3,p_4,q_5,q_6)$ satisfies systems (3-53) and (3-73), then we have $e_1=e_2=e_3=e_4=0$ and by the equations e_5 and e_6 of system (3-73) we obtain $b=\beta=0$. As in the proof of Theorem C we obtain that both linear centers in (3-49) become $\dot{x}=-y$, $\dot{y}=x$, in contradiction that they have limit cycles. So we can conclude that systems (3-53) and (3-73) have at most one solution and therefore planar discontinuous PWLC in $\mathcal{F}_E$ have at most one crossing limit cycle with four points on (E) and one crossing limit cycle with two points on (E) simultaneously.

In Proposition 3.10 we prove that the upper bound provided in statement (b) of Theorem E is reached, that is there are PWLC in $\mathcal{F}_E$ such that have one crossing limit cycle with four points on (E) and one crossing limit cycle with two points on (E) simultaneously.

This completes the proof of statement (b) of Theorem E. $\square$

Proof of statement (c) of Theorem E Here we consider the notations of the proof of Theorem C for the family $\mathcal{F}_H$, and therefore we consider the planar discontinuous PWLC (3-55) and the first integrals (3-56). In order that the discontinuous PWLC (3-55) has crossing limit cycles with four points on (H), namely $(x_1,y_1),(x_2,y_2),(x_3,y_3),(x_4,y_4)$ and one crossing limit cycle with two points on (H), namely $(x_5,y_5),(x_6,y_6)$, we must study the solutions $(p_1,p_2,p_3,p_4,p_5,p_6)$ of systems (3-58) and (3-74)

$$\begin{aligned}
e_5:\ 4(x_5^2-x_6^2)+8(c_2(x_5-x_6)-d_2y_5+b_2x_5y_5+d_2y_6-b_2x_6y_6)+(y_5^2-y_6^2)l_1 &= 0,\\
e_6: 4(x_6^2-x_5^2)+8(b_1x_6y_6-b_1x_5y_5+y_5d_1-x_5c_1+x_6c_1-y_6d_1)+(y_6^2-y_5^2)l_2 &= 0,\\
E_5: x_5^2-y_5^2-1 &= 0,\\
E_6: x_6^2-y_6^2-1 &= 0.
\end{aligned} \tag{3-74}$$

We assume that systems (3-58) and (3-74) have two real solutions where each real solution provides one crossing limit cycle with four points on (H) and one crossing limit cycle with two points on (H). By Theorem C the discontinuous PWLC (3-55) has at most 1 crossing limit cycle with four points on (H), then we have that if there are two real solutions of systems (3-58) and (3-74) they are of the form $(p_1,p_2,p_3,p_4,p_5,p_6)$ and $(p_1,p_2,p_3,p_4,q_5,q_6)$, with p_i and q_j as (3-59) for $i=1,2,3,4,5,6$ and $j=5,6$.

Considering the first solution $(p_1,p_2,p_3,p_4,p_5,p_6)$ of systems (3-58) and (3-74) we obtain the same expressions that in the proof of Theorem C for $d_1,d_2,d_3,c_2,c_1,\omega_2$ changing (m_1,n_1) by (k_5,λ_5) and (m_2,n_2) by (k_6,λ_6), respectively.

Now we assume that the point $(p_1,p_2,p_3,p_4,q_5,q_6)$ satisfies systems (3-58) and (3-74), then we have $e_1=e_2=e_3=e_4=0$, and by the equation e_5 of system (3-74) we obtain $b_2=0$ and with this the linear system in the region R_H^2 becomes a saddle which is a contradiction, because we are working with linear centers in each regions R_H^i for $i=1,2,3$. Therefore the discontinuous PWLC (3-55) has at most one crossing limit cycle with four points on (H) and one crossing limit cycle with two points on (H) simultaneously.

In Proposition 3.10 we prove that the upper bound provided in statement (c) of Theorem E is reached, that is there are PWLC in $\mathcal{F}_H$ such that have one crossing limit cycle with four points on (H) and one crossing limit cycle with two points on (H) simultaneously.

This completes the proof of statement (c) of Theorem E. $\square$

In this section we get provide the upper bounds for the maximum number of crossing limit cycles with two and with four points on Σ can have the PWLC in the families $\mathcal{F}_P, \mathcal{F}_E$ and $\mathcal{F}_H$ simultaneously. Moreover in Proposition 3.10 we proved that these upper bounds are reached.

In this section we do not consider the planar discontinuous PWLC in the family $\mathcal{F}_{LV}$, because they do not have crossing limit cycles that intersect the discontinuity curve (LV) in two points. However in this family there are two types of crossing limit cycles like it was defined in Section 3.3. Then we study the family $\mathcal{F}_{LV}$ in the following section.

3.5 Crossing limit cycles of types 1 and 2 simultaneously for planar discontinuous PWLC in $\mathcal{F}_{LV}$

In this case we study the maximum number of crossing limit cycles of types 1 and 2 that planar discontinuous PWLC in the family $\mathcal{F}_{LV}$ can have simultaneously.

Theorem F *There are planar discontinuous PWLC that belong to the family $\mathcal{F}_{LV}$ such that have one crossing limit cycle of type 1 and three crossing limit cycle of type 2 simultaneously.*

Proof. In order to have a crossing limit cycle of type 1 and one crossing limit cycle of type 2, simultaneously, we must study the real solutions $(p_1,q_1,p_2,q_2,p_3,q_3,p_4,q_4)$, of systems (3-62) and (3-35) respectively, where $p_i=(x_i,0)$ and $q_i=(0,y_i)$, with $x_1,x_2,x_3,y_1,y_2,y_3>0$ and $x_4,y_4<0$.

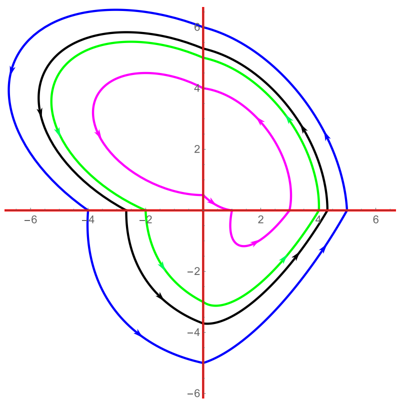

Figure 3.17: *One crossing limit cycle of type 1 and three crossing limit cycles of type 2 of the discontinuous PWLC formed by the linear centers (3-75), (3-76), (3-77) and (3-78) separated by (LV).*

In the region R^1_{LV} we consider the linear differential center

$$\dot{x}=\frac{193}{134}-\frac{x}{3}-\frac{58}{67}y, \qquad \dot{y}=-\frac{149}{134}+x+\frac{y}{3}, \tag{3-75}$$

this system has the first integral $H_1(x,y) = 201x^2 + x(134y - 447) + 3y(58y - 193))$. In the region R^2_{LV} we have the linear differential center

$$\dot{x} = \frac{9}{2} - \frac{x}{2} - 2y, \qquad \dot{y} = -\frac{1}{4} + x + \frac{y}{2}, \tag{3-76}$$

which has the first integral $H_2(x,y) = 2x(2x-1) + 4y(x-9) + 8y^2$. In the region R^3_{LV} we have the linear differential center

$$\dot{x} = 1.068079.. + \frac{\sqrt{3}}{4}x - 1.448022..y, \qquad \dot{y} = -3.860171.. + x - \frac{\sqrt{3}}{4}y, \tag{3-77}$$

which has the first integral $H_3(x,y) = x^2 + x(-7.720342.. - 0.866025..y) + y(-2.136159.. + 1.448022..y)$. And in the region R^4_{LV} we have the linear differential center

$$\dot{x} = \frac{51831 - 595\sqrt{16909}}{35912} + \frac{x}{2} + \frac{6775 - 119\sqrt{16909}}{17956}y, \qquad \dot{y} = -2 + x - \frac{y}{2}, \tag{3-78}$$

which has the first integral $H_4(x,y) = 17956x^2 - 17956x(4+y) + y(-51831 + 595\sqrt{16909} + (-6775 + 119\sqrt{16909})y)$. The unique real solutions for systems (3-62) and (3-35) are $(p_1,q_1,p_2,q_2,p_3,q_3,p_4,q_4)$ with $p_1 = (1,0)$, $q_1 = (0,1/2)$, $p_2 = (3,0)$, $q_2 = (0,4)$, $p_3 = (5,0)$, $q_3 = (0,6)$, $p_4 = (-4,0)$ and $q_4 = (0,-5)$; $(p_1,q_1,p_2,q_2,l_3,m_3,l_4,m_4)$ with $l_3 = ((149 + 3\sqrt{16909})/134, 0)$, $m_3 = (0,5)$, $l_4 = (-2,0)$ and $m_4 = (0,-3)$; and $(p_1,q_1,p_2,q_2,\lambda_3,\eta_3,\lambda_4,\eta_4)$, where $\lambda_3 = (\ 4.319114..,0)$, $\eta_3 = (\ 0,53/10)$, $\lambda_4 = (\ -2.672755..,0)$ and $\eta_4 = (\ 0,-3.703965..)$. See these crossing limit cycles of types 1 and 2 in Figure 3.17. □

In this case we use the same ideas of the proof of statement (ii) of Theorem D and that case we only get a lower bound for the maximum number of crossing limit cycles of type 2, then in this case the configuration $(1,3)$ of crossing limit cycles, this is 1-crossing limit cycle of type 1 and 3-crossing limit cycles of type 2, is a lower bound of the maximum number of crossing limit cycles of types 1 and 2 that can have the PWLC in the family $\mathcal{F}_{LV}$ simultaneously.

3.6 Discussions and conclusions

In this chapter we study on the upper bounds for the maximum number of crossing limit cycles with either two or four points on the discontinuity curve Σ, when Σ is any conic, this is, the numbers $\mathcal{N}^m_\Sigma$, $m = 2,4$ and $\Sigma \in \{(DL),(PL),(LV),(H)(P),(E)\}$.

With regard to case $m = 2$ in the papers [24, 27, 28, 20] it was determined the number $\mathcal{N}^2_\Sigma$ for $\Sigma \in \{(DL),(PL),(LV),(P),(E)\}$, see Theorem 3.3. We completed this study with Theorem B since we proved that the number $\mathcal{N}_H^2$ is equal to two and moreover we verify that this upper bound is reached. Therefore, with Theorems 3.3 and B the study on the numbers $\mathcal{N}_\Sigma^2$ for any conic it is completed.

When $m = 4$, we have two cases. First we proved that $\mathcal{N}_\Sigma^4$ for $\Sigma \in \{(PL),(P),(E),(H)\}$ it is equal to one, and moreover we proved that this upper bound it is reached in each case. Secondly for the family $\mathcal{F}_{LV}$ we observed that there are two types of crossing limit cycles which we denote crossing limits cycles of types 1 and 2. In statement (i) of Theorem D we proved that the maximum number of crossing limit cycles of type 1 for PWLC in $\mathcal{F}_{LV}$ it is one, and moreover we proved that this upper bound it is reached. In statement (ii) of Theorem D we only got a lower bound for the maximum number of crossing limit cycles of type 2 for a planar discontinuous PWLC with four zones separated by (LV) and formed by four arbitrary linear centers in each region R_{LV}^i, as (3-60). In this case we must determine the real solutions (x_1,y_1,x_2,y_2) that satisfy the *closing equations*

$$\begin{aligned} e_1 &: H_1(x_1,0) - H_1(0,y_1) = 0,\\ e_2 &: H_2(0,y_1) - H_2(x_2,0) = 0,\\ e_3 &: H_3(x_2,0) - H_3(0,y_2) = 0,\\ e_4 &: H_4(0,y_2) - H_4(x_1,0) = 0. \end{aligned} \tag{3-79}$$

Which are equivalent to system

$$\begin{aligned} e_1 &: 4x_1(2c_1+x_1) - y_1(-8d_1 + y_1(4b_1^2+\omega_1^2)) = 0,\\ e_2 &: 4x_2(2c_2+x_2) - y_1(-8d_2 + y_1(4b_2^2+\omega_2^2)) = 0,\\ e_3 &: 4x_2(2c_3+x_2) - y_2(-8d_3 + y_2(4b_3^2+\omega_3^2)) = 0,\\ e_4 &: 4x_1(2c_4+x_1) - y_2(-8d_4 + y_2(4b_4^2+\omega_4^2)) = 0. \end{aligned} \tag{3-80}$$

Due to the total of parameters and unknown variables it is difficult to apply the usual techniques such as Grobner basis, resultant theory or Bezout inequality. Therefore we only get to provide a lower bound for the maximum number in this case.

In Section 3.4 we analyze the possibility of having crossing limit cycles with two and four points simultaneously on the discontinuity curve, in Theorem E we provided the number of crossing limit cycles with two and four points on the discontinuity curve Σ that the PWLC in the families $\mathcal{F}_P, \mathcal{F}_E$ and $\mathcal{F}_H$ can have simultaneously. And in Proposition 3.10 we proved that the upper bound provided in Theorem E it is reached in each case.

Finally in Theorem F in Section 3.5 we provided a lower bound for the maximum number of crossing limit cycles of types 1 and 2 that the PWLC in the family $\mathcal{F}_{LV}$ can have simultaneously.

We observed that the numbers $\mathcal{N}_\Sigma^m$, $m = 2,4$ for a PWLC when the discontinuity curve is the conic Σ, with $\Sigma \in \{(DL),(PL),(LV),(E),(H),(P)\}$ it change depending the shape of the discontinuity curve, then as it was already analyzed what happens if

$\Sigma \in \{(DL), (PL), (LV), (E), (H), (P)\}$, in the following chapter we study the numbers $\mathcal{N}_{\Sigma}^{4}$ when Σ is a reducible cubic curve.

CHAPTER 4

Crossing limit cycles for PWLC separated by a reducible cubic curve

4.1 Introduction

In Chapter 3 was studied the maximum number of crossing limit cycles for PWLC separated by any conic. The objective of this chapter is to study the existence of crossing limit cycles of the discontinuous PWLC in $\mathbb{R}^2$ separated by a reducible cubic curve formed either by a circle and a straight line, or by a parabola and a straight line.

We observe that we have three options for crossing limit cycles of discontinuous PWLC separated by such reducible cubic curves here considered. First we have the crossing limit cycles which intersect in two points of the discontinuity curve. In [24] was proved that the class of linear differential centers separated by a straight line have no crossing limit cycles, then we can consider that those two intersection points on the discontinuity curve are on the circle or on the parabola and these two options were considered in the Section 3.2 of Chapter 3. Second the crossing limit cycles intersect the discontinuity curve in exactly four points, here we consider that at least one of the four points is on the straight line, because the case which the four points are only on the circle or on the parabola was studied in Section 3.3 of Chapter 3. Finally we have the crossing limit cycles such that intersect the discontinuity curve in six points, which we could study in future works.

In this work we only study the crossing limit cycles with four points on discontinuity curve and we have two cases, first when the discontinuity curve is formed by a circle and a straight line and second when the discontinuity curve is formed by a parabola and a straight line.

In Section 4.2 we consider the PWLC formed by linear differential centers separated by the cubic

$$\Sigma_k = \left\{(x,y) \in \mathbb{R}^2 : (x-k)(x^2+y^2-1) = 0,\ k \in \mathbb{R},\ k \geq 0\right\}.$$

And in Section 4.3 we consider the PWLC formed by linear differential centers separated by the cubic

$$\tilde{\Sigma}_k = \left\{(x,y) \in \mathbb{R}^2 : (y-k)(y-x^2) = 0,\ k \in \mathbb{R}\right\}.$$

We denote by $\mathcal{N}^4_{\Sigma_k}$ and $\mathcal{N}^4_{\tilde{\Sigma}_k}$ the number of crossing limit cycles with four points on the discontinuity curve of a PWLC when the discontinuity curve is the reducible cubic either Σ_k or $\tilde{\Sigma}_k$, respectively.

4.2 Crossing limit cycles intersecting the discontinuity curve Σ_k

Let $\mathcal{F}_{\Sigma_{k+}}$ be the family of PWLC separated by Σ_k with $k > 1$. Let $\mathcal{F}_{\Sigma_1}$ be the family of PWLC separated by Σ_k with $k = 1$. For $k \geq 1$ we have the following regions in the plane

$$\begin{aligned}
R^1_{\Sigma_{k+}} &= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\},\\
R^2_{\Sigma_{k+}} &= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > 1 \text{ and } x < k\},\\
R^3_{\Sigma_{k+}} &= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > 1 \text{ and } x > k\}.
\end{aligned}$$

And finally let $\mathcal{F}_{\Sigma_{k-}}$ be the family of PWLC separated by Σ_k with $0 \leq k < 1$. Here we have the following regions in the plane

$$\begin{aligned}
R^1_{\Sigma_{k-}} &= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1, \text{ and } x > k\},\\
R^2_{\Sigma_{k-}} &= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > 1, \text{ and } x > k\},\\
R^3_{\Sigma_{k-}} &= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > 1 \text{ and } x < k\},\\
R^4_{\Sigma_{k-}} &= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1 \text{ and } x < k\}.
\end{aligned}$$

In the family $\mathcal{F}_{\Sigma_{k-}}$, we have three types of crossing limit cycles. First crossing limit cycles such that are formed by parts of orbits of the four linear differential centers considered, namely crossing limit cycles of type 1, see Figure 4.3, second we have crossing limit cycles which intersect the regions $R^1_{\Sigma_{k-}}$, $R^2_{\Sigma_{k-}}$ and $R^4_{\Sigma_{k-}}$ or crossing limit cycles that intersect the regions $R^1_{\Sigma_{k-}}, R^4_{\Sigma_{k-}}$ and $R^3_{\Sigma_{k-}}$, namely crossing limit cycles of type 2^+ or crossing limit cycles of type 2^-, respectively, see Figure 4.4. Without loss of generality we only study the crossing limit cycles of type 2^+ because the analysis for the crossing limit cycles of type 2^- is the same, moreover we observe that these two cases can not occur simultaneously, because the orbits of linear differential system in the region $R^4_{\Sigma_{k-}}$ would not be nested. And finally we have the crossing limit cycles such that are formed by parts of orbits of the three linear differential centers in the regions $R^1_{\Sigma_{k-}}$, $R^2_{\Sigma_{k-}}$ and $R^3_{\Sigma_{k-}}$, or crossing limit cycles formed by parts of orbits of the three linear differential centers in the regions $R^2_{\Sigma_{k-}}$, $R^3_{\Sigma_{k-}}$ and $R^4_{\Sigma_{k-}}$, namely crossing limit cycles of type 3^+ and crossing limit cycles of type 3^-, respectively, see Figure 4.5. Without loss of generality in Theorem G we study the crossing limit cycles of type 3^+ because the study by the crossing limit cycles of type 3^- is the same. We observe that these types of crossing limit cycles can not appear simultaneously, because the orbits of linear differential system in the region $R^3_{\Sigma_{k-}}$ would not be nested.

Then in the following Theorem we provide examples of PWLC in $\mathcal{F}_{\Sigma_{k-}}$ with crossing limit cycles of types 1, 2^+ and 3^+ separately and PWLC in $\mathcal{F}_{\Sigma_{k-}}$ such that have simultaneously crossing limit cycles of types 1 and 2^+ or of types 1 and 3^+.

4.2.1 Statement of the main result

Theorem G *The following statements hold.*

(i) There are PWLC in the families $\mathcal{F}_{\Sigma_{k+}}$ and $\mathcal{F}_{\Sigma_1}$ formed by three linear differential centers that have four crossing limit cycles, see Figures 4.1 and 4.2, respectively.

(ii) There are PWLC in $\mathcal{F}_{\Sigma_{k-}}$, that have five crossing limit cycles of type 1, see Figure 4.3.

(iii) There are PWLC in $\mathcal{F}_{\Sigma_{k-}}$, that have four crossing limit cycles of type 2^+, see Figure 4.4.

(iv) There are PWLC in $\mathcal{F}_{\Sigma_{k-}}$, that have three crossing limit cycles of type 3^+, see Figure 4.5.

Theorem G is proved in Section 4.5.1.

4.3 Crossing limit cycles intersecting the discontinuity curve $\tilde{\Sigma}_k$

Let $\mathcal{F}_{\tilde{\Sigma}_{k-}}$ be the family of PWLC separated by $\tilde{\Sigma}_k$ with $k < 0$. In this case, we have following three regions in the plane

$$
\begin{aligned}
R^1_{\tilde{\Sigma}_{k-}} &= \{(x,y) \in \mathbb{R}^2 : y > x^2\},\\
R^2_{\tilde{\Sigma}_{k-}} &= \{(x,y) \in \mathbb{R}^2 : y < x^2 \text{ and } y > k\},\\
R^3_{\tilde{\Sigma}_{k-}} &= \{(x,y) \in \mathbb{R}^2 : y < x^2 \text{ and } y < k\}.
\end{aligned}
$$

Let $\mathcal{F}_{\tilde{\Sigma}_0}$ be the family of PWLC separated by $\tilde{\Sigma}_k$ with $k = 0$. When the discontinuity curve is $\tilde{\Sigma}_0$ we have following four regions in the plane

$$
\begin{aligned}
R^1_{\tilde{\Sigma}_0} =& \{(x,y) \in \mathbb{R}^2 : y > x^2 \},\\
R^2_{\tilde{\Sigma}_0} =& \{(x,y) \in \mathbb{R}^2 : y < x^2 \text{ and } y > 0,\ x < 0\},\\
R^3_{\tilde{\Sigma}_0} =& \{(x,y) \in \mathbb{R}^2 : y < x^2 \text{ and } y < 0\},\\
R^4_{\tilde{\Sigma}_0} =& \{(x,y) \in \mathbb{R}^2 : y < x^2 \text{ and } y > 0,\ x > 0\}.
\end{aligned}
$$

Here we have two types of crossing limit cycles, first crossing limit cycles formed by parts of orbits of the four linear differential centers considered, namely crossing limits cycles of type 4, see Figure 4.7. Second crossing limit cycles of type 5, see Figure 4.8, which intersect only three regions, in this case we have two options, first we have the case where the crossing limit cycles are formed by parts of orbits of the linear differential centers in the regions $R^1_{\tilde{\Sigma}_0}, R^3_{\tilde{\Sigma}_0}$ and $R^4_{\tilde{\Sigma}_0}$ and second the crossing limit cycles that intersect only the three regions $R^1_{\tilde{\Sigma}_0}$, $R^2_{\tilde{\Sigma}_0}$ and $R^3_{\tilde{\Sigma}_0}$, without loss of generality we can consider the first case because the study by the second is the same. Here we observe that it is not possible to have crossing limit cycles of type 5 that satisfy those two cases simultaneously, because the orbits of linear differential system in the region $R^3_{\tilde{\Sigma}_0}$ would not be nested. Therefore in the following Theorem we study the PWLC in $\mathcal{F}_{\tilde{\Sigma}_0}$ which have crossing limit cycles of types 4 and 5, respectively.

Let $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ be the family of PWLC separated by $\tilde{\Sigma}_k$ with $k > 0$, in this case we have the following five regions in the plane

$$
\begin{aligned}
R^1_{\tilde{\Sigma}_{k+}} =& \{(x,y) \in \mathbb{R}^2 : y < x^2 \text{ and } y > k,\ x > \sqrt{k}\},\\
R^2_{\tilde{\Sigma}_{k+}} =& \{(x,y) \in \mathbb{R}^2 : y > x^2 \text{ and } y > k\},\\
R^3_{\tilde{\Sigma}_{k+}} =& \{(x,y) \in \mathbb{R}^2 : y < x^2 \text{ and } y > k,\ x < -\sqrt{k}\},\\
R^4_{\tilde{\Sigma}_{k+}} =& \{(x,y) \in \mathbb{R}^2 : y < x^2 \text{ and } y < k\},\\
R^5_{\tilde{\Sigma}_{k+}} =& \{(x,y) \in \mathbb{R}^2 : x^2 < y < k\}.
\end{aligned}
$$

Here we have six types of crossing limit cycles. First we have crossing limit cycles such that are formed by parts of orbits of the four linear differential centers in the regions $R^1_{\tilde{\Sigma}_{k+}}$, $R^2_{\tilde{\Sigma}_{k+}}$, $R^5_{\tilde{\Sigma}_{k+}}$ and $R^4_{\tilde{\Sigma}_{k+}}$, or crossing limit cycles formed by parts of orbits of the four linear differential centers in the regions $R^2_{\tilde{\Sigma}_{k+}}$, $R^3_{\tilde{\Sigma}_{k+}}$, $R^4_{\tilde{\Sigma}_{k+}}$ and $R^5_{\tilde{\Sigma}_{k+}}$, namely crossing limit cycles of type 6^+ and crossing limit cycles of type 6^-, respectively, see Figure 4.17. In Theorem H we study the crossing limit cycles of type 6^+ because the study for the case of crossing limit cycles of type 6^- is the same. Second we have crossing limit cycles type 7, see Figure

4.10, which intersect the three regions $R^2_{\tilde{\Sigma}_{k+}}$, $R^5_{\tilde{\Sigma}_{k+}}$ and $R^4_{\tilde{\Sigma}_{k+}}$. Third we have the crossing limit cycles of type 8, see Figure 4.11, which intersect the regions $R^1_{\tilde{\Sigma}_{k+}}$, $R^2_{\tilde{\Sigma}_{k+}}$, $R^3_{\tilde{\Sigma}_{k+}}$ and $R^4_{\tilde{\Sigma}_{k+}}$. And finally we have the crossing limit cycles such that are formed by parts of orbits of the three linear differential centers in the regions $R^1_{\tilde{\Sigma}_{k+}}$, $R^2_{\tilde{\Sigma}_{k+}}$ and $R^4_{\tilde{\Sigma}_{k+}}$, or crossing limit cycles formed by parts of orbits of the three linear differential centers in the regions $R^2_{\tilde{\Sigma}_{k+}}$, $R^3_{\tilde{\Sigma}_{k+}}$ and $R^4_{\tilde{\Sigma}_{k+}}$, namely crossing limit cycles of type 9^+ and crossing limit cycles of type 9^-, respectively, see Figure 4.12. Without loss of generality in Theorem H we study the crossing limit cycles of type 9^+ because the study by the crossing limit cycles of type 9^- is the same.

We observe that there are no crossing limit cycles that intersect the five regions R^i_H for $i = 1,2,3,4,5$.

4.3.1 Statement of the main result

Theorem H *The following statements hold.*

(i) There are PWLC in $\mathcal{F}_{\tilde{\Sigma}_{k-}}$ that have four crossing limit cycles with four points on $\tilde{\Sigma}_k$, see Figure 4.6.

(ii) There are PWLC in $\mathcal{F}_{\tilde{\Sigma}_0}$ that have four crossing limit cycles of type 4, see Figure 4.7.

(iii) There are PWLC in $\mathcal{F}_{\tilde{\Sigma}_0}$ that have three crossing limit cycles of type 5, see Figure 4.8.

(iv) There are PWLC in $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ that have five crossing limit cycles of type 6^+, see Figure 4.9.

(v) There are PWLC in $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ that have three crossing limit cycles of type 7, see Figure 4.10.

(vi) There are PWLC in $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ that have four crossing limit cycles of type 8, see Figure 4.11.

(vii) There are PWLC in $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ that have three crossing limit cycles of type 9^+, see Figure 4.12.

Theorem H is proved in Section 4.5.1.

4.4 Simultaneity

The next results provide lower bounds for the maximum number of crossing limit cycles that can appear simultaneously in each family, $\mathcal{F}_{\Sigma_k}$, $\mathcal{F}_{\tilde{\Sigma}_k}$, with $k \in \mathbb{R}$.

First we analyze the family $\mathcal{F}_{\Sigma_k}$, $k \in \mathbb{R}$. We observe that in $\mathcal{F}_{\Sigma_{k+}}$ and $\mathcal{F}_{\Sigma_1}$ only there are crossing limit cycles of one type, respectively. Then we study the family $\mathcal{F}_{\Sigma_{k-}}$ which has three types of crossing limit cycles, namely crossing limit cycles of types $1, 2^+$ and 3^+. In order to obtain PWLC in the family $\mathcal{F}_{\Sigma_{k-}}$ which have simultaneously two types of crossing limit cycles we observe that we would have three possible combinations between the three different crossing limit cycles types nevertheless we observe that the crossing limit cycles of types 2^+ and 3^+ can not appear simultaneously, because the orbits of linear differential system in the region $R^1_{\Sigma_{k-}}$ would not be nested. For this same reason there are no PWLC in $\mathcal{F}_{\Sigma_{k-}}$ with three types of crossing limit cycles simultaneously. Then we have that PWLC with crossing limit cycles of types 1 and 2^+ are analyzed in statement (i) of Theorem I and PWLC with crossing limit cycles of types 1 and 3^+ are analyzed in statement (ii) of Theorem I.

In the family $\mathcal{F}_{\tilde{\Sigma}_k}$, with $k \in \mathbb{R}$, we observe that in the family $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ only there are crossing limit cycles of one type. In the family $\mathcal{F}_{\tilde{\Sigma}_0}$ we have two types of crossing limit cycles, namely crossing limit cycles of types 4 and 5. In statement (iii) we analyze the PWLC in $\mathcal{F}_{\tilde{\Sigma}_0}$ which have crossing limit cycles of types 4 and 5 simultaneously.

In the family $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ we have six different crossing limit cycles types, namely types $6^+, 6^-, 7, 8, 9^+$ and 9^-. Then we would have fifteen possible combinations of pairs of crossing limit cycles, we will analyze each one. PWLC with crossing limit cycles of types 6^+ and 6^- are analyzed in statement (iv) of Theorem. The study for PWLC with crossing limit cycles of types 6^+ and 7, or 6^+ and 8, or 6^+ and 9^+ is the same for PWLC with crossing limit cycles of types 6^- and 7, or 6^- and 8, or 6^- and 9^-, respectively, and they are in statements (v), (vi) and (vii) of Theorem I, respectively. The crossing limit cycles of types 6^- and 9^+ can not appear simultaneously because the orientation of these crossing limit cycles in region $R^4_{\tilde{\Sigma}_{k+}}$ would not be well defined, similarly happens with the crossing limit cycles of types 6^+ and 9^-. PWLC with crossing limit cycles of types 7 and 8 are analyzed in statement $(viii)$ of Theorem I. It is not possible to have crossing limit cycles of type 7 and 9^+, or 7 and 9^- simultaneously, because the orbits of linear differential system in the region $R^2_{\tilde{\Sigma}_{k+}}$ would not be nested. PWLC with crossing limit cycles of types 8 and 9^+ are analyzed in statement (ix) of Theorem I, the case where appear crossing limit cycles of types 8 and 9^-, simultaneously is the same. Finally we observe that it is not possible to have simultaneously crossing limit cycles of types 9^+ and 9^-, because the orbits of linear differential system in the region $R^4_{\tilde{\Sigma}_{k+}}$ would not be nested.

We observe that we would have twenty possible combinations of triplets between the six different crossing limit cycles types above, but we have fourteen combinations that include couples 6^+ and 9^-, 6^- and 9^+, 7 and $9^{\pm}$, or 9^+ and 9^- and as it was said before these combinations are not possible. Therefore we have six options, first we observed that crossing limit cycles of types 6^+, 6^- and 7, or 6^+, 6^- and 8 can not appear simultaneously because the orientation of these crossing limit cycles in region $R^4_{\tilde{\Sigma}_{k+}}$ would not be well defined. Second we have that there are PWLC with crossing limit cycles of types 6^+, 7 and 8, this case is in statement (x) of Theorem I, the case where appear crossing limit cycles of types 6^-, 7 and 8 is the same. Finally we have the PWLC with crossing limit cycles of types 6^+, 8 and 9^+, this case is in statement (xi) of Theorem I and the case where appear crossing limit cycles of types 6^-, 7 and 9^- is the same. By the previous analysis we observed that it is not possible to have PWLC in $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ with four, five or six types of crossing limit cycles simultaneously.

4.4.1 Statement of the main result

Theorem I *The following statements hold.*

(i) *There are PWLC in $\mathcal{F}_{\Sigma_{k-}}$, that have four crossing limit cycles of type* 1 *and two crossing limit cycles of type* 2^+*, see Figure* 4.13.

(ii) *There are PWLC in $\mathcal{F}_{\Sigma_{k-}}$, that have four crossing limit cycles of type* 1 *and one crossing limit cycle of type* 3^+*, see Figure* 4.14.

(iii) *There are PWLC in $\mathcal{F}_{\tilde{\Sigma}_0}$ that have simultaneously four crossing limit cycles of type* 4 *and two crossing limit cycles of type* 5*, see Figure* 4.16.

(iv) *There are PWLC in $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ that have simultaneously four crossing limit cycles of type* 6^+ *and four crossing limit cycles of type* 6^-*, see Figure* 4.17.

(v) *There are PWLC in $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ that have simultaneously four crossing limit cycles of type* 6^+ *and two crossing limit cycles of type* 7*, see Figure* 4.18.

(vi) *There are PWLC in $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ that have simultaneously three crossing limit cycles of type* 6^+ *and four crossing limit cycle of type* 8*, see Figure* 4.19.

(vii) *There are PWLC in $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ that have simultaneously four crossing limit cycles of type* 6^+ *and two crossing limit cycles of type* 9^+*, see Figure* 4.20.

(viii) *There are PWLC in $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ that have simultaneously three crossing limit cycles of type* 7 *and four crossing limit cycle of type* 8*, see Figure* 4.21.

(ix) *There are PWLC in $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ that have simultaneously four crossing limit cycles of type* 8 *and two crossing limit cycles of type* 9^+*, see Figure* 4.22.

(x) *There are PWLC in $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ that have simultaneously two crossing limit cycles of type* 6^+*, two crossing limit cycles of type* 7 *and four crossing limit cycles of type* 8*, see Figure* 4.23.

(xi) There are PWLC in $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ that have simultaneously four crossing limit cycles of type 6^+*, three crossing limit cycles of type* 8 *and two crossing limit cycles of type* 9^+*, see Figure 4.24.*

Theorem I is proved in Section 4.5.1.

By the numerical computations made for the families $\mathcal{F}_{\Sigma_k}$, $\mathcal{F}_{\tilde{\Sigma}_k}$ with $k \in \mathbb{R}$ and the illustrated examples of Theorems *G*, *H* and *I* we propose the following problem.

Open problem *The numbers $\mathcal{N}^4_{\Sigma_k}$ and $\mathcal{N}^4_{\tilde{\Sigma}_k}$ provided in Theorems G, H and I for the families $\mathcal{F}_{\Sigma_k}$, $\mathcal{F}_{\tilde{\Sigma}_k}$ with $k \in \mathbb{R}$ are the maximum numbers of crossing limit cycles in each family.*

4.5 Proof of the main results of this chapter

In this section we provide the proofs of Theorems G, H and Theorem I.

4.5.1 Proof of Theorem G

Proof of statement (i) for the family $\mathcal{F}_{\Sigma_{k+}}$ of Theorem G. By Lemma 3.1 we can consider the following PWLC

$$\begin{aligned}
\dot{x} &= -b_1 x - \frac{4b_1^2+\omega_1^2}{4} y + d_1, \qquad \dot{y} = x + b_1 y + c_1, \text{ in } R^1_{\Sigma_{k+}}, \\
\dot{x} &= -b_2 x - \frac{4b_2^2+\omega_2^2}{4} y + d_2, \qquad \dot{y} = x + b_2 y + c_2, \text{ in } R^2_{\Sigma_{k+}}, \\
\dot{x} &= -b_3 x - \frac{4b_3^2+\omega_3^2}{4} y + d_3, \qquad \dot{y} = x + b_3 y + c_3, \text{ in } R^3_{\Sigma_{k+}}.
\end{aligned} \tag{4-1}$$

And the linear differential centers in (4-1) have the first integrals

$$H_i(x,y) = 4(x+b_i y)^2 + 8(c_i x - d_i y) + y^2\omega_i^2, \text{ with } i = 1,2,3,$$

respectively. In order to have a crossing limit cycle, which intersects Σ_{k+} in four different points $p_1 = (k, y_1)$, $p_2 = (x_2, y_2)$, $p_3 = (x_3, y_3)$ and $p_4 = (k, y_4)$, with $p_2, p_3 \in \mathbb{S}^1$, where $\mathbb{S}^1 = \{(x,y) : x^2+y^2 = 1\}$. These points must satisfy the *closing equations* (3-4)

$$\begin{aligned}
e_1 : H_2(k,\ y_1) - H_2(x_2, y_2) &= 0, \\
e_2 : H_1(x_2, y_2) - H_1(x_3, y_3) &= 0, \\
e_3 : H_2(x_3, y_3) - H_2(\ k, y_4) &= 0, \\
e_4 : H_3(k,\ y_4) - H_3(\ k, y_1) &= 0, \\
x_2^2 + y_2^2 &= 1, \\
x_3^2 + y_3^2 &= 1.
\end{aligned} \tag{4-2}$$

For to build the example, we will impose the existence of periodic solutions and we will determine the parameters in (4-1) with the established conditions. First we fix the constant $k=2$ and we assume that there is a real solution, namely $q^1=(y_1^1,x_2^1,y_2^1,x_3^1,y_3^1,y_4^1)=(3,\cos(\pi/2),\sin(\pi/2),\cos(-\pi/3),\sin(-\pi/3),-5/2)$, then by equations e_i with $i=1,2,3,4$ in (4-2) we have the parameters

$$
\begin{aligned}
d_2 &= 1+b_2(3+2b_2)+c_2+\frac{\omega_2^2}{2};\\
d_1 &= -\frac{1}{16}(-2+\sqrt{3})(-4+4b_1(2\sqrt{3}+b_1)-16c_1+\omega_1^2);\\
c_2 &= \frac{70-8\sqrt{3}+4b_2(10-5\sqrt{3}+31b_2-4\sqrt{3}b_2)+(31-4\sqrt{3})\omega_2^2}{8(-8+\sqrt{3})};\\
d_3 &= \frac{1}{16}(4b_3(8+b_3)+\omega_3^2),
\end{aligned}
$$

respectively. Now by the equation e_4 we have

$$y_4=\frac{1}{2}(1-2y_1),$$

then we suppose that the point $q^2=(y_1^2,x_2^2,y_2^2,x_3^2,y_3^2,y_4^2)=(16/5,\cos(3\pi/5),\sin(3\pi/5),\cos(-2\pi/5),\sin(-2\pi/5),-27/10)$ is also a real solution of system (4-2), then by the equations e_1,e_2 and e_3 in (4-2) we obtain the following parameters

$$
\begin{aligned}
\omega_2 &= -\frac{2}{\sqrt{3894-523\sqrt{3}+225\sqrt{15}+50\sqrt{2(5+\sqrt{5})}}}\sqrt{(-635+25\sqrt{3}+675\sqrt{5}}\\
&\quad -75\sqrt{15}+75\sqrt{2(5+\sqrt{5})}+5(1468+34\sqrt{3}+100\sqrt{5}-50\sqrt{15}+5\sqrt{2(5+\sqrt{5})}\\
&\quad (-68+\sqrt{3}-8\sqrt{5}+\sqrt{15}))b_2+(-3894+523\sqrt{3}+25\sqrt{5}(70-9\sqrt{3})\\
&\quad -50\sqrt{2(5+\sqrt{5})})b_2^2);\\
c_1 &= \frac{(-2+\sqrt{3})\sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)}(-4+8\sqrt{3}b_1+4b_1^2+\omega_1^2)}{8(-1+\sqrt{5}-2\sqrt{2(5+\sqrt{5})}}+\sqrt{6(5+\sqrt{5})}));\\
b_2 &= 3.119845..,
\end{aligned}
$$

respectively. Now we fix the points $x_2=\cos(4\pi/7)$, $y_2=\sin(4\pi/7)$ and by equation e_6 we have

$$y_3=-\sqrt{1-x_3^2},$$

then by the equations e_1, e_2 and e_3 we have

$$\begin{aligned} y_1 &= 3.144465..; \\ \omega_1 &= -9.702226..\sqrt{0.042492.. + 0.031501..b_1 - 0.042492..b_1^2}; \\ x_3 &= 0.365470.., \end{aligned}$$

respectively. These conditions generate the real solution $q^3 = (3.144465.., \cos(4\pi/7), \sin(4\pi/7), 0.365470.., -0.930823.., -2.644465..)$. We build a fourth solution fixing the points $x_2 = -0.018219..$; $y_2 = 0.999834..$; therefore by the equations e_1, e_2 and e_3 we obtain $y_1 = 3.012016..$; $x_3 = 0.489429..$; $b_1 = 0.608380..$, respectively. With these conditions we have the real solution $q^4 = (3.012016.., -0.018219.., 0.999834.., 0.489429.., -0.872042.., -2.512016..)$. With these four real solutions we determined all the parameters that appear in system (4-2), even more in this particular case the parameters $b_3, c_3, \omega_3 \in \mathbb{R}$, then we fix them, $b_3 = 1$; $c_3 = 1/4$; $\omega_3 = 1$. Therefore we obtain the PWLC formed by the following linear differential centers

$$\begin{aligned} \dot{x} &= 0.977474.. - 0.608380..x - 1.451017..y, & \dot{y} &= -3.008357.. + x + 0.608380..y; \\ \dot{x} &= 9.710162.. - 3.119845..x - 10.075224..y, & \dot{y} &= -20.799821.. + x + 3.119845..y; \\ \dot{x} &= \frac{37}{16} - x - \frac{5}{4}y, & \dot{y} &= \frac{1}{4} + x + y, \end{aligned} \tag{4-3}$$

in the regions $R^1_{\Sigma_2}$ $R^2_{\Sigma_2}$ and $R^3_{\Sigma_2}$, respectively.

The linear differential centers in (4-3) have the first integrals

$$\begin{aligned} H_1(x,y) &= x^2 + x(-6.016714.. + 1.216760..y) + y(-1.954949.. + 1.451017..y), \\ H_2(x,y) &= x^2 + x(-41.599643.. + 6.239690..y) + y(-19.420324.. + 10.075224..y), \\ H_3(x,y) &= 2x + 4x^2 - \frac{37}{2}y + 8xy + 5y^2, \end{aligned}$$

respectively.

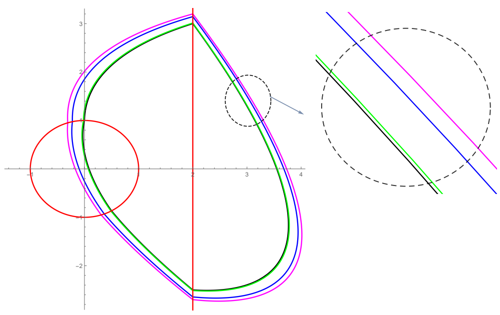

Figure 4.1: *Four crossing limit cycles of the discontinuous PWLC (4-3). These limit cycles are traveled in counterclockwise.*

In this case system (4-2) is equivalent to system

$$\begin{aligned}
79.199286..+x_2^2+6.940944..y_1-10.075224..y_1^2-19.420324..y_2+10.075224..y_2^2&\\
+x_2(-41.599643..+6.239690..y_2)&=0,\\
x_2^2-x_3^2-1.954949..y_2+1.451017..y_2^2+x_2(-6.016714..+1.216760..y_2)&\\
+x_3(6.016714..-1.216760..y_3)+1.954949..y_3-1.451017..y_3^2)&=0,\\
79.199286..+x_3^2-19.420324..y_3+10.075224..y_3^2&\\
+x_3(-41.599643..+6.239690..y_3)+6.940944..y_4-10.075224..y_4^2&=0,\\
(y_1-y_4)\left(-\frac{5}{2}+5y_1+5y_4\right)&=0,\\
x_2^2+y_2^2=1,\quad x_3^2+y_3^2&=1.
\end{aligned} \tag{4-4}$$

Taking into account that the solutions $q^i=(y_1^i,x_2^i,y_2^i,x_3^i,y_3^i,y_4^i)$ of system (4-4) must satisfy $y_4^i<y_1^i$ we have that the unique reals solutions are the points q^1,q^2,q^3 and q^4 which provide four crossing limit cycles of the PWLC (4-3). See these crossing limit cycles in Figure 4.1.

This completes the proof of statement (i) for the family $\mathcal{F}_{\Sigma_{k^+}}$ of Theorem G. □

Proof of statement (i) ***for the family*** $\mathcal{F}_{\Sigma_1}$ ***of Theorem G.*** Following the steps illustrated in the previous case we obtain a discontinuous PWLC which is formed by the following linear differential centers in each region. First in the region $R^1_{\Sigma_1}$ we have

$$\dot{x}=2.185588..-\frac{3}{20}x-6.201094..y,\quad \dot{y}=-6.726549..+x+\frac{3}{20}y. \tag{4-5}$$

This linear differential center has the first integral $H_1(x,y)=x^2+x(-13.453098..+3y/10)+y(-4.371176..+6.201094..y)$. In the region $R^2_{\Sigma_1}$ we consider the linear differential center

$$\dot{x}=-0.263120..-0.874044..x-4.914345..y,\qquad \dot{y}=-23.305757..+x+0.874044..y, \tag{4-6}$$

which has the first integral $H_2(x,y)=x^2+x(-46.611514..+1.748088..y)+y(0.526241..+4.914345..y)$. And in the region $R^3_{\Sigma_1}$ we have the linear differential center

$$\dot{x}=\frac{21}{16}-x-\frac{5}{4}y,\quad \dot{y}=\frac{1}{4}+x+y, \tag{4-7}$$

which has the first integral $H_3(x,y)=2x+4x^2-21y/2+8xy+5y^2$.

In order to have a crossing limit cycle, which intersects Σ_1 in four different points $p_1=(1,y_1)$, $p_2=(x_2,y_2)$, $p_3=(x_3,y_3)$ and $p_4=(1,y_4)$, with $p_2,p_3\in\mathbb{S}^1$, these points must satisfy the *closing equations* given in (4-2). Then for the PWLC formed by the

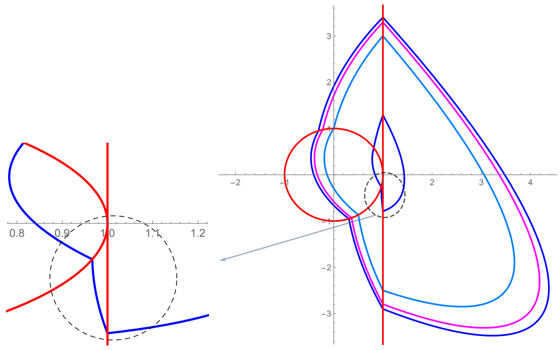

Figure 4.2: *Four crossing limit cycles of the discontinuous PWLC formed by (4-5), (4-6) and (4-7) and separated by* Σ_1. *These limit cycles are traveled in counterclockwise.*

centers (4-5), (4-6) and (4-7) we have that system (4-2) is equivalent to system

$$
\begin{aligned}
45.611514.. + x_2^2 - 2.274330..y_1 - 4.914345..y_1^2 + 0.526241..y_2 \\
+4.914345..y_2^2 + x_2(-46.611514.. + 1.748088..y_2) = 0, \\
x_2^2 - x_3^2 + x_2\left(-13.453098.. + \frac{3}{10}y_2\right) - 4.371176..y_2 \\
+6.201094..y_2^2 + x_3\left(13.453098.. - \frac{3}{10}y_3\right) + 4.371176..y_3 - 6.201094..y_3^2 = 0, \\
45.611514.. + x_3^2 + 0.526241..y_3 + 4.914345..y_3^2 + x_3(-46.611514.. \\
+1.748088..y_3) - 2.274330..y_4 - 4.914345..y_4^2 = 0, \\
(y_1 - y_4)\left(-\frac{5}{2} + 5y_1 + 5y_4\right) = 0, \\
x_2^2 + y_2^2 = 1, \quad x_3^2 + y_3^2 = 1,
\end{aligned}
\tag{4-8}
$$

Therefore the unique real solutions $q^i = (y_1^i, x_2^i, y_2^i, x_3^i, y_3^i, y_4^i)$ for system (4-8) that satisfy the condition $y_4^i < y_1^i$, are the points $q^1 = (3, \cos(\pi/2), \sin(\pi/2), \cos(-\pi/3), \sin(-\pi/3), -5/2)$; $q^2 = (17/5, \cos(3\pi/5), \sin(3\pi/5), \cos(-2\pi/5), \sin(-2\pi/5), -29/10)$; $q^3 = (3.294676.., \cos(4\pi/7), \sin(4\pi/7), 0.362651.., -0.931924.., -2.794676..)$ and $q^4 = (1.287554.., 0.814865.., 0.579649.., 0.966364.., -0.257177.., -0.787554)$, which generated four crossing limit cycles. See these crossing limit cycles of the PWLC formed by (4-5), (4-6) and (4-7) in Figure 4.2.

This completes the proof of statement (i) for the family $\mathcal{F}_{\Sigma_1}$ of Theorem G. □

Proof of statement (ii) ***of Theorem*** **G.** We consider the PWLC such that in the region $R^1_{\Sigma_{k^-}}$ it has the linear differential center

$$
\dot{x} = 0.309248.. - 0.237408..x - 0.439335..y, \qquad \dot{y} = -0.478770.. + x + 0.237408..y,
\tag{4-9}
$$

this system has the first integral $H_1(x,y) = x^2 + x(-0.957540.. + 0.474817..y) +$

$(-0.618496..+0.439335..y)y$. In the region $R^2_{\Sigma_{k^-}}$ we have the linear differential center

$$\dot{x} = 0.396090.. - 0.335276..x - 0.180370..y, \qquad \dot{y} = -0.861570.. + x + 0.335276..y, \tag{4-10}$$

which has the first integral $H_2(x,y) = x^2 + x(-1.723140.. + 0.670553..y) + (-0.792181.. + 0.180370..y)y$. In the region $R^3_{\Sigma_{k^-}}$ we have the linear differential center

$$\dot{x} = 0.242967.. + 0.112091..x - 0.194871..y, \qquad \dot{y} = 0.375114.. + x - 0.112091..y, \tag{4-11}$$

this system has the first integral $H_3(x,y) = x^2 + x(0.750229.. - 0.224182..y) + (-0.485935.. + 0.194871..y)y$. And in the region $R^4_{\Sigma_{k^-}}$ we have the linear differential center

$$\dot{x} = 0.394133.. + 0.278957..x - 0.25146..y, \qquad \dot{y} = 0.516804.. + x - 0.278957..y, \tag{4-12}$$

which has the first integral $H_4(x,y) = x^2 + x(1.033609.. - 0.557914..y) + (-0.788267.. + 0.251469..y)y$.

In order to have a crossing limit cycle of type 1, which intersects the discontinuity curve Σ_{k^-} in four different points $p_1 = (k, y_1)$, $p_2 = (x_2, y_2)$, $p_3 = (k, y_3)$ and $p_4 = (x_4, y_4)$, with $p_2, p_4 \in \mathbb{S}^1$, then these points must satisfy the system

$$\begin{aligned} H_1(k,\ y_1) &= H_1(x_2, y_2), \\ H_2(x_2, y_2) &= H_2(k,\ y_3), \\ H_3(k,\ y_3) &= H_3(x_4, y_4), \\ H_4(x_4, y_4) &= H_4(k, y_1), \\ x_2^2 + y_2^2 &= 1, \\ x_4^2 + y_4^2 &= 1, \end{aligned} \tag{4-13}$$

Considering $k = 0$ and the previous PWLC, system (4-13) is equivalent to system

$$\begin{aligned} x_2^2 + 0.618497..y_1 - 0.439336..y_1^2 + x_2(-0.957541.. + 0.474817..y_2) \\ -0.618497..y_2 + 0.439336..y_2^2 = 0, \\ 4x_2^2 - 3.168726..y_2 + 0.721481..y_2^2 + x_2(-6.892562.. + 2.682214..y_2) \\ +3.168726..y_3 - 0.721481..y_3^2 = 0, \\ x_4^2 + 0.485936..y_3 - 0.194871..y_3^2 + x_4(0.750229.. - 0.224183..y_4) \\ -0.485936..y_4 + 0.194871..y_4^2 = 0, \\ 4x_4^2 + 3.153071..y_1 - 1.005879..y_1^2 + x_4(4.134439.. - 2.231658..y_4) \\ -3.153071..y_4 + 1.005879..y_4^2 = 0, \\ x_2^2 + y_2^2 = 1, \quad x_4^2 + y_4^2 = 1. \end{aligned} \tag{4-14}$$

Therefore discontinuous PWLC formed by the linear differential centers (4-9), (4-10),

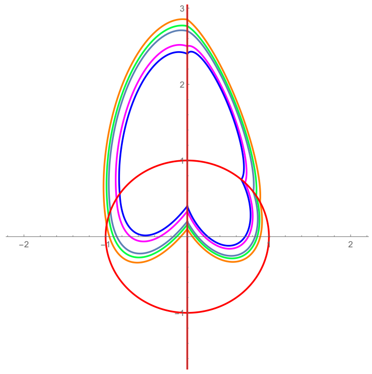

Figure 4.3: *Five crossing limit cycles of type* 1 *of the discontinuous PWLC formed by the centers (4-9), (4-10), (4-11) and (4-12). These limit cycles are traveled in counterclockwise.*

(4-11) and (4-12) has five crossing limit cycles of type 1, because system (4-14) has five real solutions $q^i = (y_1^i, x_2^i, y_2^i, y_3^i, x_4^i, y_4^i)$, for $i = 1,2,3,4,5$ that satisfy the conditions $-1 < y_1^i < 1 < y_3^i$; $x_2^i > 0$ and $x_4^i < 0$. Where $q^1 = (1/3,\ \cos(\pi/4),\ \sin(\pi/4),\ 5/2,\ \cos(5\pi/6),\ \sin(5\pi/6))$; $q^2 = (2/5,\ \cos(27\pi/10),\ \sin(27\pi/10),\ 12/5,\ \cos(81\pi/100),\ \sin(81\pi/100))$; $q^3 = (1/5,\ \cos(\pi/5),\ \sin(\pi/5),\ 27/10,\ \cos(89\pi/100),\ \sin(89\pi/100))$; $q^4 = (1/10,\ \cos(3\pi/20),\ \sin(3\pi/20),\ 57/20,\ \cos(19\pi/20),\ \sin(19\pi/20))$ and $q^5 = (0.157052..,\ 0.843891..,\ 0.536513..,\ 2.764619..,\ -0.962848..,\ 0.270041..)$. See these five crossing limit cycles of type 1 in Figure 4.3.

This completes the proof of statement (ii) of Theorem G. □

Proof of statement (iii) ***of Theorem G.*** We consider the discontinuous PWLC formed by the following linear differential centers

$$
\begin{aligned}
&\dot{x} = -0.045605.. + 0.048166..x - 0.671455..y, && \dot{y} = -0.418364.. + x - 0.048166..y;\\
&\dot{x} = 0.058276.. + \frac{x}{100} - 0.178664..y, && \dot{y} = -0.763833.. + x - \frac{y}{100};\\
&\dot{x} = \frac{901}{50000} - \frac{x}{50} - \frac{901}{2500}y, && \dot{y} = \frac{1}{10} + x + \frac{y}{50},
\end{aligned}
\tag{4-15}
$$

in the regions $R^1_{\Sigma_{k^-}}$, $R^2_{\Sigma_{k^-}}$ and $R^4_{\Sigma_{k^-}}$, respectively. The linear differential centers in (4-15) have the first integrals

$$
\begin{aligned}
H_1(x,y) &= x^2 + x(-0.836729.. - 0.096332..y) + (0.091210.. + 0.671455..y)y,\\
H_2(x,y) &= x^2 + x\left(-1.527667.. - \frac{y}{50}\right) + (-0.116553.. + 0.178664..y)y,\\
H_4(x,y) &= 4x^2 + \frac{4}{25}x(5+y) + \frac{901y(-1+10y)}{6250},
\end{aligned}
$$

respectively. In order to have a crossing limit cycle of type 2^+, which intersects Σ_k in four different points $p_1 = (x_1, y_1)$, $p_2 = (k, y_2)$, $p_3 = (k, y_3)$ and $p_4 = (x_4, y_4)$, with $p_1, p_4 \in \mathbb{S}^1$ and $0 \le k < 1$, these points must satisfy the system

$$\begin{aligned} H_1(x_1,y_1) &= H_1(k,y_2),\\ H_4(k,y_2) &= H_4(k,y_3),\\ H_1(k,y_3) &= H_1(x_4,y_4),\\ H_2(x_4,y_4) &= H_2(x_1,y_1),\\ x_1^2+y_1^2 &= 1,\\ x_4^2+y_4^2 &= 1, \end{aligned} \tag{4-16}$$

Then for the PWLC (4-15) we have that system (4-16) becomes

$$\begin{aligned} 4x_1^2 + x_1(-3.346917.. - 0.385331..y_1) + y_1(0.364840.. + 2.685822..y_1)&\\ +(-0.364840.. - 2.685822..y_2)y_2 &= 0,\\ (y_2-y_3)\left(-\frac{901}{6250} + \frac{901}{625}(y_2+y_3)\right) &= 0,\\ -4x_4^2 + y_3(0.364840.. + 2.685822..y_3) + x_4(3.346917.. + 0.385331..y_4)&\\ +(-0.364840.. - 2.685822..y_4)y_4 &= 0,\\ -4x_1^2 + 4x_4^2 + x_1\left(6.110671.. + \frac{2}{25}y_1\right) + (0.466214.. - 0.714659..y_1)y_1&\\ +x_4\left(-6.110671.. - \frac{2}{25}y_4\right) + (-0.466214.. + 0.714659..y_4)y_4 &= 0,\\ x_1^2+y_1^2 = 1, \quad x_4^2+y_4^2 &= 1, \end{aligned} \tag{4-17}$$

where $k = 0$. Therefore the unique real solutions $q^i = (x_1^i, y_1^i, y_2^i, y_3^i, x_4^i, y_4^i)$ for sys-

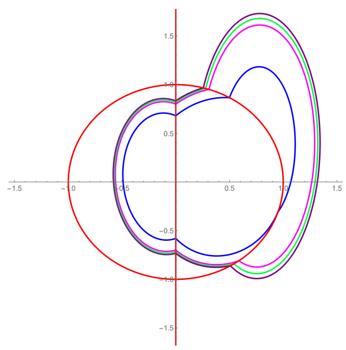

Figure 4.4: *Four crossing limit cycles of type 2^+ of the discontinuous PWLC (4-15). These limit cycles are traveled in counterclockwise.*

tem (4-17) that satisfy the conditions $-1 < y_3^i < y_2^i < 1$; $x_1^i > 0$ and $x_4^i > 0$ are the points $q^1 = (\cos(2\pi/5),\ \sin(2\pi/5),\ 8/10,\ -7/10,\ \cos(-3\pi/10),\ \sin(-3\pi/10))$; $q^2 = (\cos(\pi/3),\ \sin(\pi/3),\ 17/25,\ -29/50,\ \cos(-\pi/10),\ \sin(-\pi/10))$; $q^3 =$

$(\cos(41\pi/100),\ \sin(41\pi/100),\ 0.819235..,\ -0.719235..,\ 0.541860..,\ -0.840468..)$ and $q^4 = (0.256532.., 0.966535.., 0.833667.., -0.733667.., 0.508672.., -0.860960..)$. These four real solutions generated four crossing limit cycles of type 2^+. See these crossing limit cycles of the PWLC (4-15) in Figure 4.4.

This completes the proof of statement (iii) of Theorem G. □

Proof of statement (iv) ***of Theorem G.*** We consider the following discontinuous PWLC formed by the linear differential centers

$$\begin{aligned}
\dot{x} &= 1.018312.. + \frac{51}{40}x + 9.463668..y, & \dot{y} &= -5.008011.. - x - \frac{51}{40}y;\\
\dot{x} &= 0.712799.. - 0.278320..x - 0.250791..y, & \dot{y} &= -1.026464.. + x + 0.278320..y;\\
\dot{x} &= \frac{969}{1280} + \frac{x}{8} - \frac{17}{64}y, & \dot{y} &= \frac{1}{8} + x - \frac{y}{8},
\end{aligned} \tag{4-18}$$

in the regions $R^1_{\Sigma_{k^-}}$, $R^2_{\Sigma_{k^-}}$ and $R^3_{\Sigma_{k^-}}$, respectively. The linear differential centers in (4-18) have the first integrals

$$\begin{aligned}
H_1(x,y) =& 4x^2 + x\left(40.064090.. + \frac{51}{5}y\right) + y(8.146500.. + 37.854675..y),\\
H_2(x,y) =& x^2 + x(-2.052928.. + 0.556641..y) + (-1.425599.. + 0.250791..y)y,\\
H_3(x,y) =& x + 4x^2 - xy + \frac{17}{160}y(-57 + 10y).
\end{aligned}$$

respectively. In order to have a crossing limit cycle of type 3^+, which intersects the

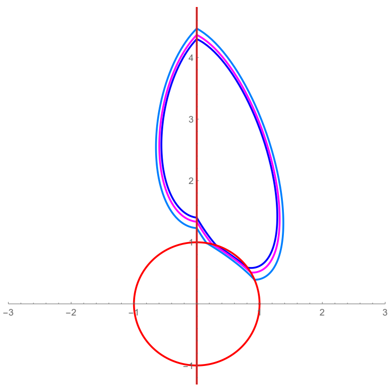

Figure 4.5: *Three crossing limit cycles of type* 3^+ *of the discontinuous PWLC (4-18). These limit cycles are traveled in counterclockwise.*

discontinuity curve Σ_k in four different points $p_1 = (k, y_1)$, $p_2 = (x_2, y_2)$, $p_3 = (k, y_3)$ and $p_4 = (x_4, y_4)$ with $0 \le k < 1$ and $p_2, p_4 \in \mathbb{S}^1$, these points must satisfy the system

$$
\begin{aligned}
H_2(x_1,y_1) = &\ H_2(k,\ y_2),\\
H_3(k,\ y_2) = &\ H_3(k,\ y_3),\\
H_2(k,\ y_3) = &\ H_2(x_4,y_4),\\
H_1(x_4,y_4) = &\ H_1(x_1,y_1),\\
x_1^2+y_1^2 = &\ 1,\\
x_4^2+y_4^2 = &\ 1,
\end{aligned}
\tag{4-19}
$$

Considering $k=0$, system (4-19) is equivalent to system

$$
\begin{aligned}
4x_1^2+y_1(-5.702397..+1.003165..y_1)+x_1(-8.211712..+2.226564..y_1)&\\
+5.702397..y_2-1.003165..y_2^2=0&,\\
(y_2-y_3)(-57+10y_2+10y_3)=0&,\\
x_4^2+(1.425599..-0.250791..y_3)y_3+x_4(-2.052928..+0.556641..y_4)&\\
-1.425599..y_4+0.250791..y_4^2=0&,\\
x_1^2-x_4^2+x_1\left(10.016022..+\frac{51}{20}y_1\right)+y_1(2.036625..+9.463668..y_1)&\\
+x_4\left(-10.016022..-\frac{51}{20}y_4\right)+(-2.036625..-9.463668..y_4)y_4=0&,\\
x_1^2+y_1^2=1,\quad x_4^2+y_4^2=1&.
\end{aligned}
\tag{4-20}
$$

Therefore discontinuous PWLC (4-18) has three crossing limit cycles of type 3^+, because system (4-20) has three real solutions $q^i=(x_1^i,y_1^i,y_2^i,y_3^i,x_4^i,y_4^i)$, for $i=1,2,3$ that satisfy the conditions $0<x_4^i<x_1^i$ and $1<y_3^i<y_2^i$. Where $q^1=(\cos(\pi/5)$, $\sin(\pi/5)$, $43/10$, $7/5$ $\cos(2\pi/5)$, $\sin(2\pi/5))$; $q^2=(\cos(16\pi/125)$, $\sin(16\pi/125)$, $447/100$, $123/100$, $\cos(9\pi/50)$, $\cos(9\pi/50))$ and $q^3=(\cos(17\pi/100)$, $\sin(17\pi/100)$, $4.366812..$, $1.333187..$, $0.242211..$, $0.970223..)$. See these three crossing limit cycles of type 3^+ in Figure 4.5.

This completes the proof of statement (iv) of Theorem *G*. □

4.5.2 Proof of Theorem *H*

Proof of statement (i) ***of Theorem H.*** We consider the following PWLC formed by the linear differential centers

$$
\begin{aligned}
&\dot{x}=-124.644504..+\frac{111}{50}x-6.045715..y, && \dot{y}=-148.901657..+x-\frac{111}{50}y;\\
&\dot{x}=0.236087..+0.003662..x-0.009243..y, && \dot{y}=-0.402647..+x-0.003662..y;\\
&\dot{x}=1+\frac{x}{5}-0.102500..y, && \dot{y}=-\frac{9}{20}+x-\frac{y}{5},
\end{aligned}
\tag{4-21}
$$

in the regions $R^1_{\tilde{\Sigma}_{k-}}$, $R^2_{\tilde{\Sigma}_{k-}}$ and $R^3_{\tilde{\Sigma}_{k-}}$, respectively. The linear differential centers in (4-21) have the first integrals

$$H_1(x,y) = x^2 + x\left(-297.803314.. - \frac{111}{25}y\right) + y(249.289008.. + 6.045715..y),$$
$$H_2(x,y) = x^2 + x(-0.805295.. - 0.007324..y) + (-0.472175.. + 0.009243..y)y,$$
$$H_3(x,y) = x^2 + x\left(-\frac{9}{10} - \frac{2}{5}y\right) + (-2 + 0.102500..y)y,$$

respectively.

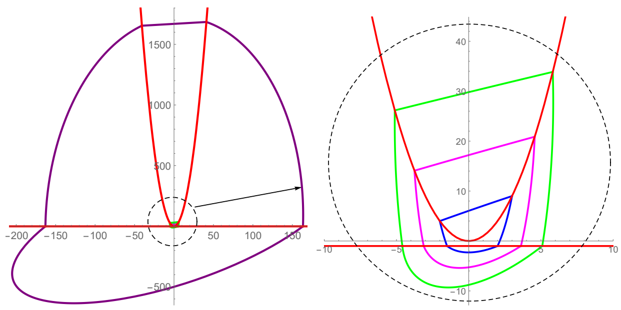

Figure 4.6: *Four crossing limit cycles of the discontinuous PWLC (4-21). These limit cycles are traveled in counterclockwise.*

For PWLC in the family $\mathcal{F}_{\tilde{\Sigma}_{k-}}$ we have crossing limit cycles which intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, x_2^2)$, $p_3 = (x_3, k)$ and $p_4 = (x_4, k)$ with $k < 0$, if these points satisfy the system

$$\begin{aligned} H_1(x_1,x_1^2) &= H_1(x_2,x_2^2),\\ H_2(x_2,x_2^2) &= H_2(x_3,k),\\ H_3(x_3,\ k) &= H_3(x_4,k),\\ H_2(x_4,\ k) &= H_2(x_1,x_1^2). \end{aligned} \tag{4-22}$$

Then for the PWLC (4-21) and $\tilde{\Sigma}_{k-}$ considering $k = -1$, system (4-22) becomes

$$\begin{aligned} x_1(-1191.213259.. + x_1(1001.156032.. + x_1(-44425 + 24.182863..x_1)))\\ +x_2(1191.213259.. + x_2(-1001.156032.. + (-44425 - 24.182863..x_2)x_2)) = 0,\\ -1.925675.. + x_2(-3.221182.. + x_2(2.111297.. + (-0.029297..\\ +0.036973..x_2)x_2)) + (3.191885.. - 4x_3)x_3 = 0,\\ (x_3 - x_4)\left(-\frac{1}{2} + x_3 + x_4\right) = 0,\\ 1.925675.. + x_1(3.221182.. + x_1(-2.111297.. + (0.029297..\\ -0.036973..x_1)x_1)) + x_4(-3.191885.. + 4x_4) = 0. \end{aligned} \tag{4-23}$$

Taking into account that the solutions (x_1,x_2,x_3,x_4) must satisfy $x_2 < x_1$ and $x_3 < x_4$, system (4-23) has four real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i)$, with $i = 1,2,3,4$. Namely, $q^1 = (3,\ -2,\ -3/2,\ 2)$; $q^2 = (457/100,\ -3.753677..,\ -3.116713..,\ 3.616713..)$; $q^3 = (5.820000..,\ -5.115260..,\ -4.592690..,\ 5.092690..)$ and $q^4 = (\ 41.045251..,\ -40.667957..,\ -162.945374..,\ 163.445374..)$. Which provide four crossing limit cycles of the PWLC (4-21). See these four crossing limit cycles in Figure 4.6.

Here we observe that there is a duality between the crossing limit cycles that intersect the discontinuity curve $\tilde{\Sigma}_{-1}$ and the crossing limit cycles that intersect the discontinuity curve Σ_2 for the family $\mathcal{F}_{\Sigma_{k+}}$ studied in statement (i) of Theorem *G*, where we also got four crossing limit cycles, see Figures 4.1 and 4.6.

This completes the proof of statement (i) of Theorem *H*. □

Proof of statement (ii) ***of Theorem H.*** We consider the following PWLC

$$\begin{aligned} &\dot{x} = \frac{11}{10} + \frac{4}{5}x - \frac{4}{5}y, && \dot{y} = 1 + x - \frac{4}{5}y, \text{ in } R^1_{\tilde{\Sigma}_0},\\ &\dot{x} = \frac{17}{75} - \frac{3}{10}x - \frac{17}{150}y, && \dot{y} = -\frac{61}{20} + x + \frac{3}{10}y, \text{ in } R^2_{\tilde{\Sigma}_0},\\ &\dot{x} = \frac{1}{6} + x - \frac{25}{16}y, && \dot{y} = -\frac{1}{4} + x - y, \text{ in } R^3_{\tilde{\Sigma}_0},\\ &\dot{x} = \frac{133}{36} + \frac{x}{10} - \frac{7}{45}y, && \dot{y} = \frac{543}{20} + x - \frac{y}{10}, \text{ in } R^4_{\tilde{\Sigma}_0}. \end{aligned} \tag{4-24}$$

The linear differential centers in (4-24) have the first integrals

$$\begin{aligned} H_1(x,y) &= 5x(2+x) - (11+8x)y + 4y^2,\\ H_2(x,y) &= 150x^2 + 17(-4+y)y + 15x(-61+6y),\\ H_3(x,y) &= 4x^2 - 2x(1+4y) + \frac{y}{12}(-16+75y),\\ H_4(x,y) &= 90x^2 + 9x(543-2y) + 7y(-95+2y), \end{aligned}$$

respectively.

In order to have a crossing limit cycle of type 4, which intersects the discontinuity curve $\tilde{\Sigma}_0$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, x_2^2)$, $p_3 = (x_3, 0)$ and $p_4 = (x_4, 0)$, these points must satisfy system

$$\begin{aligned} H_1(x_1,x_1^2) &= H_1(x_2,x_2^2),\\ H_2(x_2,x_2^2) &= H_2(x_3,0),\\ H_3(x_3,\ 0) &= H_3(x_4,0),\\ H_4(x_4,\ 0) &= H_4(x_1,x_1^2). \end{aligned} \tag{4-25}$$

Considering the PWLC (4-24) system (4-25) becomes

$$\begin{aligned}(x_1-x_2)(-1+x_1+x_2)(-5+2(-1+x_1)x_1+2(-1+x_2)x_2)&=0,\\ 2x_2(-915+x_2(82+x_2(90+17x_2)))+30(61-10x_3)x_3&=0,\\ 4(x_3-x_4)\left(-\frac{1}{2}+x_3+x_4\right)&=0,\\ 2x_1(-4887+x_1(575+2(9-7x_1)x_1))+18x_4(543+10x_4)&=0.\end{aligned} \tag{4-26}$$

In this case we have that the solutions $q^i=(x_1^i,x_2^i,x_3^i,x_4^i)$ must satisfy $x_2^i<0<x_1^i$ and $x_3^i<0<x_4^i$ then we have four real solutions $q^1=(3,\ -2,\ -3/2,2)$; $q^2=(4,\ -3,\ -2,\ 5/2)$; $q^3=(5,\ -4,\ 27/10,\ 16/5)$ and $q^4=(10.440607..,-9.440607..,\ -19.555603..,\ 20.055606..)$ of system (4-26), which provide four crossing limit cycles of type 4 of the PWLC (4-24). See these four crossing limit cycles in Figure 4.7.

Here we observe that there is a duality between the crossing limit cycles of type 4 that intersect the discontinuity curve $\tilde{\Sigma}_0$ and the crossing limit cycles that intersect the discontinuity curve Σ_1 for the family $\mathcal{F}_{\Sigma_1}$ studied in statement (i) of Theorem *G*, where we also got four crossing limit cycles, see Figures 4.2 and 4.7.

This completes the proof of statement (ii) of Theorem *H*. □

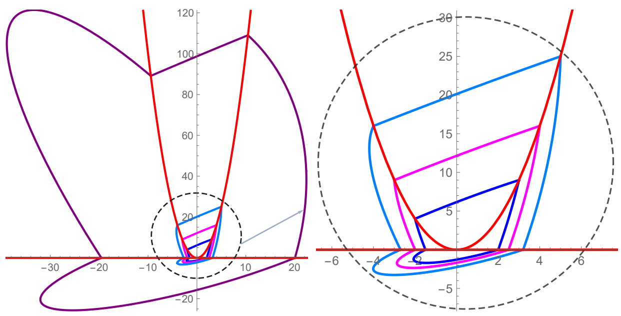

Figure 4.7: *Four crossing limit cycles of type 4 of the discontinuous PWLC (4-24). These limit cycles are traveled in counterclockwise.*

Proof of statement (iii) ***of Theorem H.*** In this case we consider the following PWLC formed by the linear differential centers

$$\begin{aligned}&\dot{x}=0.100318..-\frac{2}{5}x+0.161744..y && \dot{y}=0.260062..-x+\frac{2}{5}y;\\ &\dot{x}=1-x-\frac{13}{4}y, && \dot{y}=-\frac{31}{30}+x+y;\\ &\dot{x}=-0.399222..+0.378090..x-0.144616..y, && \dot{y}=-1.020635..+x-0.378090..y,\end{aligned} \tag{4-27}$$

in the regions $R^1_{\tilde{\Sigma}_0}$, $R^3_{\tilde{\Sigma}_0}$ and $R^4_{\tilde{\Sigma}_0}$, respectively. The linear differential centers in (4-27) have the first integrals

$$H_1(x,y)=x^2+x\left(-0.520124..-\frac{4}{5}y\right)+(0.200636..+0.161744..y)y,$$
$$H_3(x,y)=-\frac{124}{15}x-8y+9y^2+4(x+y)^2,$$
$$H_4(x,y)=4(x-0.378090..y)^2+8(-1.020635..x+0.399222..y)+0.006657..y^2,$$

respectively. In order to have a crossing limit cycle of type 5, which intersects the discontinuity curve $\tilde{\Sigma}_0$ in four different points $p_1=(x_1,x_1^2)$, $p_2=(x_2,x_2^2)$, $p_3=(x_3,0)$ and $p_4=(x_4,0)$, with $0<x_2<x_1$ and $0<x_3<x_4$, these points must satisfy system

$$\begin{aligned} H_1(x_1,x_1^2)= &\ H_1(x_2,x_2^2),\\ H_4(x_2,x_2^2)= &\ H_4(x_3,0),\\ H_3(x_3,\ 0)= &\ H_3(x_4,0),\\ H_4(x_4,\ 0)= &\ H_4(x_1,x_1^2). \end{aligned} \tag{4-28}$$

Considering the PWLC (4-27) system (4-28) becomes

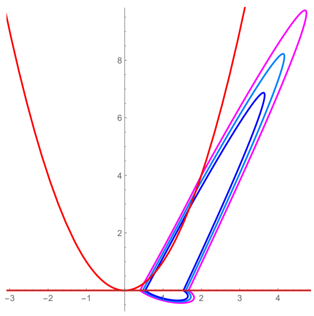

Figure 4.8: *Three crossing limit cycles of type 5 of the discontinuous PWLC (4-27). These limit cycles are traveled in counterclockwise.*

$$\begin{aligned} -2.080498..x_1+4.802546..x_1^2-3.199999..x_1^3+0.646977..x_1^4\\ +x_2(2.080498..-4.802546..x_2+3.199999..x_2^2-0.646977..x_2^3)=0,\\ x_2(-2591625737556+x_2(2283329836763+50x_2(-19201143493\\ +3672147700x_2)))-324x_3(-7998844869+3918560960x_3)=0,\\ 4(x_3-x_4)\left(-\frac{31}{15}+x_3+x_4\right)=0,\\ x_1(2591625737556+x_1(-2283329836763+50(19201143493\\ -3672147700x_1)x_1))+324x_4(-7998844869+3918560960x_4)=0. \end{aligned} \tag{4-29}$$

In this case system (4-29) has three real solutions $q^i=(x_1^i,x_2^i,x_3^i,x_4^i)$, where $q^1=(2,$

$1/2,\ 2/5,\ 5/3)$; $q^2 = (93/50,\ 63/100,\ 47/100,\ 479/300)$ and $q^3 = (17/10,\ 0.785691..,\ 0.534387..,\ 1.532279..)$ which provide three crossing limit cycles of type 5 of the PWLC (4-27). See these three crossing limit cycles in Figure 4.8.

This completes the proof of statement (iii) of Theorem H. $\square$

Proof of statement (iv) ***of Theorem*** H**.** We consider the following PWLC

$$
\begin{aligned}
\dot{x} &= -0.678037.. + 0.111302..x - 0.025436..y, & \dot{y} &= -3.106005.. + x - 0.111302..y;\\
\dot{x} &= -0.133244.. + 0.232759..x - 0.058573..y, & \dot{y} &= -0.290609.. + x - 0.232759..y;\\
\dot{x} &= 3.074032.. + 0.434135..x - 2.713559..y, & \dot{y} &= -3.035258.. + x - 0.434135..y;\\
\dot{x} &= 1.427543.. + 0.059092..x - 0.651180..y, & \dot{y} &= -1.450367.. + x - 0.059092..y,
\end{aligned}
\tag{4-30}
$$

in the regions $R^1_{\tilde{\Sigma}_{k+}}$, $R^2_{\tilde{\Sigma}_{k+}}$, $R^4_{\tilde{\Sigma}_{k+}}$ and $R^5_{\tilde{\Sigma}_{k+}}$, respectively. The linear differential centers in

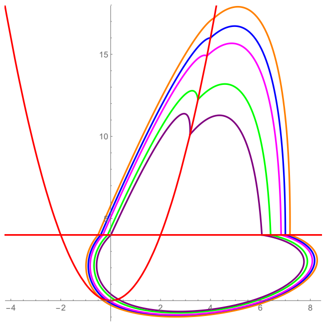

Figure 4.9: *Five crossing limit cycles of type 6^+ of the discontinuous PWLC (4-30). These limit cycles are traveled in counterclockwise.*

(4-30) have the first integrals

$$
\begin{aligned}
H_1(x,y) =& x^2 + x(-6.212010.. - 0.222604..y) + (1.356074.. + 0.025436..y)y,\\
H_2(x,y) =& x^2 + x(-0.581218.. - 0.465518..y) + (0.266488.. + 0.058573..y)y,\\
H_4(x,y) =& x^2 + x(-6.070516.. - 0.868271..y) + y(-6.148064.. + 2.713559..y),\\
H_5(x,y) =& x^2 + x(-2.900734.. - 0.118185..y) + (-2.855087.. + 0.651180..y)y,
\end{aligned}
$$

respectively. In order to have a crossing limit cycle of type 6^+, which intersects the discontinuity curve $\tilde{\Sigma}_{k+}$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, k)$, $p_3 = (x_3, x_3^2)$

and $p_4=(x_4,k)$, these points must satisfy system

$$\begin{aligned} H_1(x_4,\ k)=&\ H_1(x_1,x_1^2),\\ H_2(x_1,x_1^2)=&\ H_2(x_2,k),\\ H_5(x_2,\ k)=&\ H_5(x_3,x_3^2),\\ H_4(x_3,x_3^2)=&\ H_4(x_4,k). \end{aligned} \tag{4-31}$$

Considering PWLC (4-30) and $k=4$, system (4-31) becomes

$$\begin{aligned} -8.012495..+x_1(-2.324875..+x_1(5.065954..+(-1.862075..\\ +0.234292..x_1)x_1))+(9.773178..-3.999999..x_2)x_2=0,\\ -1.001459..+(-3.373476..+x_2)x_2+x_3(2.900734..+x_3(1.855087..\\ +(0.118185..-0.651180..x_3)x_3))=0,\\ -75.298768..+x_3(-24.282066..+x_3(-20.592258..\\ +x_3(-3.473086..+10.854237..x_3)))+(38.174413..-4x_4)x_4=0,\\ 23.325149..+x_1(24.848040..+x_1(-9.424297..+(0.890418..\\ -0.101747..x_1)x_1))+x_4(-28.409714..+4x_4)=0, \end{aligned} \tag{4-32}$$

In this case system (4-32) has five real solutions $q^i=(x_1^i,x_2^i,x_3^i,x_4^i)$ that satisfy the conditions $-2<x_2^i<2<x_1^i$ and $-2<x_3^i<2<x_4^i$. We have $q^1=(4,\ -2/5,\ -1/5,\ 7)$; $q^2=(193/50,\ -31/100,\ -1/20,\ 683/100)$; $q^3=(7/2,\ -3/25,\ 9/50,\ 641/100)$; $q^4=(159/50,\ 1/100,\ 3/10,\ 303/50)$ and $q^5=(4.149236..,\ -0.507154..,\ -0.449658..,\ 7.185104..)$, which provide five crossing limit cycles of type 6^+ of the PWLC (4-30). See these crossing limit cycles in Figure 4.9.

Here we observe that there is a duality between the crossing limit cycles of type 6^+ that intersect the discontinuity curve $\tilde{\Sigma}_{k-}$ and the crossing limit cycles of type 1 for the family $\mathcal{F}_{\Sigma_{k-}}$ that intersect the discontinuity curve Σ_0 studied in statement (ii) of Theorem *G*, where we also got five crossing limit cycles, see Figures 4.3 and 4.9.

This completes the proof of statement (iv) of Theorem *H*. $\square$

Proof of statement (v) ***of Theorem H.*** We consider the following PWLC

$$\begin{aligned} &\dot{x}=3+\frac{x}{4}-\frac{17}{16}y, && \dot{y}=\frac{21}{20}+x-\frac{y}{4},\ \text{in } R^2_{\tilde{\Sigma}_{k+}},\\ &\dot{x}=3.601959..-x-5.323060..y, && \dot{y}=-\frac{36}{25}+x+y,\ \text{in } R^4_{\tilde{\Sigma}_{k+}},\\ &\dot{x}=\frac{11827667}{24434928}-\frac{91445}{6205696}x-\frac{8433175}{97739712}y, && \dot{y}=\frac{26369}{1108160}+x+\frac{91445}{6205696}y,\ \text{in } R^5_{\tilde{\Sigma}_{k+}}. \end{aligned} \tag{4-33}$$

The linear differential centers in (4-33) have the first integrals

$$\begin{aligned}
H_2(x,y) &= \frac{2}{5}x(21+10x) - 2(12+x)y + \frac{17}{4}y^2,\\
H_4(x,y) &= x^2 + x\left(-\frac{72}{25} + 2y\right) + y(-7.203918.. + 5.323060..y),\\
H_5(x,y) &= 977397120x^2 + 63x(738332 + 457225y) + 10y(-94621336 + 8433175y),
\end{aligned}$$

respectively. In order to have a crossing limit cycle of type 7, which intersects the discontinuity curve $\tilde{\Sigma}_{k^+}$ in four different points $p_1 = (x_1, k)$, $p_2 = (x_2, k)$, $p_3 = (x_3, x_3^2)$ and $p_4 = (x_4, x_4^2)$, these points must satisfy system

$$\begin{aligned}
H_2(x_1,\ k) &= H_2(x_2,\ k),\\
H_5(x_2,\ k) &= H_5(x_3, x_3^2),\\
H_4(x_3, x_3^2) &= H_4(x_4, x_4^2),\\
H_5(x_4, x_4^2) &= H_5(x_1,\ k).
\end{aligned} \tag{4-34}$$

In this case considering $k = 4$, system (4-34) becomes

$$\begin{aligned}
&4(x_1 - x_2)\left(\frac{1}{10} + x_1 + x_2\right) = 0,\\
&-2435545440 + 4032x_2(40113 + 242410x_2) - x_3(46514916 + 5x_3(6236752\\
&\qquad + 5x_3(1152207 + 3373270x_3))) = 0,\\
&x_3\left(-\frac{288}{25} + x_3(-24.815674.. + x_3(8 + 21.292240..x_3))\right)\\
&\qquad + x_4\left(-\frac{288}{25} + x_4(24.815674.. + (-8 - 21.292240..x_4)x_4)\right) = 0,\\
&2435545440 - 4032x_1(40113 + 242410x_1) + x_4(46514916 + 5x_4(6236752\\
&\qquad + 5x_4(1152207 + 3373270x_4))) = 0.
\end{aligned} \tag{4-35}$$

System (4-35) has three real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i)$ that satisfy the conditions $-2 < x_2^i < x_1^i < 2$ and $-2 < x_3^i < x_4^i < 2$. They are $q^1 = (17/10, -9/5, -8/5, 3/2)$; $q^2 = (8/5, -17/10, -6/5, 6/5)$ and $q^3 = (89/50, -47/25, -1.788665.., 1.667136..)$, which provide three crossing limit cycles of type 7 of the PWLC (4-33). See these three crossing limit cycles in Figure 4.10.

This completes the proof of statement (v) of Theorem H. □

Proof of statement (vi) ***of Theorem*** H**.** We consider the following PWLC formed by the

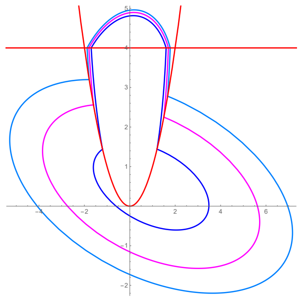

Figure 4.10: *Three crossing limit cycles of type 7 of the discontinuous PWLC (4-33). These limit cycles are traveled in counterclockwise.*

linear differential centers

$$
\begin{aligned}
\dot{x} &= -0.228658.. + 0.153388..x - 0.043263..y, & \dot{y} &= -1.233713.. + x - 0.153388..y;\\
\dot{x} &= \frac{52}{5} + x - 5y, & \dot{y} &= 2 + x - y;\\
\dot{x} &= -0.208786.. - 0.135584..x - 0.040106..y, & \dot{y} &= 1.549735 + x + 0.135584..y;\\
\dot{x} &= 2 - \frac{x}{2} - \frac{5}{4}y, & \dot{y} &= -\frac{41}{20} + x + \frac{y}{2},
\end{aligned}
\tag{4-36}
$$

in the regions $R^1_{\tilde{\Sigma}_{k+}}$, $R^2_{\tilde{\Sigma}_{k+}}$, $R^3_{\tilde{\Sigma}_{k+}}$ and $R^4_{\tilde{\Sigma}_{k+}}$, respectively. The linear differential centers in (4-36) have the first integrals

$$
\begin{aligned}
H_1(x,y) =& 15298879995x^2 + 5y(1399284923 + 132375500y) - 6x(6291478429\\
&+ 782226050y),\\
H_2(x,y) =& 4x(4+x) - \frac{8}{5}(52+5x)y + 20y^2,\\
H_3(x,y) =& 57070082030x^2 + 15y(1588730299 + 152593500y) + x(176887019081\\
&+ 15475638300y),\\
H_4(x,y) =& 4x^2 + x\left(-\frac{82}{5} + 4y\right) + y(-16+5y),
\end{aligned}
$$

respectively. In order to have a crossing limit cycle of type 8, which intersects the discontinuity curve $\tilde{\Sigma}_{k+}$ in four different points $p_1 = (x_1, k)$, $p_2 = (x_2, x_2^2)$, $p_3 = (x_3, x_3^2)$

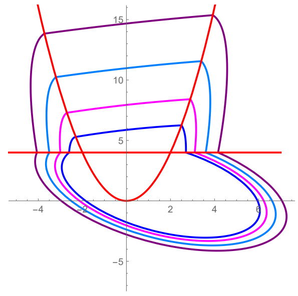

Figure 4.11: *Four crossing limit cycles of type 8 of the discontinuous PWLC (4-36). These limit cycles are traveled in counterclockwise.*

and $p_4 = (x_4, k)$, these points must satisfy system

$$\begin{aligned} H_2(x_1, x_1^2) =& \; H_2(x_2, x_2^2), \\ H_3(x_2, x_2^2) =& \; H_3(x_3,\; k), \\ H_4(x_3,\; k) =& \; H_4(x_4,\; k), \\ H_1(x_4,\; k) =& \; H_1(x_1, x_1^2). \end{aligned} \tag{4-37}$$

In this case considering $k = 4$, system (4-37) becomes

$$\begin{aligned} (x_1 - x_2)(-1 + 5x_1 + 5x_2)(-20 + x_1(-1 + 5x_1) + x_2(-1 + 5x_2)) = 0, \\ \frac{2}{28535041015}(-131946257940 + x_2(176887019081 + 5x_2(16180207303 \\ +60x_2(51585461 + 7629675x_2)))) - \frac{854345518x_3}{51046585} - 4x_3^2 = 0, \\ 4(x_3 - x_4)\left(-\frac{1}{10} + x_3 + x_4\right) = 0, \\ \frac{8}{15298879995}(19287869230 + x_1(18874435287 - 5x_1(2229530461 \\ +10x_1(-46933563 + 6618775x_1)))) - \frac{1196239064x_4}{80946455} + 4x_4^2 = 0. \end{aligned} \tag{4-38}$$

System (4-38) has four real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i)$ that satisfy the conditions $x_3^i < -2 < 2 < x_4^i$ and $x_2^i < -2 < 2 < x_1^i$. They are $q^1 = (5/2, -23/10, -13/5, 27/10)$; $q^2 = (29/10,\; -27/10,\; -3,\; 31/10)$; $q^3 = (17/5,\; -16/5,\; -7/2, 18/5)$ and $q^4 = (98/25,\; -93/25, -203/50, 104/25)$ which provide four crossing limit cycles of type 8 of the PWLC (4-36). See these four crossing limit cycles in Figure 4.11.

Here we observe that there is a duality between the crossing limit cycles for family $\mathcal{F}_{\tilde{\Sigma}_{k-}}$ studied in statement (i) Theorem H, the crossing limit cycles of type 4 for the family $\mathcal{F}_{\tilde{\Sigma}_0}$ studied in statement (ii) of Theorem H and crossing limit cycles of type 8 for the family $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ studied in statement (vi) of Theorem H. In these three cases we got four crossing limit cycles. See Figures 4.6, 4.7 and 4.11.

This completes the proof of statement (vi) of Theorem H. □

Proof of statement (vii) ***of Theorem*** H**.** We consider the following PWLC

$$
\begin{aligned}
&\dot{x} = \frac{243469}{1620885} + \frac{1826}{77185}x - \frac{9088}{324177}y, && \dot{y} = -\frac{614289}{154370} + x - \frac{1826}{77185}y, \text{ in } R^1_{\tilde{\Sigma}_{k+}},\\
&\dot{x} = -0.229652.. + \frac{7}{5}x - 0.020472..y, && \dot{y} = -1.718896.. + x - \frac{7}{5}y, \text{ in } R^2_{\tilde{\Sigma}_{k+}},\\
&\dot{x} = 1 + \frac{9}{10}x - \frac{53}{50}y, && \dot{y} = -\frac{1}{2} + x - \frac{9}{10}y, \text{ in } R^4_{\tilde{\Sigma}_{k+}}.
\end{aligned}
\tag{4-39}
$$

The linear differential centers in (4-39) have the first integrals

$$
\begin{aligned}
H_1(x,y) =& 21x(-614289 + 77185x) - 2(243469 + 38346x)y + 45440y^2,\\
H_2(x,y) =& x^2 + x\left(-3.437793.. - \frac{14}{5}y\right) + (0.459305.. + 0.020472..y)y,\\
H_4(x,y) =& 4\left(x - \frac{9}{10}y\right)^2 + y^2 - 4(x + 2y),
\end{aligned}
$$

respectively.

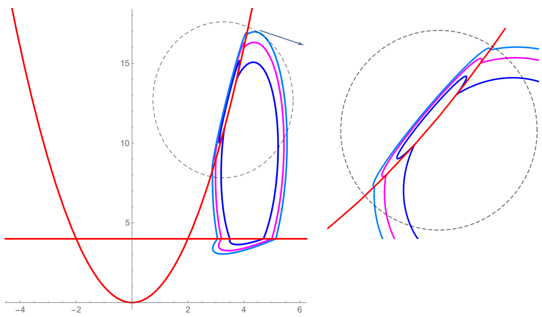

Figure 4.12: *Three crossing limit cycles of type* 9^+ *of the discontinuous PWLC (4-39). These limit cycles are traveled in counterclockwise.*

In order to have a crossing limit cycle of type 9^+, which intersects the discontinuity curve $\tilde{\Sigma}_{k+}$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, x_2^2)$, $p_3 = (x_3, k)$ and $p_4 = (x_4, k)$, these points must satisfy system

$$
\begin{aligned}
H_2(x_1,x_1^2) =& \; H_2(x_2,x_2^2),\\
H_1(x_2,x_2^2) =& \; H_1(x_3,\ k),\\
H_4(x_3,\ k) =& \; H_4(x_4,\ k),\\
H_1(x_4,\ k) =& \; H_1(x_1,x_1^2).
\end{aligned}
\tag{4-40}
$$

Considering $k=4$, system (4-40) becomes

$$\begin{aligned} x_1\left(-13.751172..+x_1\left(5.837222..+\left(-\frac{23}{25}+0.081889..x_1\right)x_1\right)\right)\\ +x_2\left(13.751172..+x_2\left(-5.837222..+\left(-\frac{23}{25}-0.081889..x_2\right)x_2\right)\right)=0,\\ x_2(12900069-x_2(1133947-76692x_2+45440x_2^2))-3(406904+7(628897\\ -77185x_3)x_3)=0,\\ 4(x_3-x_4)\left(-\frac{41}{5}+x_3+x_4\right)=0,\\ x_1(12900069-x_1(1133947-76692x_1+45440x_1^2))-3(406904+7(628897\\ -77185x_4)x_4)=0,\end{aligned} \tag{4-41}$$

And we have that system (4-41) has three real solutions $q^i=(x_1^i,x_2^i,x_3^i,x_4^i)$ that satisfy the conditions $2<x_2^i<x_1^i$ and $2<x_3^i<x_4^i$. They are $q^1=(4,3,16/5,5)$; $q^2=(15/4,33/10,7/2,47/10)$ and $q^3=(41/10,2.879320..,3.058075..,5.141924..)$ which provide three crossing limit cycles of type 9^+ of the PWLC (4-39). See these three crossing limit cycles in Figure 4.12.

Here we observe that there is a duality between the crossing limit cycles of type 3^+ for family $\mathcal{F}_{\Sigma_{k-}}$ studied in statement (iv) of Theorem G, the crossing limit cycles of type 5 for the family $\mathcal{F}_{\tilde{\Sigma}_0}$ studied in statement (iii) of Theorem H and crossing limit cycles of type 9^+ for the family $\mathcal{F}_{\tilde{\Sigma}_{k+}}$ studied in statement (vii) of Theorem H. In these three cases we got three crossing limit cycles. See Figures 4.5, 4.8 and 4.12.

This completes the proof of statement (vii) of Theorem *H*. □

4.5.3 Proof of Theorem *I*

Proof of statement (i) ***of Theorem*** ***I*.** We consider the following discontinuous PWLC formed by the linear differential centers

$$\begin{aligned} \dot{x}&=0.244909..-0.132672..x-0.724279..y, & \dot{y}&=-0.471887..+x+0.132672..y;\\ \dot{x}&=0.668802..-0.514522..x-0.636209..y, & \dot{y}&=-0.985653..+x+0.514522..y;\\ \dot{x}&=-0.081198..-0.207828..x-0.061343..y, & \dot{y}&=-0.124956..+x+0.207828..y;\\ \dot{x}&=0.211524..-0.634777..x-0.705080..y, & \dot{y}&=-0.356652..+x+0.634777..y,\end{aligned} \tag{4-42}$$

in the regions $R^1_{\Sigma_{k-}}$, $R^2_{\Sigma_{k-}}$, $R^3_{\Sigma_{k-}}$ $R^4_{\Sigma_{k-}}$, respectively. The linear differential centers in (4-42) have the first integrals

$$
\begin{aligned}
H_1(x,y) =& x^2 + x(-0.943775.. + 0.265344..y) + (-0.489818.. + 0.724279..y)y,\\
H_2(x,y) =& x^2 + (-1.337605.. + 0.636209..y)y + x(-1.971307.. + 1.029044..y),\\
H_3(x,y) =& x^2 + x(-0.249913.. + 0.415657..y) + (0.162397.. + 0.061343..y)y,\\
H_4(x,y) =& x^2 + (-0.423048.. + 0.705080..y)y + x(-0.713304.. + 1.269555..y),
\end{aligned}
$$

respectively. In order to have crossing limit cycles of types 1 and 2^+, simultaneously, such that the crossing limit cycles of type 1 intersect the discontinuity curve Σ_0 in four different points $p_1 = (0, y_1)$, $p_2 = (x_2, y_2)$, $p_3 = (0, y_3)$ and $p_4 = (x_4, y_4)$, with $-1 < y_1 < 1 < y_3$ and $x_4 < 0 < x_2$ and $p_2, p_4 \in \mathbb{S}^1$; and the crossing limit cycles of type 2^+ intersect the discontinuity curve Σ_0 in four different points $p_5 = (x_5, y_5)$, $p_6 = (0, y_6)$, $p_7 = (0, y_7)$ and $p_8 = (x_8, y_8)$, with $-1 < y_7 < y_6 < 1$ and $x_5, x_8 > 0$, with $p_5, p_8 \in \mathbb{S}^1$. These points must satisfy systems (4-13) and (4-16), respectively. Considering the PWLC (4-42) systems (4-13) and (4-16) become

$$
\begin{aligned}
& x_2^2 + x_2(-0.943775.. + 0.265344..y_2) - 0.489818..y_2 + 0.724279..y_2^2\\
&\qquad + 0.489818..y_1 - 0.724279..y_1^2 = 0,\\
& 4x_2^2 - 5.350421..y_2 + 2.544838..y_2^2 + x_2(-7.885229.. + 4.116178..y_2)\\
&\qquad + 5.350421..y_3 - 2.544838..y_3^2 = 0,\\
& x_4^2 - 0.162397..y_3 - 0.061343..y_3^2 + x_4(-0.249913.. + 0.415657..y_4)\\
&\qquad + 0.162397..y_4 + 0.061343..y_4^2 = 0,\\
& 4x_4^2 - 1.692192..y_4 + 2.820321..y_4^2 + x_4(-2.853217.. + 5.078222..y_4)\\
&\qquad + 1.692192..y_1 - 2.820321..y_1^2 = 0,\\
& 4x_5^2 - 1.959275..y_5 + 2.897117..y_5^2 + x_5(-3.775101.. + 1.061377..y_5)\\
&\qquad + 1.959275..y_6 - 2.897117..y_6^2 = 0,\\
& (y_6 - y_7)(-1.692192.. + 2.820321..(y_6 + y_7)) = 0,\\
& x_8^2 + 0.489818..y_7 - 0.724279..y_7^2 + x_8(-0.943775.. + 0.265344..y_8)\\
&\qquad - 0.489818..y_8 + 0.724279..y_8^2) = 0,\\
& x_5^2 - x_8^2 - 1.337605..y_5 + 0.636209..y_5^2 + x_5(-1.971307.. + 1.029044..y_5)\\
&\qquad + x_8(1.971307.. - 1.029044..y_8) + 1.337605..y_8 - 0.636209..y_8^2 = 0,\\
& x_2^2 + y_2^2 = 1, \quad x_4^2 + y_4^2 = 1, \quad x_5^2 + y_5^2 = 1, \quad x_8^2 + y_8^2 = 1.
\end{aligned}
\tag{4-43}
$$

We have four real solutions $q^i = (y_1^i, x_2^i, y_2^i, y_3^i, x_4^i, y_4^i, x_5^i, y_5^i, y_6^i, y_7^i, x_8^i, y_8^i)$ with $i = 1,2,3,4$, for system (4-43) that satisfy the above conditions, namely $q^1 = (-1/3$, $\cos(-\pi/6)$, $\sin(-\pi/6)$, $3/2$, $\cos(2\pi/3)$, $\sin(2\pi/3)$, $\cos(\pi/3)$, $\sin(\pi/3)$, $7/10$, $-1/10$, 1, $0)$; $q^2 = (-0.654342..$, $\cos(-\pi/3)$, $\sin(-\pi/3)$, $12/5$, $\cos(79\pi/100)$,

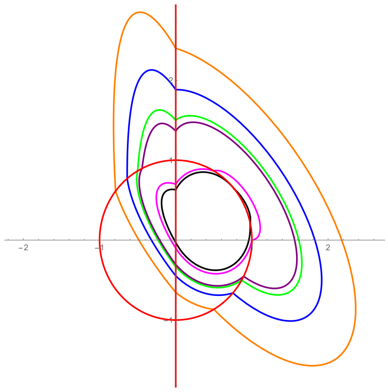

Figure 4.13: *Four crossing limit cycles of type* 1 *and two crossing limit cycles of type* 2^+ *(black and magenta) of the discontinuous PWLC (4-42). These limit cycles are traveled in counterclockwise.*

$\sin(79\pi/100)$, $\cos(11\pi/50)$, $\sin(11\pi/50)$, $63/100$, $-3/100$, $0.975733..$, $0.216981..)$; $q^3 = (-0.447098..$, $\cos(-23\pi/100)$, $\sin(-23\pi/100)$, $1.882264..$, $\cos(18\pi/25)$, $\sin(18\pi/25)$, $-0.654342..$, $\cos(11\pi/50)$, $\sin(11\pi/50)$, $63/100$, $-3/100$, $0.975733..$, $0.216981..)$; $q^4 = (-0.305568..$, $\cos(-3\pi/20)$, $\sin(-3\pi/20)$, $1.365012..$, $-0.441883..$, $0.897073..$, $\cos(11\pi/50)$, $\sin(11\pi/50)$, $63/100$, $-3/100$, $0.975733..$, $0.216981..)$, these four solutions generated four crossing limit cycles of type 1 and two crossing limit cycles of type 2^+. See these crossing limit cycles of the PWLC (4-42) in Figure 4.13.

Here we observed that we obtain a total of six crossing limit cycles between limit cycles of type 1 and of type 2^+, moreover these six crossing limit cycles have the configuration $(4,2)$, this is, 4-crossing limit cycle of type 1 and 2-crossing limit cycles of type 2^+. Clearly this lower bound for the maximum number of crossing limit cycles of types 1 and 2^+ simultaneously, could be also obtained with the configurations $(3,3)$ or $(2,4)$. But after several numeric computations we could not build a third limit cycle of type 2^+, previously fixing two limit cycles of type 1, so we only get those lower bound with the configuration $(4,2)$.

This completes the proof of statement (i) of Theorem *I*. □

Proof of statement (ii) ***of Theorem I.*** We consider the discontinuous PWLC formed by the following linear differential centers

$$
\begin{aligned}
\dot{x} &= 0.078341.. + 0.855624..x + 1.571418..y, & \dot{y} &= -0.065526.. - x - 0.855624..y;\\
\dot{x} &= 0.496667.. + 0.078616..x - 0.193136..y, & \dot{y} &= -0.471461.. + x - 0.078616..y;\\
\dot{x} &= 5.276135.. + 0.212817..x - 1.851275..y, & \dot{y} &= -5.383865.. + x - 0.212817..y;\\
\dot{x} &= 0.484115.. + 0.548314..x - 0.303113..y, & \dot{y} &= 0.569064.. + x - 0.548314..y,
\end{aligned}
\tag{4-44}
$$

in the regions $R^1_{\Sigma_{k^-}}$, $R^2_{\Sigma_{k^-}}$, $R^3_{\Sigma_{k^-}}$ and $R^4_{\Sigma_{k^-}}$, respectively. The linear differential centers in

(4-44) have the first integrals

$$
\begin{aligned}
H_1(x,y) =&x^2+y(0.156682..+1.571418..y)+x(0.131053..+1.711249..y),\\
H_2(x,y) =&x^2+x(-0.942922..-0.157232..y)+(-0.993334..+0.193136..y)y,\\
H_3(x,y) =&x^2+x(-10.767731..-0.425635..y)+y(-10.552270..+1.851275..y),\\
H_4(x,y) =&x^2+x(1.138128..-1.096628..y)+(-0.968231..+0.303113..y)y,
\end{aligned}
$$

respectively. In order to have crossing limit cycles of types 1 and 3^+, simultaneously, such that the crossing limit cycles of type 1 intersect the discontinuity curve Σ_0 in four different points $p_1=(0,y_1)$, $p_2=(x_2,y_2)$, $p_3=(0,y_3)$ and $p_4=(x_4,y_4)$,with $-1<y_1<1<y_3$ and $x_4<0<x_2$ and $p_2,p_4\in\mathbb{S}^1$; and the crossing limit cycles of type 3^+ intersect the discontinuity curve Σ_0 in four different points $p_5=(x_5,y_5)$, $p_6=(0,y_6)$, $p_7=(0,y_7)$ and $p_8=(x_8,y_8)$, with $1<y_7<y_6$, $x_5,x_8>0$ and $p_5,p_8\in\mathbb{S}^1$, these points must satisfy systems (4-13) and (4-19), respectively. Considering the PWLC (4-44) systems (4-13) and (4-19) become

$$
\begin{aligned}
-0.524214..x_2-4x_2^2-6.844999..x_2y_2+(y_1-y_2)(0.626729..&\\
+6.285673..(y_1+y_2))&=0,\\
-3.771690..x_2+4x_2^2-0.628929..x_2y_2+(y_2-y_3)(-3.973339..&\\
+0.772547..(y_2+y_3))&=0,\\
43.070926..x_4-4x_4^2+1.702543..x_4y_4+(y_3-y_4)(-42.209082..&\\
+7.405102..(y_3+y_4))&=0,\\
4.552513..x_4+4x_4^2-4.386514..x_4y_4-(y_1-y_4)(-3.872927..&\\
+1.212454..(y_1+y_4))&=0,\\
-3.771690..x_5+4x_5^2-0.628929..x_5y_5+(y_5-y_6)(-3.973339..&\\
+0.772547..(y_5+y_6))&=0,\\
(y_6-y_7)(-42.209082..+7.405102..(y_6+y_7))&=0,\\
3.771690..x_8-4x_8^2+0.628929..x_8y_8+(y_7-y_8)(-3.973339..&\\
+0.772547..(y_7+y_8))&=0,\\
-4x_5^2+4x_8^2+x_5(-0.524214..-6.844999..y_5)+(-0.626729..-6.285673..y_5)y_5&\\
+y_8(0.626729..+6.285673..y_8)+x_8(0.524214..+6.844999..y_8)&=0,\\
x_2^2+y_2^2=1,\quad x_4^2+y_4^2=1,\quad x_5^2+y_5^2=1,\quad x_8^2+y_8^2&=1.
\end{aligned}
\tag{4-45}
$$

We have four real solutions $q^i=(y_1^i,x_2^i,y_2^i,y_3^i,x_4^i,y_4^i,x_5,y_5,y_6,y_7,x_8,y_8)$ with $i=1,2,3,4$, for system (4-45) that satisfy the above conditions, namely $q^1=(4/5,\ 1,\ 0,\ 26/5,$ $\cos(3\pi/5),\ \sin(3\pi/5),\ \cos(\pi/5),\ \sin(\pi/5),\ 43/10,\ 7/5,\ \cos(2\pi/5),\ \sin(2\pi/5))$; $q^2=(53/100,\ \cos(-13\pi/100),\ \sin(-13\pi/100),\ 557/100,\ \cos(17\pi/25),\ \sin(17\pi/25),$

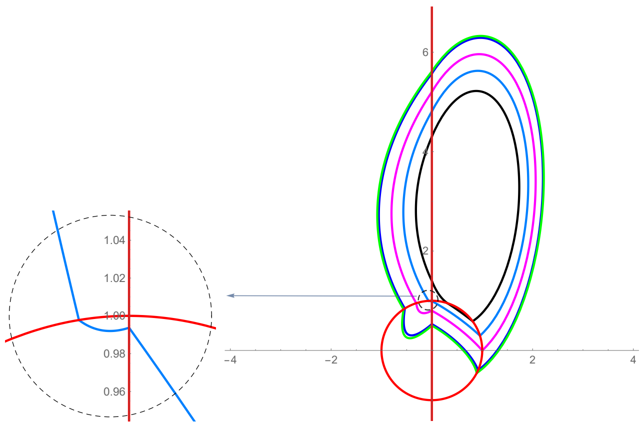

Figure 4.14: *Four crossing limit cycles of type* 1 *and one crossing limit cycle of type* 3^+ *(black) of the discontinuous PWLC (4-44). These limit cycles are traveled in counterclockwise.*

$\cos(\pi/5)$, $\sin(\pi/5)$, $43/10$, $7/5$, $\cos(2\pi/5)$, $\sin(2\pi/5))$; $q^3 = (1/2$, $\cos(-3\pi/20)$, $\sin(-3\pi/20)$, $5.611962..$, $\cos(17239\pi/25000)$, $\sin(17239\pi/25000)$, $\cos(\pi/5)$, $\sin(\pi/5)$, $43/10$, $7/5$, $\cos(2\pi/5)$, $\sin(2\pi/5))$; $q^4 = (\ 0.993727..$, $\cos(12\pi/125)$, $\sin(12\pi/125)$, $4.808026..$, $-0.066301..$, $0.997799..$, $\cos(\pi/5)$, $\sin(\pi/5)$, $43/10$, $7/5$, $\cos(2\pi/5)$, $\sin(2\pi/5))$, these four solutions generated four crossing limit cycles of type 1 and one crossing limit cycle of type 3^+. See these crossing limit cycles of the PWLC (4-44) in Figure 4.14.

Here we observed that we obtain a total of five crossing limit cycles between limit cycles of type 1 and of type 3^+, moreover these five crossing limit cycles have the configuration $(4,1)$, this is, 4-crossing limit cycle of type 1 and 1-crossing limit cycles of type 3^+. In order to obtain a result similar to the previous statement, this is, an example with a configuration $(4,2)$, we tried to build a second cycle of type 3^+ but when building this second cycle we lost a cycle of type 1, so we only got a configuration $(3,2)$.

If we consider the PWLC formed by the linear differential centers

$$\begin{aligned}
&\dot{x} = -0.128852.. - 0.332114..x - 0.791281..y, && \dot{y} = -0.143708.. + x + 0.332114..y;\\
&\dot{x} = 0.597908.. + 0.108856..x - 0.227688..y, && \dot{y} = -0.530777.. + x - 0.108856..y;\\
&\dot{x} = 0.716356.. + 0.457342..x - 0.251353..y, && \dot{y} = -0.189975.. + x - 0.457342..y;\\
&\dot{x} = 1.857676.. - \frac{4}{5}x - 0.688147..y, && \dot{y} = -1.219907.. + x + \frac{4}{5}y,
\end{aligned} \tag{4-46}$$

in the regions $R^1_{\Sigma_{k^-}}$, $R^2_{\Sigma_{k^-}}$, $R^3_{\Sigma_{k^-}}$ and $R^4_{\Sigma_{k^-}}$, respectively.

It is possible verify that we obtain the configuration $(3,2)$, see Figure 4.15. But after several numeric computations we could not build a third limit cycle of type 3^+, previously fixing two limit cycles of type 1, so we only get those lower bound by the

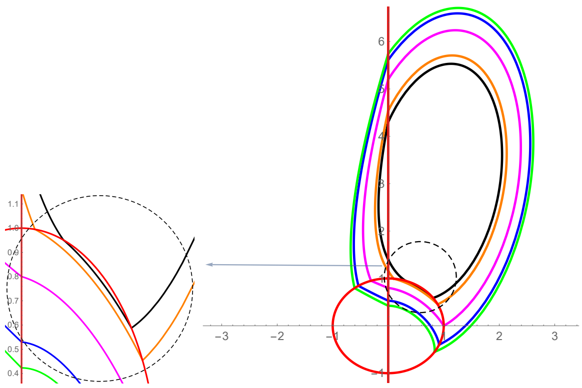

Figure 4.15: *Three crossing limit cycles of type* 1 *and two crossing limit cycle of type* 3^+ *(black and orange) of the discontinuous PWLC (4-46). These limit cycles are traveled in counterclockwise.*

maximum number of types 1 and 3^+, simultaneously, with the configurations $(4,1)$ and $(3,2)$.

This completes the proof of statement (ii) of Theorem *I*. $\square$

Proof of statement (iii) ***of Theorem I.*** We consider the following PWLC formed by the linear differential centers

$$
\begin{aligned}
&\dot{x}=45.736851..-\frac{x}{2}-7.515818..y, && \dot{y}=-1146.321640..+x+\frac{y}{2};\\
&\dot{x}=-0.320594..-0.199436..x-0.051960..y, && \dot{y}=0.460058..+x+0.199436..y;\\
&\dot{x}=2+\frac{x}{20}-\frac{13}{200}y, && \dot{y}=-\frac{23}{4}+x-\frac{y}{20};\\
&\dot{x}=-0.457007..+0.276952..x-0.076768..y, && \dot{y}=-4.377702..+x-0.276952..y,
\end{aligned}
\tag{4-47}
$$

in the regions $R^1_{\tilde{\Sigma}_0}$, $R^2_{\tilde{\Sigma}_0}$, $R^3_{\tilde{\Sigma}_0}$ and $R^4_{\tilde{\Sigma}_0}$, respectively. The linear differential centers in (4-47) have the first integrals

$$
\begin{aligned}
H_1(x,y)=&x^2+x(-2292.643280..+y)+y(-91.473702..+7.515818..y),\\
H_2(x,y)=&x^2+x(0.920117..+0.398872..y)+(0.641188..+0.051960..y)yx^2\\
&+x(0.920117..+0.398872..y)+(0.641188..+0.051960..y)y,\\
H_3(x,y)=&2x(-23+2x)-\frac{2}{5}(40+x)y+\frac{13}{50}y^2,\\
H_4(x,y)=&x^2+x(-8.755405..-0.553904..y)+(0.914014..+0.076768..y)y,
\end{aligned}
$$

respectively. In order to have crossing limit cycles of type 4 and 5, simultaneously, such that the crossing limit cycles of type 4 intersect the discontinuity curve $\tilde{\Sigma}_0$ in four different points $p_1=(x_1,x_1^2)$, $p_2=(x_2,x_2^2)$, $p_3=(x_3,0)$ and $p_4=(x_4,0)$, with $x_2<0<x_1$ and $x_3<0<x_4$, and the crossing limit cycles of type 5 intersect the discontinuity curve

$\tilde{\Sigma}_0$ in four different points $p_5 = (x_5, x_5^2)$, $p_6 = (x_6, x_6^2)$, $p_7 = (x_7, 0)$ and $p_8 = (x_8, 0)$, with $0 < x_6 < x_5$ and $0 < x_7 < x_8$, these points must satisfy systems (4-25) and (4-28), respectively. Considering the PWLC (4-47) systems (4-25) and (4-28) become

$$\begin{aligned}
&x_1(-9170.573120.. + x_1(-361.894811.. + x_1(3.999999.. + 30.063275..x_1)))\\
&\quad + x_2(9170.573120.. + x_2(361.894811.. + (-3.999999.. - 30.063275..x_2)x_2)) = 0,\\
&x_2(3.680468.. + x_2(6.564754.. + (1.595489.. + 0.207843..x_2)x_2))\\
&\quad -3.680468..x_3 - 4x_3^2 = 0,\\
&(x_3 - x_4)(-23 + 2x_3 + 2x_4) = 0,\\
&x_1(35.021620.. + x_1(-7.656056.. + (2.215618.. - 0.307072..x_1)x_1))\\
&\quad -35.021620..x_4 + 4x_4^2 = 0,\\
&x_5(-9170.573120.. + x_5(-361.894811.. + x_5(3.999999.. + 30.063275..x_5)))\\
&\quad + x_6(9170.573120.. + x_6(361.894811.. + (-3.999999.. - 30.063275..x_6)x_6)) = 0,\\
&x_6(-35.021620.. + x_6(7.656056.. + (-2.215618.. + 0.307072..x_6)x_6))\\
&\quad +35.021620..x_7 - 4x_7^2 = 0,\\
&(x_7 - x_8)(-23 + 2x_7 + 2x_8) = 0,\\
&x_5(35.021620.. + x_5(-7.656056.. + (2.215618.. - 0.307072..x_5)x_5))\\
&\quad -35.021620..x_8 + 4x_8^2 = 0.
\end{aligned} \tag{4-48}$$

In this case system (4-48) has four real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i, x_5, x_6, x_7, x_8)$,

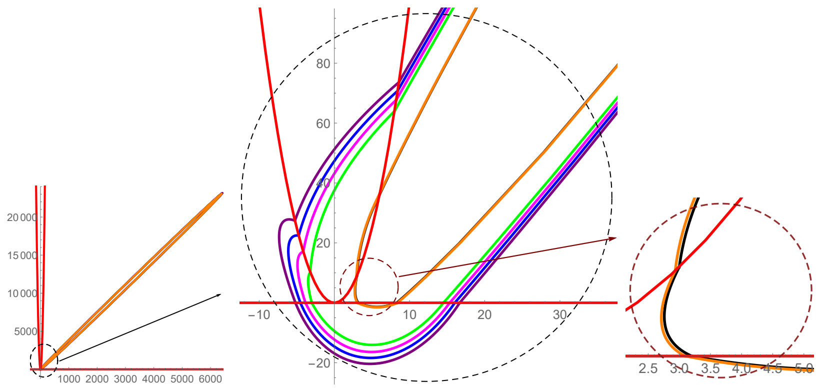

Figure 4.16: *Four crossing limit cycles of type 4 and two crossing limit cycles of type 5 (black and orange) of the discontinuous PWLC (4-47). These limit cycles are traveled in counterclockwise.*

that satisfy the necessary conditions to have crossing limit cycles of types 4 and 5. Namely, $q^1 = (8,\ -16/5,\ -3,\ 29/2,\ 6,\ 3,\ 16/5,\ 83/10)$; $q^2 = (823/100,\ -413/100,\ -96/25,\ 767/50,\ 6,\ 3,\ 16/5,\ 83/10)$; $q^3 = (841/100,\ -4.737905..,\ -4.516438..,\ 16.016438..,\ 6.040228..,\ 2.934482..,\ 3.093430..,\ 8.406569..)$ and $q^4 = (429/50,\ -5.236369..,\ -5.170738..,\ 16.670738..,\ 6.040228..,\ 2.934482..,\ 3.093430..,$

8.406569..). These solutions provide four crossing limit cycles of type 4 and two crossing limit cycles of type 5 of the PWLC (4-47). See these crossing limit cycles in Figure 4.16.

Here we observed that we obtain a total of six crossing limit cycles between limit cycles of type 4 and of type 5, moreover these six crossing limit cycles have the configuration $(4,2)$, this is, 4-crossing limit cycle of type 4 and 2-crossing limit cycles of type 5. We know that this lower bound for the maximum number of crossing limit cycles of types 4 and 5 simultaneously, could be also obtained with the configuration $(3,3)$. But if we previously fixing two limit cycles of each type after several numeric computations we could not build a third limit cycle of type 5, then we only get those lower bound with the configuration $(4,2)$.

This completes the proof of statement (iii) of Theorem *I*. □

Proof of statement (iv) ***of Theorem I.*** We consider the following discontinuous PWLC formed by the linear differential centers

$$
\begin{aligned}
&\dot{x} = 0.751960..-0.008805..x-0.043938..y, && \dot{y} = -1.117055..+x+0.008805..y;\\
&\dot{x} = -\frac{4701043}{7161144}-\frac{122761}{156650025}x+\frac{91946}{31330005}y, && \dot{y} = -\frac{42715283}{313300050}-x+\frac{122761}{156650025}y;\\
&\dot{x} = 0.041424..-0.228644..x-0.115044..y, && \dot{y} = 2.030027..+x+0.228644..y;\\
&\dot{x} = 6.094659..-0.970562..x-1.475325..y, && \dot{y} = -4.066695+x+0.970562..y;\\
&\dot{x} = -0.014046..-0.011408..x+0.000796..y, && \dot{y} = -0.900270..-x+0.011408..y,
\end{aligned}
\tag{4-49}
$$

in the regions $R^1_{\tilde{\Sigma}_{k^+}}$, $R^2_{\tilde{\Sigma}_{k^+}}$, $R^3_{\tilde{\Sigma}_{k^+}}$, $R^4_{\tilde{\Sigma}_{k^+}}$ and $R^5_{\tilde{\Sigma}_{k^+}}$, respectively. The linear differential centers in (4-49) have the first integrals

$$
\begin{aligned}
H_1(x,y) =& x^2+x(-2.234111..+0.017610..y)+(-1.503920..+0.043938..y)y,\\
H_2(x,y) =& 626600100x^2+x(170861132-982088y)+5y(-164536505+367784y),\\
H_3(x,y) =& x^2+x(4.060055..+0.457288..y)+(-0.082848..+0.115044..y)y,\\
H_4(x,y) =& x(-544800479242800689018\,3+669831938277330213420x)\\
&-160y(510294348343124366\,27-812642257076495750\,0x)\\
&+988220002292252000000y^2,\\
H_5(x,y) =& 17172023317192110696x^2+x(30918934250652233287\\
&-391817091205831000y)+6y(-80400672913407451\\
&+2279188834700000y),
\end{aligned}
$$

respectively. In order to have simultaneously crossing limit cycles of types 6^+ and 6^-, such that the crossing limit cycles of type 6^+ intersect the discontinuity curve $\tilde{\Sigma}_{k^+}$ in four different points $p_1 = (x_1,x_1^2)$, $p_2 = (x_2,k)$, $p_3 = (x_3,x_3^2)$ and $p_4 = (x_4,k)$, with

$-2 < x_2 < 2 < x_1$ and $-2 < x_3 < 2 < x_4$, and the crossing limit cycles of type 6^- intersect the discontinuity curve $\tilde{\Sigma}_{k^+}$ in four different points $p_5 = (x_5, x_5^2)$, $p_6 = (x_6, k)$, $p_7 = (x_7, x_7^2)$ and $p_8 = (x_8, k)$, with $x_5 < -2 < x_7 < 2$ and $x_6 < -2 < x_8 < 2$, these points must satisfy systems (4-31) and

$$\begin{aligned} H_3(x_5, x_5^2) = & \; H_3(x_6,\ k), \\ H_4(x_6,\ k) = & \; H_4(x_7, x_7^2), \\ H_5(x_7, x_7^2) = & \; H_5(x_8,\ k), \\ H_2(x_8,\ k) = & \; H_2(x_5, x_5^2), \end{aligned} \tag{4-50}$$

respectively. Considering the PWLC (4-49) and $k = 4$, systems (4-31) and (4-50) become

$$\begin{aligned}
&170861132x_1 - 196082425x_1^2 - 982088x_1^3 + 1838920x_1^4 - 60(-54355123 \\
&\qquad + 2782213x_2 + 10443335x_2^2) = 0, \\
&-1710814021790578824 + 2935166588582890928 7x_2 \\
&\qquad + 17172023317192110696x_2^2 - 3091893425065223328 7x_3 \\
&\qquad - 16689619279711665990x_3^2 + 391817091205831000x_3^3 \\
&\qquad - 13675133008200000x_3^4 = 0, \\
&-5448004792428006890183x_3 - 7494877635212659646900x_3^2 + \\
&\qquad 1300227611322393200000x_3^3 + 988220002292252000000x_3^4 \\
&\qquad + 21(802253250346853687680 - 1176639748278257572 3x_4 \\
&\qquad + 31896758965587153020x_4^2) = 0, \\
&-21.250638.. + 8.936444..x_1 + 2.015680..x_1^2 - 0.070440..x_1^3 \\
&\qquad - 0.175755..x_1^4 - 8.654682..x_4 + 4x_4^2 = 0, \\
&-6.037269.. + 16.240221..x_5 + 3.668606..x_5^2 + 1.829154..x_5^3 \\
&\qquad + 0.460177..x_5^4 - 23.556840..x_6 - 4x_6^2 = 0, \\
&168473182572839274412 80 + 2470943471384340901 83x_6 \\
&\qquad - 66983193827733021342 0x_6^2 - 5448004792428006890183x_7 \\
&\qquad - 7494877635212659646900x_7^2 + 1300227611322393200000x_7^3 \\
&\qquad + 988220002292252000000x_7^4 = 0, \\
&3091893425065223328 7x_7 + 16689619279711665990x_7^2 \\
&\qquad - 391817091205831000x_7^3 + 13675133008200000x_7^4 \\
&\qquad - 21(-814673343709799 44 + 1397698375515662347x_8 \\
&\qquad + 817715396056767176x_8^2) = 0, \\
&-170861132x_5 + 196082425x_5^2 + 982088x_5^3 - 1838920x_5^4 \\
&\qquad + 60(-54355123 + 2782213x_8 + 10443335x_8^2) = 0.
\end{aligned} \tag{4-51}$$

We have four real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i, x_5^i, x_6^i, x_7^i, x_8^i)$ with $i = 1, 2, 3, 4$, for system (4-51) that satisfy the above conditions namely $q^1 = (5,\ 1/2,\ 9/50,\ 23/5,\ -18/5,$

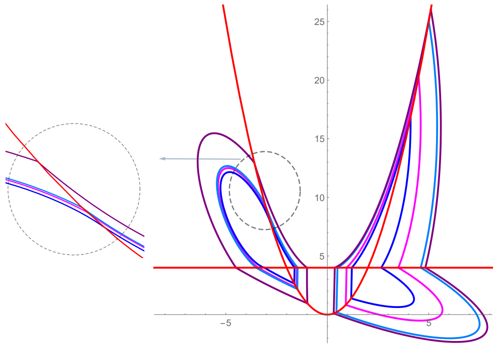

Figure 4.17: *Four crossing limit cycles of type* 6^+ *in the right hand side and four crossing limit cycles of type* 6^- *in the left hand side, of the discontinuous PWLC (4-49). These limit cycles are traveled in counterclockwise.*

$-9/2,\ -49/50,\ -1)$; $q^2 = (9/2,\ 19/20,\ 91/100,\ 7/5,\ -3,\ -17/5,\ -303/200,\ -3/2)$; $q^3 = (41/10,\ 1.208958..,\ 1.176604..,\ 2.657283..,\ -2.816357..,\ -31/10,\ -1.626433..,\ -1.613770..)$, and $q^4 = (51/10, 0.368157.., 0.315951.., 4.829311.., -3.059352.., -7/2, -1.475955.., -1.460360..)$, these four solutions generated four crossing limit cycles of type 6^+ and four crossing limit cycles of type 6^-. See these crossing limit cycles of the PWLC (4-49) in Figure 4.17.

Here we obtain a total of eight crossing limit cycles of types 6^+ and 6^- simultaneously, with a configuration $(4,4)$. And observed that it is possible obtain this lower bound with the configurations $(5,3)$ or $(3,5)$, but here we only present the example with the configuration $(4,4)$.

This completes the proof of statement (iv) of Theorem *I*. □

Proof of statement (v) ***of Theorem I.*** We consider the following discontinuous PWLC formed by the linear differential centers

$$
\begin{aligned}
\dot{x} &= 1.717686.. + 0.650612..x - 0.423688..y, & \dot{y} &= 0.850546.. + x - 0.650612..y;\\
\dot{x} &= 0.516832.. + 0.082481..x - 0.038759..y, & \dot{y} &= 0.179926.. + x - 0.082481..y;\\
\dot{x} &= 1.470269.. + 0.406982..x - 3.640154..y, & \dot{y} &= -0.122065.. + x - 0.406982..y;\\
\dot{x} &= 0.685228.. + 0.043300..x - 0.293631..y, & \dot{y} &= 0.017396.. + x - 0.043300..y,
\end{aligned} \tag{4-52}
$$

in the regions $R^1_{\tilde{\Sigma}_{k+}}$, $R^2_{\tilde{\Sigma}_{k+}}$, $R^4_{\tilde{\Sigma}_{k+}}$ and $R^5_{\tilde{\Sigma}_{k+}}$, respectively. The linear differential centers in (4-52) have the first integrals

$$
\begin{aligned}
H_1(x,y) &= x^2 + x(1.701093.. - 1.301224..y) + (-3.435373.. + 0.423688..y)y,\\
H_2(x,y) &= x^2 + x(0.359853.. - 0.164963..y) + (-1.033664.. + 0.038759..y)y,\\
H_4(x,y) &= x^2 + x(-0.244130.. - 0.813965..y) + y(-2.940538.. + 3.640154..y),\\
H_5(x,y) &= x^2 + x(0.034792.. - 0.086601..y) + (-1.370456.. + 0.293631..y)y,
\end{aligned}
$$

respectively. In order to have simultaneously crossing limit cycles of types 6^+ and 7, such that the crossing limit cycles of type 6^+ intersect the discontinuity curve $\tilde{\Sigma}_{k^+}$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, k)$, $p_3 = (x_3, x_3^2)$ and $p_4 = (x_4, k)$, with $-2 < x_2 < 2 < x_1$ and $-2 < x_3 < 2 < x_4$, and the crossing limit cycles of type 7 intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_5 = (x_5, k)$, $p_6 = (x_6, k)$, $p_7 = (x_7, x_7^2)$ and $p_8 = (x_8, x_8^2)$, with $-2 < x_6 < x_5 < 2$ and $-2 < x_7 < x_8 < 2$ these points must satisfy systems (4-31) and (4-34), respectively. Considering the PWLC (4-52) and $k = 4$, systems (4-31) and (4-34) become

$$
\begin{aligned}
14.058034.. + 1.439414..x_1 - 0.134656..x_1^2 - 0.659853..x_1^3 + 0.155036..x_1^4&\\
+\frac{6}{5}x_2 - 4x_2^2 &= 0,\\
-0.783728.. - 0.311613..x_2 + x_2^2 - 0.034792..x_3 + 0.370456..x_3^2 + 0.086601..x_3^3&\\
-0.293631..x_3^4 &= 0,\\
-185.921253.. - 0.976522..x_3 - 7.762153..x_3^2 - 3.255860..x_3^3 + 14.560616..x_3^4&\\
+13.999964..x_4 - 4x_4^2 &= 0,\\
-27.849933.. - 6.804375..x_1 + 9.741494..x_1^2 + 5.204898..x_1^3 - 1.694752..x_1^4&\\
-14.015217..x_4 + 4x_4^2 &= 0,\\
4(x_5 - x_6)\left(-\frac{3}{10} + x_5 + x_6\right) &= 0,\\
-0.783728.. - 0.311613..x_6 + x_6^2 - 0.034792..x_7 + 0.370456..x_7^2&\\
+0.086601..x_7^3 - 0.293631..x_7^4 &= 0,\\
-0.976522..x_7 - 7.762153..x_7^2 - 3.255860..x_7^3 + 14.560616..x_7^4&\\
+x_8(0.976522.. + 7.762153..x_8 + 3.255860..x_8^2 - 14.560616..x_8^3) &= 0,\\
-0.783728.. - 0.311613..x_5 + x_5^2 - 0.034792..x_8 + 0.370456..x_8^2&\\
+0.086601..x_8^3 - 0.293631..x_8^4 &= 0.
\end{aligned}
\tag{4-53}
$$

We have four real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i, x_5, x_6, x_7, x_8)$ with $i = 1,2,3,4$, for system (4-53) that satisfy the above conditions. We have $q^1 = (4, -9/5, -19/10, 7/2, 1, -7/10, -9/10, 11/10)$; $q^2 = (106/25, -39/20, -1.975633.., 51/10, 1, -7/10, -9/10, 11/10)$; $q^3 = (413/100, -469/250, -1.938820.., 4.420122.., 101/100, -71/100, -941/1000,$

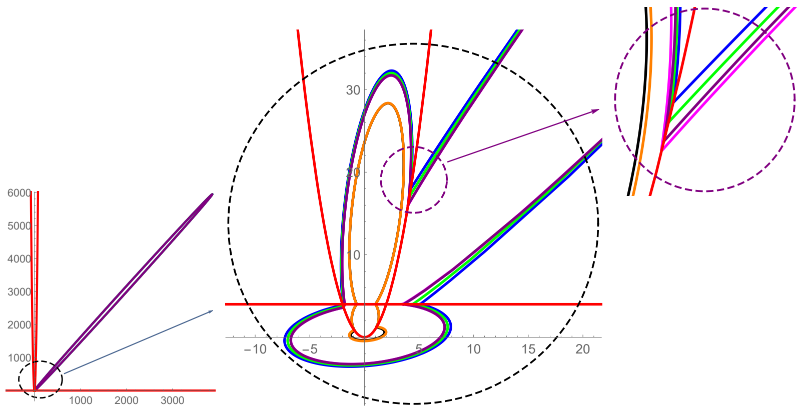

Figure 4.18: *Four crossing limit cycles of type* 6^+ *and two crossing limit cycles of type* 7 *(black and orange) of the discontinuous PWLC (4-52). These limit cycles are traveled in counterclockwise.*

1.132764..) and $q^4 = (401/100,\ -1.805407..,\ -1.902798..,\ 3.579564..,\ 101/100,\ -71/100,\ -941/1000,\ 1.132764..)$. These four real solutions generated four crossing limit cycles of type 6^+ and two crossing limit cycles of type 7. See these crossing limit cycles of the PWLC (4-52) in Figure 4.18.

Here we observed that we obtain a total of six crossing limit cycles between limit cycles of type 6^+ and of type 7, moreover these six crossing limit cycles have the configuration $(4,2)$. We observe that this lower bound for the maximum number of crossing limit cycles of types 6^+ and 7 simultaneously, could be also obtained with the configuration $(3,3)$. But if we previously fixing two limit cycles of type 6^+ after several numeric computations we could not build a third limit cycle of type 7, then we only get those lower bound with the configuration $(4,2)$.

We can also observe that there is a duality between the case studied in statement (i) of Theorem I, where we have studied simultaneously crossing limit cycles of types 1 and 2^+ and this case, where study the crossing limit cycles of types 6^+ and 7, simultaneously. In these two cases we got the configuration $(4,2)$. See Figures 4.13 and 4.18.

This completes the proof of statement (v) of Theorem *I*. □

Proof of statement (vi) ***of Theorem I.*** We consider the following discontinuous PWLC formed by the linear differential centers

$$
\begin{aligned}
&\dot{x} = 0.212208.. - 0.051128..x - 0.004724..y, && \dot{y} = -3.713538.. + x + 0.051128..y;\\
&\dot{x} = 0.592855.. - 0.098217..x - 0.044462..y, && \dot{y} = -1.739750.. + x + 0.098217..y;\\
&\dot{x} = -0.324307.. - 0.152006..x - 0.023227..y, && \dot{y} = 2.010345.. + x + 0.152006y;\\
&\dot{x} = 5.173755.. - 0.530837..x - 1.789344..y, && \dot{y} = -2.823348.. + x + 0.530837..y;\\
&\dot{x} = 0.905547.. + \frac{9}{50}x + 0.037591..y, && \dot{y} = -2.213772.. - x - \frac{9}{50}y,
\end{aligned}
\tag{4-54}
$$

in the regions $R^1_{\tilde{\Sigma}_{k+}}$, $R^2_{\tilde{\Sigma}_{k+}}$, $R^3_{\tilde{\Sigma}_{k+}}$, $R^4_{\tilde{\Sigma}_{k+}}$ and $R^5_{\tilde{\Sigma}_{k+}}$, respectively. The linear differential centers in (4-54) have the first integrals

$$
\begin{aligned}
H_1(x,y) =& 92350000x^2 + 2y(-19597489 + 218145y) + x(-685890524 + 9443461y),\\
H_2(x,y) =& x(-2350427721 + 675507095x) + 2(-400478067 + 66346510x)y + 30034700y^2,\\
H_3(x,y) =& x^2 + x(4.020691.. + 0.304014..y) + (0.648615.. + 0.023227..y)y,\\
H_4(x,y) =& 2.248715.. \times 10^{16}x^2 - 5x(2.539563.. \times 10^{16} - 4.774807.. \times 10^{15}y)\\
&+ y(-2.326860.. \times 10^{17} + 4.023727.. \times 10^{16}y),\\
H_5(x,y) =& -5.437818.. \times 10^{22}x^2 + 6x(-4.012698.. \times 10^{22} - 3.262691.. \times 10^{21}y)\\
&+ 5(-1.969681.. \times 10^{22} - 4.088345.. \times 10^{20}y)y,
\end{aligned}
$$

respectively. In order to have crossing limit cycles of types 6^+ and 8, simultaneously,

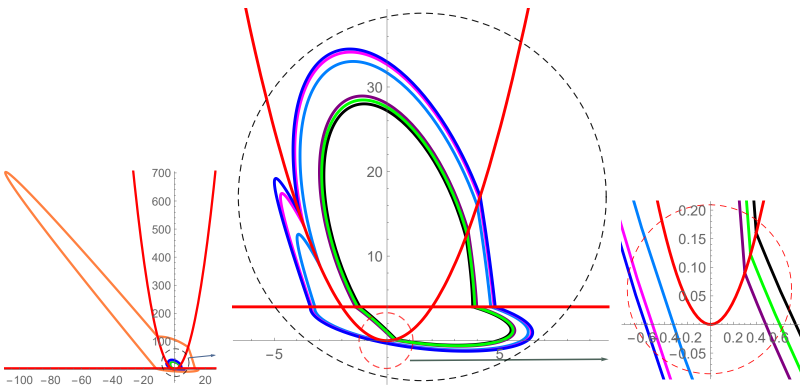

Figure 4.19: *Three crossing limit cycles of type* 6^+ *(purple, green and black) and four crossing limit cycles of type* 8 *(orange, blue, magenta and light blue) of the discontinuous PWLC (4-54). These limit cycles are traveled in counterclockwise.*

such that the crossing limit cycles of type 6^+ intersect the discontinuity curve $\tilde{\Sigma}_{k+}$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, k)$, $p_3 = (x_3, x_3^2)$ and $p_4 = (x_4, k)$, with $-2 < x_2 < 2 < x_4$ and $-2 < x_3 < 2 < x_1$, and the crossing limit cycles of type 8 intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_5 = (x_5, x_5^2)$, $p_6 = (x_6, x_6^2)$, $p_7 = (x_7, k)$ and $p_8 = (x_8, k)$, with $x_7 < -2 < 2 < x_8$ and $x_6 < -2 < 2 < x_5$, these points must satisfy systems (4-31) and (4-37), respectively. Considering the PWLC (4-54) and $k = 4$, systems (4-31) and (4-37) become

$$
\begin{aligned}
&16.125777.. - 13.918004..x_1 - 0.742843..x_1^2 + 0.785738..x_1^3 + 0.177849..x_1^4 \\
&\qquad + 10.775049..x_2 - 4x_2^2 = 0, \\
&31.383400.. + 23.470181..x_2 + 4x_2^2 - 17.710181..x_3 - 11.244381..x_3^2 - \frac{36}{25}x_3^3 \\
&\qquad - 0.150367..x_3^4 = 0, \\
&51.042105.. - 22.586789..x_3 - 37.390043..x_3^2 + 4.246697..x_3^3 + 7.157379..x_3^4 \\
&\qquad + 5.599999..x_4 - 4x_4^2 = 0, \\
&-6.488327.. + 29.708306..x_1 - 2.302329..x_1^2 - 0.409029..x_1^3 - 0.018897..x_1^4 \\
&\qquad - 28.072189..x_4 + 4x_4^2 = 0, \\
&-149799272 - 648116680x_8 + 92350000x_8^2 + 685890524x_5 - 53155022x_5^2 \\
&\qquad - 9443461x_5^3 - 436290x_5^4 = 0, \\
&-2350427721x_5 - 125449039x_5^2 + 132693020x_5^3 + 30034700x_5^4 \\
&\qquad + x_6(2350427721 + 125449039x_6 - 132693020x_6^2 - 30034700x_6^3) = 0, \\
&-11.864396.. + 16.082766..x_6 + 6.594461..x_6^2 + 1.216054..x_6^3 + 0.092909..x_6^4 \\
&\qquad - 20.946982..x_7 - 4x_7^2 = 0, \\
&(x_8 - x_7)(-7 + 5x_8 + 5x_7) = 0.
\end{aligned}
\tag{4-55}
$$

We have four real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i, x_5^i, x_6^i, x_7^i, x_8^i)$ with $i = 1, 2, 3, 4$, for system (4-55) that satisfy the above conditions. We have $q^1 = (7/2,\ -6/5,\ 2/5,\ 19/5,\ 4,\ -3,\ -16/5,\ 23/5)$; $q^2 = (18/5,\ -7/5,\ 3/10,\ 199/50,\ 41/10,\ -37/10,\ -3351/1000,\ 4751/1000)$; $q^3 = (71/20,\ -1.299400..,\ 7/20,\ 3.893976..,\ 4.132430..,\ -3.871790..,\ -17/5,\ 24/5)$ and $q^4 = (71/20,\ -1.299400..,\ 7/20,\ 3.893976..,\ 178349/20000,\ 108083/10000,\ -119/10,\ 133/10)$. These four real solutions generated three crossing limit cycles of type 6^+ and four crossing limit cycle of type 8. See these crossing limit cycles of the PWLC (4-54) in Figure 4.19.

Here we observed that we obtain a total of seven crossing limit cycles between limit cycles of type 6^+ and of type 8, moreover in this example, the seven crossing limit cycles have the configuration $(3,4)$. We observe that this lower bound for the maximum number of crossing limit cycles of types 6^+ and 8 simultaneously, could be also obtained with the configurations $(4,3)$. And we obtain a example with this configuration in the proof of statement (xi) of Theorem *I* with PWLC (4-64), see Figure 4.24.

This completes the proof of statement (vi) of Theorem *I*. □

Proof of statement (vii) ***of Theorem I.*** We consider the following discontinuous PWLC

formed by the linear differential centers

$$\dot{x} = -0.478750.. + 0.183274..x - 0.037189..y, \quad \dot{y} = -4.300673.. + x - 0.183274..y;$$
$$\dot{x} = 0.122511.. + 0.079715..x - 0.013506..y, \quad \dot{y} = -1.007263.. + x - 0.079715..y;$$
$$\dot{x} = -1.261810.. + 0.053348..x - 0.212413..y, \quad \dot{y} = -4.836606.. + x - 0.053348..y;$$
$$\dot{x} = 0.060157.. + 0.062627..x - 0.047729..y, \quad \dot{y} = -0.739728.. + x - 0.062627..y, \tag{4-56}$$

in the regions $R^1_{\tilde{\Sigma}_{k+}}$, $R^2_{\tilde{\Sigma}_{k+}}$, $R^4_{\tilde{\Sigma}_{k+}}$ and $R^5_{\tilde{\Sigma}_{k+}}$, respectively.

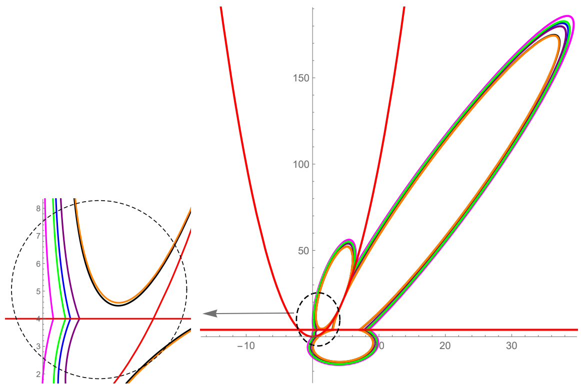

Figure 4.20: *Four crossing limit cycles of type 6^+ and two crossing limit cycles of type 9^+ (black and orange) of the discontinuous PWLC (4-56). These limit cycles are traveled in counterclockwise.*

The linear differential centers in (4-56) have the first integrals

$$\begin{aligned}
H_1(x,y) =& x^2 + x(-8.601346.. - 0.366548..y) + (0.957501401147845` + 0.037189..y)y, \\
H_2(x,y) =& x^2 + x(-2.014527.. - 0.159430..y) + (-0.245022.. + 0.013506..y)y, \\
H_4(x,y) =& x^2 + x(-9.673213.. - 0.106696..y) + (2.523620.. + 0.212413..y)y, \\
H_5(x,y) =& x^2 + x(-1.479456.. - 0.125255..y) + (-0.120314.. + 0.047729..y)y,
\end{aligned}$$

respectively. In order to have simultaneously crossing limit cycles of types 6^+ and 9^+, such that the crossing limit cycles of type 6^+ intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, k)$, $p_3 = (x_3, x_3^2)$ and $p_4 = (x_4, k)$, with $-2 < x_2 < 2 < x_4$ and $-2 < x_3 < 2 < x_1$, and the crossing limit cycles of type 9^+ intersect the discontinuity curve $\tilde{\Sigma}_{k^+}$ in four different points $p_5 = (x_5, x_5^2)$, $p_6 = (x_6, x_6^2)$, $p_7 = (x_7, k)$ and $p_8 = (x_8, k)$, with $2 < x_6 < x_5$ and $2 < x_7 < x_8$, these points must satisfy systems (4-31) and (4-40), respectively. Considering the PWLC (4-56) and $k = 4$, systems (4-31) and

(4-40) become

$$
\begin{aligned}
3.055923..+x_1(-8.058108..+x_1(3.019909..+(-0.637722..&\\
+0.054027..x_1)x_1))+(10.608997..-4x_2)x_2&=0,\\
0.282412..+(-1.980480..+x_2)x_2+x_3(1.479456..+x_3(-0.879685..&\\
+(0.125255..-0.047729..x_3)x_3))&=0,\\
-53.972411..+x_3(-38.692854..+x_3(14.094480..+(-0.426786..&\\
+0.849655..x_3)x_3))+(40.4000000..-3.999999..x_4)x_4&=0,\\
17.700131..+x_1(34.405384..+x_1(-7.8300056..+(1.466193..&\\
-0.148756..x_1)x_1))+x_4(-40.270159..+4x_4)&=0,\\
-8.058108..x_5+3.019909..x_5^2-0.637722..x_5^3+0.054027..x_5^4+x_6(8.058108..&\\
-3.019909..x_6+0.637722..x_6^2-0.054027..x_6^3)&=0,\\
-17.700131..-34.405384..x_6+7.830005..x_6^2-1.466193..x_6^3+0.148756..x_6^4&\\
+40.270159..x_7-4x_7^2&=0,\\
4(x_7-x_8)(-10.100000..+x_7+x_8)&=0,\\
17.700131..+34.405384..x_5-7.830005..x_5^2+1.466193..x_5^3-0.148756..x_5^4&\\
-40.270159..x_8+4x_8^2&=0.
\end{aligned}
\tag{4-57}
$$

We have four real solutions $q^i=(x_1^i,x_2^i,x_3^i,x_4^i,x_5,x_6,x_7,x_8)$ with $i=1,2,3,4$, for system (4-57) that satisfy the above conditions. We have $q^1=(6,1/2,4/10,8,5,14/5,3,71/10)$; $q^2=(317/50,\ 19/100,\ 1/25,\ 423/50,\ 5,\ 14/5,\ 3,\ 71/10)$; $q^3=(291/50,\ 0.664193..,\ 3/5,\ 7.554404..,\ 487/100,\ 3.986608..,\ 3.058022..,\ 7.041977..)$ and $q^4=(61/10,\ 0.409425..,\ 0.293958..,\ 8.128324..,\ 487/100,\ 3.986608..,\ 3.058022..,\ 7.041977..)$ These four real solutions generated four crossing limit cycles of type 6^+ and two crossing limit cycles of type 9^+. See these crossing limit cycles of the PWLC (4-56) in Figure 4.20.

Here we obtain a total of six crossing limit cycles between limit cycles of type 6^+ and of type 9^+, moreover these six crossing limit cycles have the configuration $(4,2)$. We observed that this lower bound for the maximum number of crossing limit cycles of types 6^+ and 9^+ simultaneously, could be also obtained with the configuration $(3,3)$. But if we build two crossing limit cycles of type 6^+ and two of type 9^+, simultaneously, we have that all the parameters that appear in system (4-40) are determined, where this system is such that generated limit cycles of type 9^+, then it is no possible to build a third crossing limit cycle of type 9^+ and therefore we can not obtain the configuration $(3,3)$.

This completes the proof of statement (vii) of Theorem *I*. □

Proof of statement $(viii)$ ***of Theorem I.*** We consider the following discontinuous PWLC

formed by the linear differential centers

$$
\begin{aligned}
&\dot{x} = -0.147861.. + 0.083875..x - 0.018000..y, && \dot{y} = -3.106437.. + x - 0.083875..y;\\
&\dot{x} = \frac{7769951}{9492348} + \frac{176465}{2373087}x - \frac{204250}{2373087}y, && \dot{y} = \frac{6997939}{47461740} + x - \frac{176465}{2373087}y;\\
&\dot{x} = -0.284659.. - 0.174915..x - 0.046689..y, && \dot{y} = 1.660380.. + x + 0.174915..y;\\
&\dot{x} = -\frac{3871251}{31913000} + \frac{3}{10}x - \frac{4335}{31913}y, && \dot{y} = -\frac{19}{20} + x - \frac{3}{10}y;\\
&\dot{x} = 0.206531.. + 0.150466..x - 0.054352..y, && \dot{y} = 0.451143.. + x - 0.150466..y,
\end{aligned}
\tag{4-58}
$$

in the regions $R^1_{\tilde{\Sigma}_{k+}}$, $R^2_{\tilde{\Sigma}_{k+}}$, $R^3_{\tilde{\Sigma}_{k+}}$, $R^4_{\tilde{\Sigma}_{k+}}$ and $R^5_{\tilde{\Sigma}_{k+}}$, respectively. The linear differential

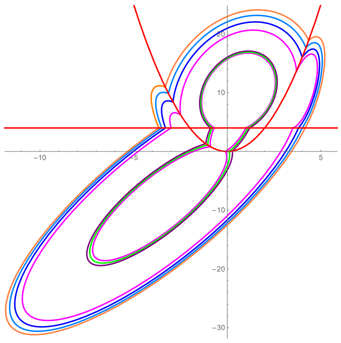

Figure 4.21: *Three crossing limit cycles of type 7 (purple, green and black) and four crossing limit cycles of type 8 of the discontinuous PWLC (4-58). These limit cycles are traveled in counterclockwise.*

centers in (4-58) have the first integrals

$$
\begin{aligned}
H_1(x,y) =& (58546435625x^2 + 4y(4328392296 + 263466775y) - 15x(24249448597\\
& + 654747306y),\\
H_2(x,y) =& x(6997939 + 23730870x) - 5(7769951 + 705860x)y + 2042500y^2,\\
H_3(x,y) =& 1.054579.. \times 10^{-58}(3.792980.. \times 10^{58}x^2 + y(2.159417.. \times 10^{58}\\
& + 1.770939.. \times 10^{57}y) + x(1.259558..10^{59} + 1.326899.. \times 10^{58}y)),\\
H_4(x,y) =& 4x^2 + \frac{2}{5}x(19 - 6y) + \frac{3y(1290417 + 722500y)}{3989125},\\
H_5(x,y) =& 16x(472818597 + 524021995x) - 75(46176919 + 33641680x)y\\
& + 455712500y^2,
\end{aligned}
$$

respectively. In order to have crossing limit cycles of types 7 and 8, simultaneously, such that the crossing limit cycles of type 7 intersect the discontinuity curve $\tilde{\Sigma}_{k+}$ in four different points $p_1 = (x_1, k)$, $p_2 = (x_2, k)$, $p_3 = (x_3, x_3^2)$ and $p_4 = (x_4, x_4^2)$, with

$-2 < x_2 < x_1 < 2$ and $-2 < x_3 < x_4 < 2$, and the crossing limit cycles of type 8 intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_5 = (x_5, x_5^2)$, $p_6 = (x_6, x_6^2)$, $p_7 = (x_7, k)$ and $p_8 = (x_8, k)$, with $x_6 < -2 < 2 < x_5$ and $x_7 < -2 < 2 < x_8$, these points must satisfy systems (4-34) and (4-37), respectively. Considering the PWLC (4-58) and $k = 4$, systems (4-34) and (4-37) become

$$
\begin{aligned}
4(x_1 - x_2)\left(-\frac{3}{10} + x_1 + x_2\right) &= 0,\\
-6561675700 - 2527406448x_2 + 8384351920x_2^2 - 7565097552x_3 &\\
-4921082995x_3^2 + 2523126000x_3^3 - 455712500x_3^4 &= 0,\\
30317350x_3 + 19827751x_3^2 - 9573900x_3^3 + 2167500x_3^4 &\\
-x_4(30317350 + 19827751x_4 - 9573900x_4^2 + 2167500x_4^3) &= 0,\\
6561675700 + 2527406448x_1 - 8384351920x_1^2 + 7565097552x_4 &\\
+4921082995x_4^2 - 2523126000x_4^3 + 455712500x_4^4 &= 0,\\
86116150336 - 403026567315x_8 + 58546435625x_8^2 + 363741728955x_5 &\\
-75860004809x_5^2 + 9821209590x_5^3 - 1053867100x_5^4 &= 0,\\
6997939x_5 - 15118885x_5^2 - 3529300x_5^3 + 2042500x_5^4 + x_6(-6997939 &\\
+15118885x_6 + 3529300x_6^2 - 2042500x_6^3) &= 0,\\
-1.030050.. + 8(1.660379.. + 0.284660..x_6)x_6 + 4(1 + 0.174915..x_6)^2x_6^2 &\\
+0.064378..x_6^4 + 8(-1.138640.. - 1.660379..x_7) - 4(0.699661.. + x_7)^2 &= 0,\\
-4(x_8 - x_7)\left(-\frac{1}{2} + x_8 + x_7\right) &= 0.
\end{aligned}
\tag{4-59}
$$

We have four real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i, x_5^i, x_6^i, x_7^i, x_8^i)$ with $i = 1,2,3,4$, for system (4-59) that satisfy the above conditions. We have $q^1 = (1,\ -7/10,\ -9/10,\ -1/10,\ 37/10,\ -5/2,\ -3, 7/2)$; $q^2 = (1, -7/10, -9/10, -1/10, 4,\ -29/10,\ -33/10,\ 19/5)$; $q^3 = (11/10,\ -8/10,\ -26/25,\ 1/10,\ 21/5,\ -157/100,\ -7/2,\ 4)$ and $q^4 = (1.194602..,\ -0.894602..,\ -1.147986..,\ 0.273096..,\ 87/20,\ -3.312719..,\ -3653/1000,\ 4153/1000)$. These four real solutions generated three crossing limit cycles of type 7 and four crossing limit cycle of type 8. See these crossing limit cycles of the PWLC (4-58) in Figure 4.21.

Here we obtain a total of seven crossing limit cycles between limit cycles of type 7 and of type 8, moreover these seven crossing limit cycles have the configuration $(3,4)$. By our numerical computations we observed that this lower bound for the maximum number of crossing limit cycles of types 7 and 8 simultaneously, could not be obtained with the configuration $(4,3)$, because in the statement (v) of Theorem H we only got three crossing limit cycle of type 7.

This completes the proof of statement $(viii)$ of Theorem *I*. □

Proof of statement (ix) ***of Theorem I.*** We consider the following discontinuous PWLC

formed by the linear differential centers

$$
\begin{aligned}
&\dot{x} = -0.224106.. + 0.256615..x - 0.075244..y, && \dot{y} = -3.489877.. + x - 0.256615..y;\\
&\dot{x} = 33.031408.. - \frac{x}{2} - 5.321982..y, && \dot{y} = -816.418879.. + x + \frac{y}{2};\\
&\dot{x} = -0.151463.. - 0.173662..x - 0.047290..y, && \dot{y} = 0.297861.. + x + 0.173662..y;\\
&\dot{x} = 2 + \frac{x}{20} - \frac{13}{200}y, && \dot{y} = -\frac{111}{20} + x - \frac{y}{20},
\end{aligned}
\tag{4-60}
$$

in the regions $R^1_{\tilde{\Sigma}_{k+}}$, $R^2_{\tilde{\Sigma}_{k+}}$, $R^3_{\tilde{\Sigma}_{k+}}$ and $R^4_{\tilde{\Sigma}_{k+}}$, respectively. The linear differential centers in (4-60) have the first integrals

$$
\begin{aligned}
H_1(x,y) =& x^2 + x(-6.979755.. - 0.513231..y) + (0.448213.. + 0.075244..y)y,\\
H_2(x,y) =& x^2 + x(-1632.837759.. + y) + y(-66.062816.. + 5.321982..y),\\
H_3(x,y) =& x^2 + x(0.595723.. + 0.347324..y) + (0.302926.. + 0.047290..y)y,\\
H_4(x,y) =& 4x^2 - 16y + \frac{13}{50}y^2 - \frac{2}{5}x(111 + y),
\end{aligned}
$$

respectively.

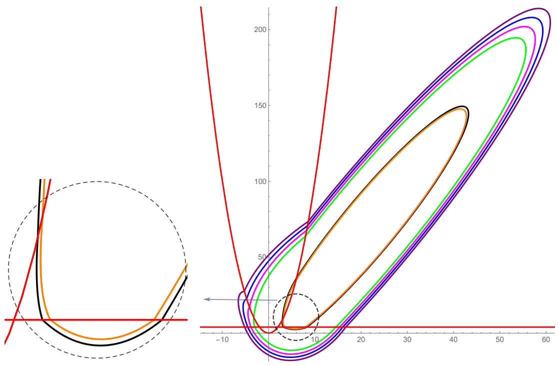

Figure 4.22: *Four crossing limit cycles of type 8 and two crossing limit cycles of type 9^+ (black and orange) of the discontinuous PWLC (4-60). These limit cycles are traveled in counterclockwise.*

In order to have simultaneously crossing limit cycles of types 8 and 9^+, such that the crossing limit cycles of type 8 intersect the discontinuity curve $\tilde{\Sigma}_{k^+}$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, x_2^2)$, $p_3 = (x_3, k)$ and $p_4 = (x_4, k)$, with $x_2 < -2 < 2 < x_1$ and $x_3 < -2 < 2 < x_4$, and the crossing limit cycles of type 9^+ intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_5 = (x_5, x_5^2)$, $p_6 = (x_6, x_6^2)$, $p_7 = (x_7, k)$ and $p_8 = (x_8, k)$, with $2 < x_6 < x_5$ and $2 < x_7 < x_8$, these points must satisfy systems (4-37) and (4-40), respectively. Considering the PWLC (4-60) and $k=4$, systems (4-37) and (4-40) become

$$
\begin{aligned}
&-6531.351039..x_1-260.251264..x_1^2+4x_1^3+21.287931..x_1^4+x_2(6531.351039..\\
&\qquad+260.251264..x_2-4x_2^2-21.287931..x_2^3)=0,\\
&-7.873414..+2.382895..x_2+5.211706..x_2^2+1.389297..x_2^3+0.189161..x_2^4\\
&\qquad-7.940084..x_3-4x_3^2=0,\\
&4(x_3-x_4)\left(-\frac{23}{2}+x_3+x_4\right)=0,\\
&11.987037..+27.919023..x_1-5.792854..x_1^2+2.052924..x_1^3-0.300976..x_1^4\\
&\qquad-36.130722..x_4+4x_4^2=0\\
&x_5(-6531.351039..+x_5(-260.251264..+x_5(4+21.287931..x_5)))\\
&\qquad+x_6(6531.351039..+x_6(260.251264..+(-4-21.287931..x_6)x_6))=0,\\
&-11.987037..+x_6(-27.919023..+x_6(5.792854..+(-2.052924..\\
&\qquad+0.300976..x_6)x_6))+(36.130722..-4x_7)x_7=0,\\
&4(x_7-x_8)\left(-\frac{23}{2}+x_7+x_8\right)=0,\\
&11.987037..+x_5(27.919023..+x_5(-5.792854..+(2.052924..\\
&\qquad-0.300976..x_5)x_5))+x_8(-36.130722..+4x_8)=0,
\end{aligned}
\tag{4-61}
$$

We have four real solutions $q^i=(x_1^i,x_2^i,x_3^i,x_4^i,x_5,x_6,x_7,x_8)$ with $i=1,2,3,4$, for system (4-61) that satisfy the above conditions. We have $q^1=(8,\,-16/5,\,-3,\,29/2,\,6,\,3,\,16/5,\,83/10)$; $q^2=(823/100,\,-4.136449..,\,-3.840062..,\,15.340062..,\,6,\,3,\,16/5,\,83/10)$; $q^3=(\,841/100,\,-4.748093..,\,-4.516514..,\,16.016514..,\,587/100,\,3.203924..,\,177/50,\,199/25)$ and $q^4=(429/50,\,-5.249123..,\,-5.170790..,\,16.670790..,\,587/100,\,3.203924..,\,177/50,\,199/25)$. These four real solutions generated four crossing limit cycles of type 8 and two crossing limit cycles of type 9^+. See these crossing limit cycles of the PWLC (4-60) in Figure 4.22.

Here we obtain a total of six crossing limit cycles between limit cycles of type 8 and of type 9^+, moreover these six crossing limit cycles have the configuration $(4,2)$. We observed that this lower bound for the maximum number of crossing limit cycles of types 8 and 9^+ simultaneously, could be also obtained with the configurations $(3,3)$. But if we build two crossing limit cycles of type 8 and two of type 9^+, simultaneously, we have that all the parameters that appear in system (4-40) are determined, where this system is such that generated limit cycles of type 9^+, then it is no possible to build a third crossing limit cycle of type 9^+ and therefore we can not obtain the configurations $(3,3)$.

We can also observe that there is a duality between the case studied in statement (iii) of Theorem I, where we have studied simultaneously crossing limit cycles of types 4 and 5 and this case, where study the crossing limit cycles of types 8 and 9^+, simultaneously. In these two cases we got the configuration $(4,2)$. See Figures 4.16 and 4.22.

This completes the proof of statement (ix) of Theorem *I*. $\square$

Proof of statement (x) ***of Theorem I.*** We consider the following discontinuous PWLC formed by the linear differential centers

$$
\begin{aligned}
&\dot{x} = -0.107128.. + 0.268308..x - 0.095415..y, && \dot{y} = -2.390037.. + x - 0.268308..y;\\
&\dot{x} = 0.492346.. + 0.144928..x - 0.061289..y, && \dot{y} = 0.429713.. + x - 0.144928..y;\\
&\dot{x} = 1.394400.. + 0.300769..x - 0.091362..y, && \dot{y} = 2.707746.. + x - 0.300769..y;\\
&\dot{x} = 0.976917.. + 0.400189..x - 4.241691..y, && \dot{y} = -0.349243.. + x - 0.400189..y;\\
&\dot{x} = 0.685228.. + 0.043300..x - 0.293631..y, && \dot{y} = 0.017396.. + x - 0.043300..y,
\end{aligned}
\tag{4-62}
$$

in the regions $R^1_{\tilde{\Sigma}_{k^+}}$, $R^2_{\tilde{\Sigma}_{k^+}}$, $R^3_{\tilde{\Sigma}_{k^+}}$, $R^4_{\tilde{\Sigma}_{k^+}}$ and $R^5_{\tilde{\Sigma}_{k^+}}$, respectively. The linear differential centers in (4-62) have the first integrals

$$
\begin{aligned}
H_1(x,y) =& x^2 + x(-4.780074.. - 0.536616..y) + (0.214257.. + 0.095415..y)y,\\
H_2(x,y) =& x^2 + x(0.859427.. - 0.289856..y) + (-0.984693.. + 0.061289..y)y,\\
H_3(x,y) =& x^2 + x(5.415492.. - 0.601538..y) + (-2.788801.. + 0.091362..y)y,\\
H_4(x,y) =& x^2 + x(-0.698486.. - 0.800378..y) + y(-1.953834.. + 4.241691..y),\\
H_5(x,y) =& x^2 + x(0.034792.. - 0.086601..y) + (-1.370456.. + 0.293631..y)y,
\end{aligned}
$$

respectively. In order to have crossing limit cycles of types 6^+, 7 and 8 simultaneously, such that the crossing limit cycles of type 6^+ intersect the discontinuity curve $\tilde{\Sigma}_{k^+}$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, k)$, $p_3 = (x_3, x_3^2)$ and $p_4 = (x_4, k)$, with $-2 < x_2 < 2 < x_1$ and $-2 < x_3 < 2 < x_4$, the crossing limit cycles of type 7 intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_5 = (x_5, k)$, $p_6 = (x_6, k)$, $p_7 = (x_7, x_7^2)$ and $p_8 = (x_8, x_8^2)$, with $x_5 < -2 < x_7 < 2$ and $x_6 < -2 < x_8 < 2$ and the crossing limit cycles of type 8 intersect the discontinuity curve $\tilde{\Sigma}_{k^+}$ in four different points $p_9 = (x_9, x_9^2)$, $p_{10} = (x_{10}, x_{10}^2)$, $p_{11} = (x_{11}, k)$ and $p_{12} = (x_{12}, k)$, with $x_{10} < -2 < 2 < x_9$ and $x_{11} < -2 < 2 < x_{12}$ these points must satisfy systems (4-31), (4-34) and (4-37) respectively. Considering the PWLC

(4-62) and $k = 4$, systems (4-31), (4-34) and (4-37) become

$$
\begin{aligned}
11.832571.. + 3.437710..x_1 + 0.061227..x_1^2 - 1.159427..x_1^3 + 0.245157..x_1^4 & \\
+1.200000..x_2 - 3.999999..x_2^2 &= 0,\\
-0.783728.. - 0.311613..x_2 + x_2^2 - 0.034792..x_3 + 0.370456..x_3^2 + 0.086601..x_3^3 & \\
-0.293631..x_3^4 &= 0,\\
-240.206876.. - 2.793946..x_3 - 3.815339..x_3^2 - 3.201513..x_3^3 + 16.966764..x_3^4 & \\
+15.600000..x_4 - 4x_4^2 &= 0,\\
9.534728.. + 19.120296..x_1 - 4.857030..x_1^2 + 2.146465..x_1^3 - 0.381662..x_1^4 & \\
-27.706159..x_4 + 4x_4^2 &= 0,\\
4(x_5 - x_6)(-0.300000.. + x_5 + x_6) &= 0,\\
-0.783728.. - 0.311613..x_6 + x_6^2 - 0.034792..x_7 + 0.370456..x_7^2 + 0.086601..x_7^3 & \\
-0.293631..x_7^4 &= 0,\\
-2.793946..x_7 - 3.815339..x_7^2 - 3.201513..x_7^3 + 16.966764..x_7^4 + x_8(2.793946.. & \\
+3.815339..x_8 + 3.201513..x_8^2 - 16.966764..x_8^3) &= 0,\\
-0.783728.. - 0.311613..x_5 + x5^2 - 0.034792..x_8 + 0.370456..x_8^2 & \\
+0.086601..x_8^3 - 0.293631..x_8^4 &= 0,\\
-3.437710..x_{10} - 0.061227..x_{10}^2 + 1.159427..x_{10}^3 - 0.245157..x_{10}^4 & \\
+x_9(3.437710.. + 0.061227..x_9 - 1.159427..x_9^2 + 0.245157..x_9^3) &= 0,\\
38.773655.. + 21.661968..x_{10} - 7.155207..x_{10}^2 - 2.406152..x_{10}^3 + 0.365448..x_{10}^4 & \\
-12.037359..x_{11} - 4x_{11}^2 &= 0,\\
4(x_{11} - x_{12})(-3.900000.. + x_{11} + x_{12}) &= 0,\\
2.383682.. - 6.926539..x_{12} + x_{12}^2 + 4.780074..x_9 - 1.214257..x_9^2 & \\
+0.536616..x_9^3 - 0.095415..x_9^4 &= 0.
\end{aligned}
\tag{4-63}
$$

We have four real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i, x_5^i, x_6^i, x_7^i, x_8^i, x_9^i, x_{10}^i, x_{11}^i, x_{12}^i)$ with $i = 1,2,3,4$, for system (4-63) that satisfy the above conditions, namely $q^1 = (4,\ -9/5,\ -19/10,\ 7/2,\ 1,\ -7/10,\ -9/10,\ 11/10,\ 5,\ -27/10,\ -5/2,\ 32/5)$; $q^2 = (2007/500,\ -181/100,\ -1.905170..,\ 3.692535..,\ 101/100,\ -71/100,\ -941/1000,\ 1.132764..,\ 511/100,\ -2.805313..,\ -139/50,\ 167/25)$; $q^3 = (2007/500,\ -181/100,\ -1.905170..,\ 3.692535..,\ 101/100,\ -71/100,\ -941/1000,\ 1.132764..,\ 26/5,\ -2.891869..,\ -3.012824..,\ 6.912824..)$ and $q^4 = (2007/500,\ -181/100,\ -1.905170..,\ 3.692535..,\ 101/100,\ -71/100,\ -941/1000,\ 1.132764..,\ 549/10,\ -52.535582..,\ -883.528310..,\ 887.428310..)$. These four real solutions generated two crossing limit cycles of type 6^+, two crossing limit cycles of type 7 and four crossing limit cycles of type 8. See these crossing limit cycles of the PWLC (4-62) in Figure 4.23.

Here we obtain a total of eight crossing limit cycles between limit cycles of

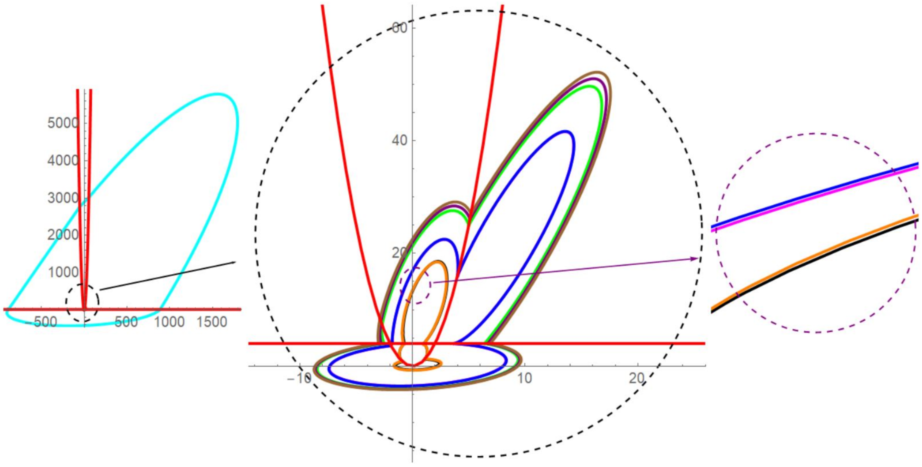

Figure 4.23: *Two crossing limit cycle of type* 6^+ *(magenta and blue), two crossing limit cycles of type* 7 *(black and orange) and four crossing limit cycles of type* 8 *(green, purple, brown and cyan) of the discontinuous PWLC (4-62). These limit cycles are traveled in counterclockwise.*

types 6^+, 7 and 8, moreover these eight crossing limit cycles have the configuration $(2,2,4)$, this is 2-crossing limit cycles of type 6^+, 2-crossing limit cycles of type 7 and 4-crossing limit of type 8. We observed that this lower bound for the maximum number of crossing limit cycles of types 6^+, 7 and 8 simultaneously, could be also obtained with other configurations. But if we build two crossing limit cycles of each type we obtain that all parameters of systems (4-31) and (4-34) are determined, and these systems are such that generated the limit cycles of types 6^+ and 7, then we can not build more than two crossing limit cycles of types 6^+ or 7 when we have previously fixed two crossing limit cycles of each type. Then we only obtain the configuration obtained here, namely $(2,2,4)$.

This completes the proof of statement (x) of Theorem *I*. $\square$

Proof of statement (xi) ***of Theorem I.*** We consider the following discontinuous PWLC formed by the linear differential centers

$$
\begin{aligned}
&\dot{x} = -0.312756.. + 0.105676..x - 0.022483..y, && \dot{y} = -4.523476.. + x - 0.105676..y;\\
&\dot{x} = -0.158662.. + 0.176712..x - 0.031977..y, && \dot{y} = -1.018470.. + x - 0.176712..y;\\
&\dot{x} = 0.893671.. + \frac{x}{10} - 0.055338..y, && \dot{y} = 1.647781.. + x - \frac{y}{10};\\
&\dot{x} = -1.521810.. + 0.129660..x - 0.102089..y, && \dot{y} = -4.531357.. + x - 0.129660..y;\\
&\dot{x} = 2.392166.. + 0.863445..x - 1.210282..y, && \dot{y} = 11.457801.. + x - 0.863445..y,
\end{aligned}
\tag{4-64}
$$

in the regions $R^1_{\tilde{\Sigma}_{k+}}$, $R^2_{\tilde{\Sigma}_{k+}}$, $R^3_{\tilde{\Sigma}_{k+}}$, $R^4_{\tilde{\Sigma}_{k+}}$ and $R^5_{\tilde{\Sigma}_{k+}}$, respectively. The linear differential centers in (4-64) have the first integrals

$$H_1(x,y) = x^2 + x(-9.046952.. - 0.211353..y) + (0.625512.. + 0.022483..y)y,$$
$$H_2(x,y) = x^2 + x(-2.03694.. - 0.353424..y) + (0.317325.. + 0.031977..y)y,$$
$$H_3(x,y) = x^2 + x\left(3.295563.. - \frac{y}{5}\right) + (-1.787342.. + 0.055338..y)y,$$
$$H_4(x,y) = x^2 + x(-9.062715.. - 0.259321..y) + (3.043621.. + 0.102089..y)y,$$
$$H_5(x,y) = x^2 + x(22.915603.. - 1.726890..y) + y(-4.784333.. + 1.210282..y),$$

respectively.

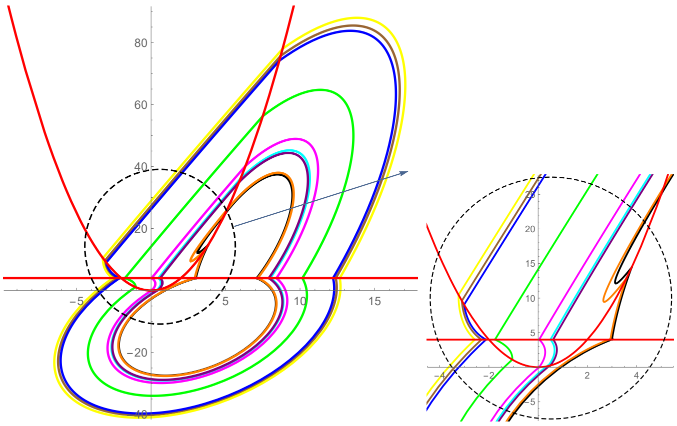

Figure 4.24: *Four crossing limit cycles of type* 6^+ *(green, magenta, cyan and purple), three crossing limit cycles of type* 8 *(yellow, brown and blue) and two crossing limit cycles of type* 9^+ *(black and orange) of the discontinuous PWLC (4-64). These limit cycles are traveled in counterclockwise.*

In order to have crossing limit cycles of types 6^+, 8 and 9^+ simultaneously, such that the crossing limit cycles of type 6^+ intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, k)$, $p_3 = (x_3, x_3^2)$ and $p_4 = (x_4, k)$, with $-2 < x_2 < 2 < x_1$ and $-2 < x_3 < 2 < x_4$, the crossing limit cycles of type 8 intersect the discontinuity curve $\tilde{\Sigma}_{k^+}$ in four different points $p_5 = (x_5, x_5^2)$, $p_6 = (x_6, x_6^2)$, $p_7 = (x_7, k)$ and $p_8 = (x_8, k)$, with $x_6 < -2 < 2 < x_5$ and $x_7 < -2 < 2 < x_8$ and the crossing limit cycles of type 9^+ intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_9 = (x_9, x_9^2)$, $p_{10} = (x_{10}, x_{10}^2)$, $p_{11} = (x_{11}, k)$ and $p_{12} = (x_{12}, k)$, with $2 < x_{10} < x_9$ and $2 < x_{11} < x_{12}$ these points must satisfy systems (4-31), (4-37) and (4-40) respectively. Considering the PWLC (4-64) and $k = 4$, systems (4-31), (4-37) and (4-40) become

$$
\begin{aligned}
&-7.123782..+x_1(-8.147767..+x_1(5.269300..\\
&+(-1.413698..+0.127911..x_1)x_1))+(13.802561..-4x_2)x_2=0,\\
&0.227189..+x_2(16.008041..+x_2)+x_3(-22.915603..\\
&+x_3(3.784333..+(1.726890..-1.210282..x_3)x_3))=0,\\
&-55.231640..+x_3(-36.250863..+x_3(16.174485..\\
&+(-1.037284..+0.408356..x_3)x_3))+\left(\frac{202}{5}-4x_4\right)x_4=0,\\
&11.447141..+x_1(36.187810..+x_1(-6.502051..+(0.845414..-0.089933..x_1)x_1))\\
&+x_4(-39.569467..+4x_4)=0,\\
&x_5(-8.147767..+x_5(5.269300..+(-1.413698..+0.127911..x_5)x_5))\\
&+x_6(8.147767..+x_6(-5.269300..+(1.413698..-0.127911..x_6)x_6))=0,\\
&25.055786..+x_6(13.182255..+x_6(-3.149369..+\left(-\frac{4}{5}+0.221355..x_6\right)x_6))\\
&+(-9.982255..-4x_7)x_7=0,\\
&4(x_7-x_8)\left(-\frac{101}{10}+x_7+x_8\right)=0,\\
&11.447141..+x_5(36.187810..+x_5(-6.502051..+(0.845414..-0.089933..x_5)x_5))\\
&+x_8(-39.569467..+4x_8)=0,\\
&x_{10}(8.147767..+x_{10}(-5.269300..+(1.413698..-0.127911..x_{10})x_{10}))\\
&+x_9(-8.147767..+x_9(5.269300..+(-1.413698..+0.127911..x_9)x_9))=0,\\
&-11.447141..+x_{10}(-36.187810..+x_{10}(6.502051..+(-0.845414..\\
&+0.089933..x_{10})x_{10}))+(39.569467..-4x_{11})x_{11}=0,\\
&4(x_{11}-x_{12})\left(-\frac{101}{10}+x_{11}+x_{12}\right)=0,\\
&2.861785..+(-9.892366..+x_{12})x_{12}+x_9(9.046952..\\
&+x_9(-1.625512..+(0.211353..-0.022483..x_9)x_9))=0.
\end{aligned}
\tag{4-65}
$$

We have four real solutions $q^i=(x_1^i,x_2^i,x_3^i,x_4^i,x_5^i,x_6^i,x_7^i,x_8^i,x_9^i,x_{10}^i,x_{11}^i,x_{12}^i)$ with $i=1,2,3,4,$ for system (4-65) that satisfy the above conditions, namely $q^1=(6,1/2,2/5,8,87/10,-31/10,-23/10,62/5,5,19/5,3,71/10)$; $q^2=(317/50,0.042569..,1/25,8.417274..,861/100,-3.007479..,-2.117234..,12.217234..,5,19/5,3,71/10)$; $q^3=(1479/250,\ -0.610424..,\ -1/2,\ 7.904488..,\ 883/100,\ -3.233408..,\ -2.568105..,\ 12.668105..,\ 51/10,\ 3.582979..,\ 2.936322..,\ 7.163677..)$, and $q^4=(15/2,\ -1.752776..,\ -1.049779..,\ 10.157706..,\ 883/100,\ -3.233408..,\ -2.568105..,\ 12.668105..,\ 51/10,\ 3.582979..,\ 2.936322..,\ 7.163677..)$ these four solutions generated four crossing limit cycles of type 6^+, three crossing limit cycles of type 8 and two crossing limit cycle of type 9^+. See these crossing limit cycles of the PWLC

(4-64) in Figure 4.24.

Here we obtain a total of nine crossing limit cycles between limit cycles of types 6^+, 8 and 9^+, moreover these nine crossing limit cycles have the configuration $(4,3,2)$. We observed that this lower bound for the maximum number of crossing limit cycles of types 6^+, 8 and 9^+ simultaneously, could be also obtained with other configurations. When we build two crossing limit cycles of each type we obtain that system (4-40) has all parameters determined, and therefore we can not build a third crossing limit cycle of type 9^+. Systems (4-31), (4-37) which generated the limit cycles of types 8 and 9^+ would still have free parameters and it is possible verify that we can have the configurations $(4,3,2)$ or $(3,4,2)$. Here we have illustrated the configuration $(4,3,2)$.

This completes the proof of statement (xi) of Theorem *I*. □

4.6 Discussions and conclusions

In this chapter we study on the numbers of crossing limit cycles with four points on the discontinuity curve Σ, when Σ is a reducible cubic curve formed either by a circle and a straight line, or by a parabola and a straight line, this is, the numbers $\mathcal{N}^4_{\Sigma_k}$ and $\mathcal{N}^4_{\tilde{\Sigma}_k}$, $k \in \mathbb{R}$.

First we study the crossing limit cycles for PWLC when the discontinuity curve is formed by a circle and a straight line, Σ_k, here we have three types of crossing limit cycles and we analyze the numbers $\mathcal{N}^4_{\Sigma_k}$ in each case. Similar to the case of PWLC separated by conic (LV) which was studied in the Chapter 3 due to total of parameters, unknown variables and the number the regions considered in these cases it is difficult to apply the usual techniques such as Grobner basis, resultant theory or Bezout inequality. Therefore in this chapter we only got provide lower bound for the maximum numbers of crossing limit cycles in each family.

With regard to the family of PWLC which discontinuity curve is formed by a parabola and a straight line, we observed that eight types of cycles arise and we study on the maximum number for each type, nevertheless similarly to the above case in Theorem H we only got provide lower bounds for the maximum numbers of crossing limit cycles in each family and of each type.

Later on in Theorem I we analyze the possibility of having PWLC with two or three types of crossing limit cycles simultaneously.

With the techniques used in this chapter we only got provide lower bounds for the maximum numbers of crossing limit cycles in each family, nevertheless by the numerical computations made for the families $\mathcal{F}_{\Sigma_k}$, $\mathcal{F}_{\tilde{\Sigma}_k}$ with $k \in \mathbb{R}$ and the illustrated examples of Theorems *G*, *H* and *I*, we believe that the numbers $\mathcal{N}^4_{\Sigma_k}$ and $\mathcal{N}^4_{\tilde{\Sigma}_k}$ are the upper bounds for the maximum numbers of crossing limit cycles in each family which we aim study in future works.

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