Zero entropy for some birational maps of $\mathbb{C}^2$*

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Abstract

This work deals with a special case of family of birational maps $f : \mathbb{C}^2 \to \mathbb{C}^2$ dynamically classified in [9]. In this work we study the zero entropy sub families of f. The sequence of degrees d_n associated to the iterates of f is found to grow periodically, linearly, quadratically or exponentially. Explicit invariant fibrations for zero entropy families and all the integrable and periodic mappings inside the family f are given.

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1 Introduction

Consider the family of fractional maps $f : \mathbb{C}^2 \to \mathbb{C}^2$ of the form:

$$f(x,y) = \left(\alpha_0 + \alpha_1 x + \alpha_2 y, \frac{\beta_0 + \beta_1 x + \beta_2 y}{\gamma_0 + \gamma_1 x + \gamma_2 y}\right),\ (\gamma_1, \gamma_2) \neq (0,0) \tag{1}$$

where the parameters $\alpha_i,\ \beta_i,\ \gamma_i,\ i \in \{0,1,2\}$ are complex numbers.

In this work the family of mappings $f(x,y)$ in (1) is required to be birational in general. The values of parameters $\alpha_i,\ \beta_i,\ \gamma_i,\ i \in \{0,1,2\}$ for which $f(x,y)$ is a birational mapping is discussed in Lemma 1 in this article. The study of the dynamics generated by birational mappings in the plane and their classification is a well discussed topic in recent years as can

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be found in [1, 3, 4, 12, 13, 15, 14, 18, 19, 20, 21, 22, 23, 26]. The family of mappings $f(x,y)$ in (1) is dynamically classified in [9]. In this work we are going to study the mapping f in case where it shows a kind of degenerate behavior for general values of parameters.

For a birational map $f(x,y)$ the sequence of degrees d_n of the iterates of f satisfies a homogeneous linear recurrence, see [16]. This is governed by the characteristic polynomial $\mathcal{X}(x)$ of a certain matrix associated to F, where $F : P\mathbb{C}^2 \to P\mathbb{C}^2$ is the extension of $f : \mathbb{C}^2 \to \mathbb{C}^2$ in the projective plane $P\mathbb{C}^2$. This further provides information regarding the quantity called as the *dynamical degree of* F and defined as

$$\delta(F) := \lim_{n\to\infty} (\deg(F^n))^{\frac{1}{n}}, \tag{2}$$

where F^n represents the iterates of F. The logarithm of $\delta(F)$ is the *algebraic entropy of* F, see [3, 4, 5, 15, 16].

Considering the embedding $(x_1, x_2) \in \mathbb{C}^2 \mapsto [1 : x_1 : x_2] \in P\mathbb{C}^2$ into projective space, the induced map $F : P\mathbb{C}^2 \to P\mathbb{C}^2$ has three components $F_i[x_0 : x_1 : x_2]$, $i = 1, 2, 3$ which are homogeneous polynomials as $F[x_0 : x_1 : x_2] = [F_1[x_0 : x_1 : x_2] : F_2[x_0 : x_1 : x_2] : F_3[x_0 : x_1 : x_2]]$, where

$$\begin{aligned}
F_1[x_0 : x_1 : x_2] &= x_0(\gamma_0 x_0 + \gamma_1 x_1 + \gamma_2 x_2),\\
F_2[x_0 : x_1 : x_2] &= (\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2)(\gamma_0 x_0 + \gamma_1 x_1 + \gamma_2 x_2),\\
F_3[x_0 : x_1 : x_2] &= x_0(\beta_0 x_0 + \beta_1 x_1 + \beta_2 x_2).
\end{aligned} \tag{3}$$

The map F has degree two as the components of F do not have common factors for general values of the parameters. Similarly the degree of each iterate of F can be found in general by iterating F and removing the common homogenous components as $F^n = F\circ\cdots\circ F$ for each $n \in \mathbb{N}$.

Birational mappings $F : P\mathbb{C}^2 \to P\mathbb{C}^2$ have an indeterminacy set $\mathcal{I}(F)$ of points where F is ill-defined as a continuous map. Hence they also have a set of curves which are sent to a single point called the *exceptional locus* of F denoted as $\mathcal{E}(F)$. Generically the mappings of the form (1) have three indeterminacy points. The exceptional locus is formed by three straight lines, each two of them intersecting on a single indeterminate point of F. We call them as *non degenerate mappings.* However in some cases exceptional locus is formed by only two straight lines. In this case these mappings are identified as *degenerate mappings.* Lemma 1 in preliminary results section discusses the conditions for birationality and degeneracy of family f in (1). This study includes all the subfamilies of $f(x,y)$ where it shows a degenerate behavior. The cases where exceptional locus is formed by three straight lines are discussed and studied in the papers [8] and [9]. Such cases are recognized as *non degenerate mappings.*

The first goal of this study is look for sequence of degrees d_n. This is done by performing a series of blow-up's in order to find the characteristic polynomial which determines the behaviour of d_n.

The second goal is to identify for which values of the parameters these mappings have zero algebraic entropy and extract dynamical consequences. For this we use the results of Diller and Favre, see [16], which characterize the growth rate of d_n with the existence of invariant fibrations. We find all the prescribed invariant fibrations in each one of this cases. We emphasize which elements of the family are integrable mappings. We also distinguish all the periodic mappings giving a pair of first integrals generically transverse.

The article is organized as follows: Section two is devoted to give some preliminary results on birational maps and family f, in order to describe the blow-up process and the Picard group. In Section three we study the subfamily $\alpha_1\gamma_2 - \alpha_2\gamma_1 = 0$, while in Section four we study the subfamily $\alpha_1\beta_2 - \alpha_2\beta_1 = 0$.

The results that we get are the following. We have named Theorem the results on the dynamical degree and the growth of d_n and Proposition the results on the zero entropy and existence of invariant fibrations. In this way in Section 3 we give Theorem 4 with Proposition 5 concerning the family $\alpha_1\gamma_2 - \alpha_2\gamma_1 = 0$. Section 4 which deals with mappings satisfying $\alpha_1\beta_2 - \alpha_2\beta_1 = 0$, is splitted in three subsections. We present Theorem 6 with Proposition 7 when $\gamma_1\gamma_2 \neq 0$, Theorem 8 with Proposition 9 ($\alpha_2 \neq 0$) and Proposition 11 ($\alpha_2 = 0$) when $\gamma_1 = 0$ and Theorem 12 with Proposition 13 when $\gamma_2 = 0$.

2 Preliminary results

Consider the mapping $F[x_0 : x_1 : x_2] : P\mathbb{C}^2 \to P\mathbb{C}^2$ in (3), then the exceptional locus of $F[x_0 : x_1 : x_2]$ is given as $\mathcal{E}(F) = \{S_0, S_1, S_2\}$, where

$$S_0 = \{x_0 = 0\}, \quad S_1 = \{\gamma_0 x_0 + \gamma_1 x_1 + \gamma_2 x_2 = 0\},$$

$$S_2 = \{(\alpha_1(\beta\gamma)_{02} - \alpha_2(\beta\gamma)_{01})\, x_0 + \alpha_1(\beta\gamma)_{12}x_1 + \alpha_2(\beta\gamma)_{12}x_2 = 0\}.$$

We have used the notation: $(\delta\epsilon)_{ij} = \delta_i\epsilon_j - \delta_j\epsilon_i$. The exceptional locus of $F^{-1}[x_0 : x_1 : x_2]$ is $\mathcal{E}(F^{-1}) = \{T_0, T_1, T_2\}$, where

$$\begin{aligned} T_0 &= \{(\gamma_0(\alpha\beta)_{12} - \gamma_1(\alpha\beta)_{02} + \gamma_2(\alpha\beta)_{01})\, x_0 - (\beta\gamma)_{12}x_1 = 0\}\,, \\ T_1 &= \{(\alpha\beta)_{12}x_0 - (\alpha\gamma)_{12}x_2 = 0\}, \qquad T_2 = \{x_0 = 0\}. \end{aligned}$$

The birational map $F[x_0 : x_1 : x_2]$ has an indeterminacy set $\mathcal{I}(F)$ of points where F is ill-defined as a continuous map. This set is given by:

$$\{[x_0 : x_1 : x_2] \in P\mathbb{C}^2 : F_1[x_0 : x_1 : x_2] = 0, F_2[x_0 : x_1 : x_2] = 0, F_3[x_0 : x_1 : x_2] = 0]\},$$

which gives:

$$\mathcal{I}(F) = \{O_1, O_2, O_3\},$$

where

$$\begin{aligned} O_0 &= [(\beta\gamma)_{12} : (\beta\gamma)_{20} : (\beta\gamma)_{01}], \\ O_1 &= [0 : \alpha_2 : -\alpha_1], \\ O_2 &= [0 : \gamma_2 : -\gamma_1], \end{aligned}$$

and $(\beta\gamma)_{ij} := \beta_i\gamma_j - \gamma_j\beta_i$ for $i, j = 0, 1, 2$.

By calling $g(x, y)$ the inverse of $f(x, y)$ given in (1) and considering $G[x_0 : x_1 : x_2]$ its extension on $P\mathbb{C}^2$, also a indeterminancy set $\mathcal{I}(G)$ exists i.e. $\mathcal{I}(G) = \{A_1, A_2, A_3\}$, where

$$\begin{aligned} A_0 &= [0 : 1 : 0], \\ A_1 &= [0 : 0 : 1], \\ A_2 &= [(\beta\gamma)_{12}\,(\alpha\gamma)_{12}, (\alpha_0\,(\beta\gamma)_{12} - \alpha_1\,(\beta\gamma)_{02} + \alpha_2\,(\beta\gamma)_{01})\,(\alpha\gamma)_{12} : (\alpha\beta)_{12}\,(\beta\gamma)_{12}]. \end{aligned}$$

We are interested in the birational mappings (1) when the corresponding F only has two distinct exceptional curves. Next lemma informs about the set of parameters which are available in this study.

Recall that a birational map is a map $f : \mathbb{C}^2 \to \mathbb{C}^2$ with rational components such that there exists an algebraic curve V and another rational map g such that $f \circ g = g \circ f = id$ in $\mathbb{C}^2 \setminus V$.

Lemma 1. *Consider the mappings*

$$f(x_1, x_2) = \left(\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2, \frac{\beta_0 + \beta_1 x_1 + \beta_2 x_2}{\gamma_0 + \gamma_1 x_1 + \gamma_2 x_2}\right), \; (\gamma_1, \gamma_2) \neq (0, 0).$$

Then:

(a) The mapping f is birational if and only if the vectors $(\beta_0, \beta_1, \beta_2)$, $(\gamma_0, \gamma_1, \gamma_2)$ are linearly independent and $((\alpha\beta)_{12}, (\alpha\gamma)_{12}) \neq (0, 0)$, $((\alpha\gamma)_{12}, (\beta\gamma)_{12}) \neq (0, 0)$, and either $((\alpha\beta)_{12}, (\beta\gamma)_{12}) \neq (0, 0)$ or $(\beta_1,\, \beta_2) = (0, 0)$.

(b) The mapping f is degenerate if and only if $(\beta\gamma)_{12} = 0$ or $(\alpha\gamma)_{12} = 0$.

Proof. The conditions in *(a)* are necessary for f to be invertible as if the vectors $(\beta_0, \beta_1, \beta_2)$, $(\gamma_0, \gamma_1, \gamma_2)$ are linearly dependent then the second component of f is a constant, also if $((\alpha\beta)_{12},\, (\alpha\gamma)_{12}) = (0, 0)$ or $((\alpha\gamma)_{12}, (\beta\gamma)_{12}) = (0, 0)$ then f only depends on $\alpha_1\, x_1 + \alpha_2\, x_2$ or on $\gamma_1 x_1 + \gamma_2 x_2$. If $((\alpha\beta)_{12}, (\beta\gamma)_{12}) = (0, 0)$ and $(\beta_1,\, \beta_2) \neq (0, 0)$ then f only depends on $\beta_1 x_1 + \beta_2 x_2$.

Now assume that conditions (a) are satisfied. Then the inverse of f which formally is

$$f^{-1}(x, y) = \left(\frac{-(\alpha\beta)_{02} + \beta_2 x + (\alpha\gamma)_{02} y - \gamma_2 xy}{(\alpha\beta)_{12} - (\alpha\gamma)_{12} y}, \frac{(\alpha\beta)_{01} - \beta_1 x + (\alpha\gamma)_{10} y + \gamma_1 xy}{(\alpha\beta)_{12} - (\alpha\gamma)_{12} y}\right),$$

is well defined. Furthermore the numerators of the determinants of the Jacobian of f and f^{-1} are

$$\alpha_1(\beta\gamma)_{02} - \alpha_2(\beta\gamma)_{01} + \alpha_1(\beta\gamma)_{12}x + \alpha_2(\beta\gamma)_{12}y \tag{4}$$

and

$$\alpha_0(\beta\gamma)_{12} - \alpha_1(\beta\gamma)_{02} + \alpha_2(\beta\gamma)_{01} - (\beta\gamma)_{12}y, \tag{5}$$

respectively. It is easily seen that conditions (a) imply that both (4) and (5) are not identically zero. Hence, $f \circ f^{-1} = f^{-1} \circ f = id$ in $\mathbb{C}^2 \setminus V$, where V is the algebraic curve determined by the common zeros of (4) and (5).

To see *(b)* we know that since S_i maps to A_i, this implies that the points A_0, A_1, A_2 are not all distinct. Since $A_0 \neq A_1$ we have two possibilities: $A_0 = A_2$ or $A_1 = A_2$. Condition $A_0 = A_2$ writes as $(\beta\gamma)_{12}\,(\alpha\gamma)_{12} = 0$ and $(\alpha\beta)_{12}(\beta\gamma)_{12} = 0$. From *(a)*, the vector $((\alpha\beta)_{12}, (\alpha\gamma)_{12}) \neq (0,0)$. Hence $(\beta\gamma)_{12}$ must be zero. In a similar way it is seen that $A_1 = A_2$ if and only if $(\alpha\gamma)_{12} = 0$. ∎

It is easy to see that F maps each S_i to A_i and that the inverse of F maps T_i to O_i for $i \in \{0,1,2\}$. To specify this behaviour we write $F : S_i \twoheadrightarrow A_i$ (also $F^{-1} : T_i \twoheadrightarrow O_i$). It is known that the dynamical degree depends on the orbits of A_0, A_1, A_2 under the action of F (see Proposition of section 2). Indeed, the key point is whether the iterates of A_0, A_1, A_2 coincide with any of the indeterminacy points of F. When we find such orbit of iterates of F that ends at some indeterminacy point of F we perform a series of blow up in order to remove the indeterminacy of F in the new extended space.

For $X = \{((x,y),[u:v]) \in \mathbb{C}^2 \times P\mathbb{C}^1 \;:\; xv = yu\}$ and $p \in \mathbb{C}^2$ we let (X, π) to be the blowing-up of $\mathbb{C}^2$ at the point p. By translating p at the origin, $\pi^{-1}p = \pi^{-1}(0,0) = \{((0,0),[u:v])\} := E_p \simeq P\mathbb{C}^1$ and $\pi^{-1}q = \pi^{-1}(x,y) = ((x,y),[x:y]) \in X$ for $q = (x,y) \neq (0,0)$. Every blow up gives a new expanded space X and a new induced map $\tilde{F} : X \to X$ is defined on it. Indeterminacy sets and exceptional locus can also be defined by considering meromorphic functions on complex manifolds X we get after a series of blow ups. Consider the Picard group of X denoted by $\mathcal{P}ic(X)$, where X is the complex manifold. For a generic line $L \in P\mathbb{C}^2$ the $\mathcal{P}ic(P\mathbb{C}^2)$ is generated by the class of L. If the base points of the blow-ups are $\{p_1, p_2, \ldots, p_k\} \subset P\mathbb{C}^2$ and $E_i := \pi^{-1}\{p_i\}$ then it is known that $\mathcal{P}ic(X)$ is generated by $\{\hat{L}, E_1, E_2, \ldots, E_k\}$, see [3, 4]. The curve $\hat{L}$ is the *strict transform* of $L \in \mathbb{C}^2$ is the adherence of $\pi^{-1}(C \setminus \{p\})$, in the Zariski topology. Furthermore $\pi : X \longrightarrow P\mathbb{C}^2$ induces a morphism of groups $\pi^* : \mathcal{P}ic(P\mathbb{C}^2) \longrightarrow \mathcal{P}ic(X)$, with the property that for any complex curve $C \subset P\mathbb{C}^2$,

$$\pi^*(C) = \hat{C} + \sum m_i\, E_i, \tag{6}$$

where m_i is the algebraic multiplicity of C at p_i. For $F \in P\mathbb{C}^2$, $\tilde{F}$ is denoted as natural extension of F on X and it induces a morphism of groups, $\tilde{F}^* : \mathcal{P}ic(X) \to \mathcal{P}ic(X)$ by considering the classes of preimages such that $\tilde{F}^*(\hat{L}) = d\,\hat{L} + \sum_{i=1}^{k} c_i\, E_i \quad , \quad c_i \in \mathbb{Z}$, where d is the degree of F. By iterating F, we get the corresponding formula by changing F by F^n and d by d_n. To know the behavior of the sequence of degrees d_n we deal with maps $\tilde{F}$ such that

$$(\tilde{F^n})^* = (\tilde{F}^*)^n. \tag{7}$$

Maps $\tilde{F}$ satisfying condition (7) are called *Algebraically Stable maps* (AS for short), (see [16]). In order to get AS maps we will use the following useful result showed by Fornaess and Sibony in [17] (see also Theorem 1.14) of [16]:

$$\text{The map } \tilde{F} \text{ is AS if and only if for every exceptional curve } C \text{ and all } n \geq 0\,,\ \tilde{F}^n(C) \notin \mathcal{I}(\tilde{F}). \tag{8}$$

It is known (see Theorem 0.1 of [16]) that one can always arrange for a birational map to be AS performing a finite number of blowing-up's. If it is the case and we call $\mathcal{X}(x) = x^k + \sum_{i=0}^{k-1} c_i\, x^i$ the characteristic polynomial of $A := (\tilde{F}^*)$, then since $\mathcal{X}(A) = 0$ and d_i is the $(1,1)$ term of A^i we get that $d_k = -(c_0 + c_1 d_1 + c_2 d_2 + \cdots + c_{k-1} d_{k-1})$, i.e., the sequence d_n satisfies a homogeneous linear recurrence with constant coefficients. The dynamical degree is then the largest real root of $\mathcal{X}(x)$. The following is a direct consequence of Theorem 0.2 of [16]. It is quiet useful in our work. Given a birational map F of $P\mathbb{C}^2$, let $\tilde{F}$ be its regularized map so that the induced map $\tilde{F}^* : \mathcal{P}ic(X) \to \mathcal{P}ic(X)$ satisfies $(\tilde{F^n})^* = (\tilde{F}^*)^n$. Then

Theorem 2. *Let $F : P\mathbb{C}^2 \to P\mathbb{C}^2$ be a birational map and let $d_n = deg(F^n)$. Then up to bimeromorphic conjugacy, exactly one of the following holds:*

- *The sequence d_n grows quadratically and $\tilde{F}$ is an automorphism preserving an elliptic fibration.*
- *The sequence d_n grows linearly and $\tilde{F}$ preserves a rational fibration. In this case $\tilde{F}$ cannot be conjugated to an automorphism.*
- *The sequence d_n is bounded and $\tilde{F}$ preserves a two generically transverse rational fibrations and $\tilde{F}$ is an automorphism.*
- *The sequence d_n grows exponentially.*

In the first three cases $\delta(F) = 1$ while in the last one $\delta(F) > 1$. Furthermore in the first and second, the invariant fibrations are unique.

Since we only deal with degenerate maps, we have to consider two subfamiles: $(\beta\gamma)_{12} = 0$ or $(\alpha\gamma)_{12} = 0$. We begin with the simplest case $(\alpha\gamma)_{12} = 0$.

3 Subfamily $(\alpha\gamma)_{12} = 0$.

Lemma 3. *Consider birational mappings*

$$f(x_1, x_2) = \left(\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2, \frac{\beta_0 + \beta_1 x_1 + \beta_2 x_2}{\gamma_0 + \gamma_1 x_1 + \gamma_2 x_2}\right), (\gamma_1, \gamma_2) \neq (0, 0)$$

with the condition $(\alpha\gamma)_{12} = \alpha_1\gamma_2 - \alpha_2\gamma_1 = 0$. *Then either*

(i) The four numbers $\alpha_1, \alpha_2, \gamma_1, \gamma_2$ *are distinct for zero.*

(ii) $\alpha_1 = 0, \gamma_1 = 0$ *and* $\alpha_2 \neq 0 \neq \gamma_2$.

(iii) $\alpha_2 = 0, \gamma_2 = 0$ *and* $\alpha_1 \neq 0 \neq \gamma_1$.

Proof. From Lemma 1 we know that $(\alpha_1, \alpha_2) \neq (0, 0)$. Then if α_1 (resp. γ_1) is zero then α_2 (resp. γ_2) is not and from $\alpha_1\gamma_2 - \alpha_2\gamma_1 = 0$ we get that γ_1 (resp. α_1) must be zero. ∎

Theorem 4. *Consider birational mappings*

$$f(x_1, x_2) = \left(\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2, \frac{\beta_0 + \beta_1 x_1 + \beta_2 x_2}{\gamma_0 + \gamma_1 x_1 + \gamma_2 x_2}\right), (\gamma_1, \gamma_2) \neq (0, 0)$$

with the condition $(\alpha\gamma)_{12} = \alpha_1\gamma_2 - \alpha_2\gamma_1 = 0$. *Then the following hold:*

(i) If $\alpha_1 \neq 0, \alpha_2 \neq 0, \gamma_1 \neq 0$ *and* $\gamma_2 \neq 0$, *then* $\delta(F) = 2$.

(ii) If $\alpha_1 = \gamma_1 = 0$, *then* $\delta(F) = \delta^*$ *and* $d_{n+2} = d_{n+1} + d_n$.

(iii) If $\alpha_2 = \gamma_2 = 0$, *then* $\delta(F) = 1$ *and* $d_n = 1 + n$.

Proof. From the hypothesis we have that: $\mathcal{E}(F) = \{S_0, S_1\}$, $\mathcal{I}(F) = \{O_0, O_1\}$, $\mathcal{E}(F^{-1}) = \{T_0, T_1\}$ and $\mathcal{I}(F^{-1}) = \{A_0, A_1\}$ with

$$S_0 = \{x_0 = 0\}, \, S_1 = \{\gamma_0 x_0 + \gamma_1 x_1 + \gamma_2 x_2 = 0\},$$

$$O_0 = [(\beta\gamma)_{12} : (\beta\gamma)_{20} : (\beta\gamma)_{01})], \, O_1 = [0 : \alpha_2 : -\alpha_1],$$

$$T_0 = \{(\beta_2(\alpha\gamma)_{01} - \beta_1(\alpha\gamma)_{12})x_0 - (\beta\gamma)_{12}x_1 = 0\}, \, T_1 = \{x_0 = 0\},$$

$$A_0 = [0 : 1 : 0], \, A_1 = [0 : 0 : 1].$$

When $\alpha_1, \alpha_2, \gamma_1$ and γ_2 are non zero we observe that $A_0 \neq O_0$ and $A_0 \neq O_1$. Hence, since $F(A_0) = [0 : \alpha_1 \gamma_1 : 0] = A_0$ and $F(A_1) = [0 : \alpha_2 \gamma_2 : 0] = A_0$ we get that F is AS. It implies that $d_n = 2^n$ and consequently $\delta(F) = 2$.

To prove (ii) we observe that $\alpha_1 = \gamma_1 = 0$ not only implies that $(\alpha_2, \gamma_2) \neq (0, 0)$ but also that $\beta_1 \neq 0$ (if not f would only depend on y and it would not be birational). Now

$A_0 = O_1 \in \mathcal{I}(F)$ and we have to blow-up this point. Let E_0 be the principal divisor at this point and consider a point $[u:v]_{E_0} \in E_0$. In order to extend F on E_0 we see $[u:v]_{E_0}$ as $\lim_{t\to 0}[tu:1:tv]$ and we are going to evaluate $F[tu:1:tv]$:

$$F[tu:1:tv] = [u(\gamma_0 u + \gamma_2 v)t : (\alpha_0 u + \alpha_2 v)(\gamma_0 u + \gamma_2 v)t : \beta_1 u + (\beta_0 u + \beta_2 v)ut].$$

Taking the limit when t tends to zero we have that when $u \neq 0$, $\tilde{F}[u:v]_{E_0} = [0:0:1]$ while $[0:1]_{E_0}$ becomes an indeterminacy point for $\tilde{F}$.

To know the action of $\tilde{F}$ on S_0 we see the point $[0:x_1:x_2]$ as $\lim_{t\to 0}[t:x_1:x_2]$. Then for $t \to 0$ (and $x_2 \neq 0$)

$$\lim_{t\to 0} F[t:x_1:x_2] = \lim_{t\to 0}[\gamma_2 x_2 t : \alpha_2\gamma_2 x_2^2 : (\beta_1 x_1 + \beta_2 x_2)t] = [\gamma_2 x_2 : \beta_1 x_1 + \beta_2 x_2]_{E_0}.$$

The above considerations imply that $\mathcal{I}(\tilde{F}) = \{O_0, [0:1]_{E_0}\}$, $\mathcal{E}(\tilde{F}) = \{\hat{S}_1, E_0\}$ with $\hat{S}_1 \twoheadrightarrow A_1$ and $E_0 \twoheadrightarrow A_1$. To follow the orbit of A_1 under $\tilde{F}$ we observe that $A_1 = [0:0:1] \in S_0$ and hence $\tilde{F}[0:0:1] = [\gamma_2 : \beta_2]_{E_0} \neq [0:1]_{E_0}$ and which is sent to A_1 again giving a two-periodic orbit. It implies that $\tilde{F} : X \longrightarrow X$ is AS. The Picard group of X is $Pic(X) = <\hat{L}, E_0>$ where L is a generic line of $P\mathbb{C}^2$. Let $\tilde{F}^*$ denote the corresponding map on $Pic(X)$, which acts just taking preimages. Hence $\tilde{F}^*(E_0) = \hat{S}_0$. In order to write $\hat{S}_0$ as a linear combination of $\hat{L}, E_0$ we are going to use (6). We have that $\pi^*(S_0) = \hat{S}_0 + E_0 = \hat{L}$ which implies that $\tilde{F}^*(E_0) = \hat{L} - E_0$. Also $\pi^*(F^{-1}(L)) = \widehat{F^{-1}}(L) + E_0 = 2\hat{L}$ which implies that $\tilde{F}^*(\hat{L}) = 2\hat{L} - E_0$. Hence the matrix of $\tilde{F}^*$ on $Pic(X) = <\hat{L}, E_0>$ is $\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$ with characteristic polynomial $z^2 - z - 1$. Hence $\delta(F) = \delta^*$.

To prove (iii) we observe, as before, that $\alpha_2 = \gamma_2 = 0$ not only implies that $(\alpha_1, \gamma_1) \neq (0,0)$ but also that $\beta_2 \neq 0$ (if not f would only depend on x and it would not be birational). Now $A_1 = O_1 = [0:0:1] \in \mathcal{I}(F)$ and we have to blow-up this point. To know the action of $\tilde{F}$ on S_1 we see the point $[\gamma_1 x_0 : -\gamma_0 x_0 : \gamma_1 x_2]$ as $\lim_{t\to 0}[\gamma_1 x_0 : t - \gamma_0 x_0 : \gamma_1 x_2]$. Similar computations as before give us that each point in $S_1 \setminus \{O_0, [0:0:1]\}$ is sent to the point $[\gamma_1 : (\alpha\gamma)_{01}]_{E_1}$. That is $\hat{S}_1$ is still exceptional for $\tilde{F}$.

Now consider a point $[u:v]_{E_1} \in E_1$. It is seen as $\lim_{t\to 0}[tu:tv:1]$. Then for $t \to 0$,

$$\lim_{t\to 0} F[tu:tv:1] = \lim_{t\to 0}[tu(\gamma_0 u + \gamma_1 v) : t(\gamma_0 u + \gamma_1 v)(\alpha_0 u + \alpha_1 v) : \beta_2 u].$$

If $\gamma_0 u + \gamma_1 v \neq 0$ and $u \neq 0$ then $\tilde{F}[u:v]_{E_1} = [u : \alpha_0 u + \alpha_1 v]_{E_1}$.

If $\gamma_0 u + \gamma_1 v = 0$, in the above computation with $[u:v]_{E_1} = [\gamma_1 : -\gamma_0]_{E_1}$ we deal with the point $[\gamma_1 t : -\gamma_0 t : 1] \in S_1$ and we have to apply $\tilde{F}$ giving $\tilde{F}[\gamma_1 t : -\gamma_0 t : 1] = [\gamma_1 : (\alpha\gamma)_{01}]_{E_1}$. We observe that $\lim_{u\to\gamma_1, v\to -\gamma_0} \tilde{F}[u:v]_{E_1} = [\gamma_1 : (\alpha\gamma)_{01}]_{E_1}$, that is $\tilde{F}$ is well defined.

But if $u = 0$, $F[0:t:1] = [0:1:0]$. It implies that $[0:1]_{E_1} \in \mathcal{I}(\tilde{F})$.

We claim that after this blow-up the map $\tilde{F}$ is AS. It is so because $S_0 \twoheadrightarrow A_0$ and A_0 is a fixed point of F and $\hat{S}_1 \twoheadrightarrow [\gamma_1 : (\alpha\gamma)_{01}]_{E_1}$ and the iterates of this point never coincide with $[0:1]_{E_1}$. The Picard group of X is now $Pic(X) =< \hat{L}, E_1 >$ where L is a generic line of $P\mathbb{C}^2$, $\tilde{F}^*(E_1) = \hat{S}_1 + E_1$, and similar computations as the ones of (ii) give the matrix

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

The characteristic polynomial is given by $(z-1)^2$. Hence $\delta(F) = 1$. Furthermore, since $d_1 = 2$ and $d_2 = 3$ we get that $d_n = 1 + n$. ∎

Proposition 5. *Assume that*

$$f(x_1, x_2) = \left(\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2, \frac{\beta_0 + \beta_1 x_1 + \beta_2 x_2}{\gamma_0 + \gamma_1 x_1 + \gamma_2 x_2}\right), (\gamma_1, \gamma_2) \neq (0,0)$$

with the condition $(\alpha\gamma)_{12} = \alpha_1\gamma_2 - \alpha_2\gamma_1 = 0$, *has zero entropy. Then after an affine change of coordinates it can be written as*

$$f(x,y) = \left(\alpha_0 + \alpha_1\, x, \frac{\beta_0 + y}{x}\right), \quad \alpha_1 \neq 0.$$

This map preserves the fibration $V(x,y) = x$ *and this fibration is unique.*

If $m(x) := \alpha_0 + \alpha_1 x$ *is periodic of period* p, *that is if* $\alpha_1^p = 1$ *for some* $p > 1$, $\alpha_1 \neq 1$, *then*

$$W(x,y) = x \cdot m(x) \cdot m(m(x)) \cdots m^{p-1}(x)$$

is a first integral of $f(x,y)$. *Also when* $\alpha_1 = 1$ *and* $\alpha_0 = 0$, f *is integrable.*

Proof. From Theorem 4 we know that the only zero entropy maps in the family are the ones with $\alpha_2 = \gamma_2 = 0$ and we also know that in this case β_2, α_1 and γ_1 are different from zero. Hence we can conjugate $f(x,y)$ with $h(x,y) = \left(\frac{\beta_2}{\gamma_1}x - \frac{\gamma_0}{\gamma_1}, \frac{1}{\beta_2}y + \frac{\beta_1}{\gamma_1}\right)$. Renaming the parameters we see that the conjugate map is of the form

$$f(x,y) = \left(\alpha_0 + \alpha_1\, x, \frac{\beta_0 + y}{x}\right), \quad \alpha_1 \neq 0.$$

Clearly this map preserves the fibration $V(x,y) = x$ and this fibration is unique from Theorem 2. If $\alpha_1^p = 1$ for some $p > 1$, $\alpha_1 \neq 1$, then $W(f(x,y)) = W(x,y)$ and the result follows. When $\alpha_1 = 1$ then we see that $f(x,y)$ is integrable if and only if $\beta_0 = 0$. ∎

Now we are going to consider the second subfamily.

4 Subfamily $(\beta\gamma)_{12} = 0$.

We are going to consider three different cases, depending on $\gamma_1\gamma_2 \neq 0$, $\gamma_1 = 0$ and $\gamma_2 = 0$. When $(\beta\gamma)_{12} = \beta_1\gamma_2 - \beta_2\gamma_1 = 0$, we have that $\mathcal{E}(F) = \{S_0, S_1\}$, $\mathcal{I}(F) = \{O_0, O_1\}$, $\mathcal{E}(F^{-1}) = \{T_0, T_1\}$ and $\mathcal{I}(F^{-1}) = \{A_0, A_1\}$ with

$$S_0 = \{x_0 = 0\}\,,\ S_1 = \{\gamma_0 x_0 + \gamma_1 x_1 + \gamma_2 x_2 = 0\}$$

$$O_0 = [0 : \gamma_2 : -\gamma_1]\,,\ O_1 = [0 : \alpha_2 : -\alpha_1]$$

$$T_0 = \{x_0 = 0\}\,,\ T_1 = \{(\alpha\beta)_{12} x_0 - (\alpha\gamma)_{12} x_2 = 0\}$$

$$A_0 = [0 : 1 : 0]\,,\ A_1 = [0 : 0 : 1].$$

4.1 $(\beta\gamma)_{12} = 0$ with $\gamma_1\gamma_2 \neq 0$

Theorem 6. *Consider birational mappings*

$$f(x_1, x_2) = \left(\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2, \frac{\beta_0 + \beta_1 x_1 + \beta_2 x_2}{\gamma_0 + \gamma_1 x_1 + \gamma_2 x_2}\right),\ (\gamma_1, \gamma_2) \neq (0, 0)$$

with the conditions $(\beta\gamma)_{12} = 0$ *and* $\gamma_1\gamma_2 \neq 0$. *Then either*

(i) $\alpha_1 \neq 0 \neq \alpha_2$ *and* $\delta(F) = 2$ *with* $d_n = 2^n$ *for all* $n \in \mathbb{N}$.

(ii) $\alpha_1 = 0$ *and the dynamical degree is* $\delta(F) = \delta^*$ *with* $d_{n+2} = d_{n+1} + d_n$ *for all* $n \in \mathbb{N}$.

(iii) $\alpha_2 = 0$ *and the dynamical degree is* $\delta(F) = 1$ *with* $d_n = 1 + n$ *for all* $n \in \mathbb{N}$.

Proof. To prove (i) we observe that $S_0 \twoheadrightarrow A_0$ and $S_1 \twoheadrightarrow A_1$ with $F(A_0) = [0 : \alpha_1\gamma_1 : 0] = A_0 \notin \mathcal{I}(F)$ and $F(A_1) = [0 : \alpha_2\gamma_2 : 0] = A_0 \notin \mathcal{I}(F)$. Thus using (8) we see that F is AS, which implies that $d_n = 2^n$ and consequently $\delta(F) = 2$.

Now consider that $\alpha_1 = 0$. It implies that $\alpha_2 \neq 0$. In this case $S_0 \twoheadrightarrow A_0 = O_1 \in \mathcal{I}(F)$. Hence we blow-up A_0 to obtain E_0. Similar computations as before says that $\tilde{F}$ sends $\hat{S}_0 \to E_0 \to \hat{T}_1$ and no new indeterminacy points are created.

Now we have to follow the orbit of A_1 under the action of $\tilde{F}$. As $A_1 \in S_0$ we find that $\tilde{F}(A_1) = [\gamma_2 : \beta_2]_{E_0}$ and $\tilde{F}[\gamma_2 : \beta_2]_{E_0} = [\gamma_1\gamma_2 : \alpha_0\gamma_2 + \alpha_2\beta_2 : \beta_1\gamma_2] \in T_1$. Observe that $\mathcal{I}(\tilde{F}) = \{O_0\}$ and $O_0 \in S_0 = T_0$. We know that the only points on T_0 which have preimages are A_0 and A_1 which implies that if the iterates of A_1 reaches O_0 for some iterate of F then O_0 should be equal to either A_0 or A_1. But the conditions on the parameters implies that $A_0 \neq O_0 \neq A_1$. This implies that O_0 has no preimages hence the iterates of A_1 cannot reach O_0. Hence we see that $\tilde{F}$ is AS.

In this case $\tilde{F}^*(\hat{L}) = 2\hat{L} - E_0$ and $\tilde{F}^*(E_0) = \hat{L} - E_0$. Hence the characteristic polynomial of the corresponding matrix is $z^2 - z - 1$. It implies that the dynamical degree is $\delta(F) = \frac{1+\sqrt{5}}{2}$ and $d_{n+2} = d_{n+1} + d_n$ for all $n \in \mathbb{N}$.

Finally to see (iii), since $\alpha_2 = 0$ we get that $\alpha_1 \neq 0$. Now we observe that S_0 collapses to $A_0 = [0:1:0] \in S_0$ and that $F[0:1:0] = [0:\alpha_1\gamma_1:0] = [0:1:0]$. Hence A_0 is a fixed point.

The other exceptional curve $S_1 \twoheadrightarrow A_1 = O_1 = [0:0:\alpha_1] = [0:0:1] \in \mathcal{I}(F)$. Hence we have to blow-up A_1 obtaining E_1. Similar computations as before says that $\tilde{F}$ sends $\hat{S}_1 \to E_1 \to \hat{T}_1$ and no new indeterminacy points are created. After this blow-up the mapping $\tilde{F}$ is AS. And we can see that $\tilde{F}^*(\hat{L}) = 2\hat{L} - E_0$ and $\tilde{F}^*(E_1) = \hat{L}$. Hence the matrix of $\tilde{F}^*$ is:

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}. \tag{9}$$

The characteristic polynomial is $(z-1)^2$, and hence the dynamical degree is 1. Since $d_1 = 2$, $d_2 = 3$ we get that the sequence of degrees is $d_n = 1 + n$ for all $n \in \mathbb{N}$. ∎

Concerning the zero entropy we see that the only case is the third one, when $\alpha_2 = 0$. The result (and the proof) we get is very similar to the one stated in Proposition 5.

Proposition 7. *Let*

$$f(x_1, x_2) = \left(\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2, \frac{\beta_0 + \beta_1 x_1 + \beta_2 x_2}{\gamma_0 + \gamma_1 x_1 + \gamma_2 x_2}\right), \ (\gamma_1, \gamma_2) \neq (0,0)$$

with the conditions $(\beta\gamma)_{12} = 0$ and $\gamma_1\gamma_2 \neq 0$ and assume that $f(x,y)$ has zero entropy. Then after an affine change of coordinates it can be written as

$$f(x,y) = \left(\alpha_0 + \alpha_1 x, \frac{\beta_0}{x+y}\right), \quad \alpha_1 \neq 0.$$

This map preserves the fibration $V(x,y) = x$ and this fibration is unique. If $m(x) := \alpha_0 + \alpha_1 x$ is periodic of period p, that is if $\alpha_1^p = 1$ for some $p > 1$, $\alpha_1 \neq 1$, then

$$W(x,y) = x \cdot m(x) \cdot m(m(x)) \cdots m^{p-1}(x)$$

is a first integral of $f(x,y)$. Also when $\alpha_1 = 1$ and $\alpha_0 = 0$, f is integrable.

4.2 $(\beta\gamma)_{12} = 0$ with $\gamma_1 = 0$.

Next Theorem gives the behaviour of d_n in this family. As we can see below after affine change of coordinates these mappings are simple, and the sequence of degrees can be deduced by elementary methods. We have adopted this point of view in the proof of item (ii). But in the first part we have preferred the blow-up approach. In fact some multiple blow-up's are implemented and it is amazing to see how the method detects the different behaviours of d_n.

Theorem 8. *Consider birational mappings*

$$f(x_1, x_2) = \left(\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2, \frac{\beta_0 + \beta_1 x_1 + \beta_2 x_2}{\gamma_0 + \gamma_1 x_1 + \gamma_2 x_2}\right), (\gamma_1, \gamma_2) \neq (0, 0)$$

with the conditions $(\beta\gamma)_{12} = 0$ *and* $\gamma_1 = 0$.

(i) Assume that $\alpha_2 \neq 0$. *Then, after an affine change of coordinates* $f(x, y)$ *can be written as*

$$f(x, y) = \left(\alpha_0 + \alpha_1 x + y, \frac{\beta_0}{\gamma_0 + y}\right), \ \alpha_1 \neq 0 \neq \beta_0 \tag{10}$$

and the following hold:

- *(a) If the onedimensional mapping* $h(y) := \frac{\beta_0}{\gamma_0 + y}$ *is not a periodic map, then the sequence of degrees is* $d_n = 1 + n$.
- *(b) If* $h(y)$ *is a k- periodic map and* $1 + \alpha_1^k + \alpha_1^{2k} + \cdots + \alpha_1^{nk} \neq 0$ *for all* $n \in \mathbb{N}$, *then* $d_n = 1 + n$ *for all* $n \leq k - 1$ *and* $d_n = k$ *for all* $n \geq k$.
- *(c) If* $h(y)$ *is a k-periodic map and* $1 + \alpha_1^k + \alpha_1^{2k} + \cdots + \alpha_1^{nk} = 0$ *for some* $n \in \mathbb{N}$, *then* d_n *is a* $(n+1)k-$*periodic sequence.*

(ii) Assume that $\alpha_2 = 0$. *Then, after an affine change of coordinates* $f(x, y)$ *can be written as*

$$f(x, y) = \left(\alpha_0 + \alpha_1 x, \frac{\beta_0}{\gamma_0 + y}\right) \ , \ \alpha_1 \neq 0 \neq \beta_0 \tag{11}$$

and the following hold:

- *(a) If the onedimensional mapping* $h(y) := \frac{\beta_0}{\gamma_0 + y}$ *is not a periodic map, then* $d_n = 2$ *for all* $n \in \mathbb{N}$.
- *(b) If* $h(y)$ *is a k- periodic map then* d_n *is a* $k-$*periodic sequence.*

Proof. We notice that since $\gamma_1 = 0\,,\ \gamma_2 \neq 0$ we can conjugate $f(x, y)$ with

$$\psi(x, y) = \left(\frac{\alpha_2}{\gamma_2} x, \frac{1}{\gamma_2} y + \frac{\beta_2}{\gamma_2}\right)$$

and renaiming the coefficients if necessary, we get the desired map (10). Now we have that

$$S_0 = \{x_0 = 0\}\,,\ S_1 = \{\gamma_0 x_0 + x_2 = 0\}\,,\ A_0 = [0:1:0]\,,\ A_1 = [0:0:1],$$

and

$$T_0 = \{x_0 = 0\}, T_1 = \{x_2 = 0\}, O_0 = [0:1:0], O_1 = [0:1:-\alpha_1].$$

Since $A_0 = O_0$ we have to blow-up this point getting E_0. Then

$$\tilde{F}[u:v]_{E_0} = [\gamma_0 u + v : \beta_0 u]_{E_0}\ ,\ [u:v]_{E_0} \neq [1:-\gamma_0]_{E_0}$$

and

$$\hat{S}_0 \twoheadrightarrow [1:0]_{E_0}.$$

The point $[1:-\gamma_0]_{E_0}$ is now an indeterminacy point of $\tilde{F}$. Hence, if $\tilde{F}^p[1:0]_{E_0} \neq [1:-\gamma_0]_{E_0}$ for all $p \in \mathbb{N}$, since $\hat{S}_1 \twoheadrightarrow A_1 \in S_0$, $\tilde{F}(A_1) = [1:0]_{E_0}$ and we get that $\tilde{F}$ is AS. It can be seen that the matrix of $\tilde{F}^* : Pic(X) \to Pic(X) =< \hat{L}, E_0 >$ is

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}. \tag{12}$$

The characteristic polynomial is $(z-1)^2$, and hence the dynamical degree is 1. Since $d_1 = 2$, $d_2 = 3$ we get that the sequence of degrees is $d_n = 1 + n$ for all $n \in \mathbb{N}$.

Now assume that there exists some $p \in \mathbb{N}$ such that $\tilde{F}^p[1:0]_{E_0} = [1:-\gamma_0]_{E_0}$. We claim that in this case $\tilde{F} : E_0 \to E_0$ is a $(p+2)-$periodic map. To prove the claim we distinguish between the case $\gamma_0 = 0$ (which gives a 2-periodic map and corresponds to $p = 0$) and the case $\gamma_0 \neq 0$. We have that

$$[1:0]_{E_0} \longrightarrow^{\tilde{F}^p} [1:-\gamma_0]_{E_0} \longrightarrow^{\tilde{F}} [0:1]_{E_0} \longrightarrow^{\tilde{F}} [1:0]_{E_0}.$$

Hence $\tilde{F}^{p+2}$ which in fact is a Moebius map, fixes at least three different points. It clearly implies that $\tilde{F}^{p+2}$ is the identity map. Since the restriction of $\tilde{F}$ at E_0 is exactly the map $h(y) = \frac{\beta_0}{\gamma_0 + y}$ extended to the projective line we can assert that $\tilde{F}^p[1:0]_{E_0} = [1:-\gamma_0]_{E_0}$ if ad only of $h(y)$ is a $(p+2)-$periodic map. Hence (a) is proved.

Following the process, if $\tilde{F}^p[1:0]_{E_0} = [1:-\gamma_0]_{E_0}$, we have to blow-up all the points $\tilde{F}^j[1:0]_{E_0}$ for $j = 0, 1, \ldots, p$. We call E_{0j} the corresponding principal divisors, getting:

$$E_{00} \longrightarrow E_{01} \longrightarrow E_{02} \longrightarrow \cdots \longrightarrow E_{0p}. \tag{13}$$

Calling again $\tilde{F}$ the map at this new variety we are going to see which is the image of S_0 and which is the image of E_{0p}.

A point of coordinate k in E_{00} is seen as $\lim_{t\to 0}[t:1:kt^2]$. Then, for any point in S_0 different from the indeterminacy points and for $t \sim 0$, we have that:

$$F(t, x_1, x_2) \sim [x_2 t : (\alpha_1 x_1 + \alpha_2 x_2)x_2 : \beta_0 t^2] = \left[\frac{t}{\alpha_1 x_1 + \alpha_2 x_2} : 1 : \frac{\beta_0}{x_2(\alpha_1 x_1 + \alpha_2 x_2)} t^2\right].$$

Naming $T := \frac{t}{\alpha_1 x_1+\alpha_2 x_2}$ this point looks like $\left[T : 1 : \frac{\beta_0(\alpha_1 x_1+\alpha_2 x_2)}{x_2} T^2\right]$, that is

$$\tilde{F}[0 : x_1 : x_2] = \frac{\beta_0(\alpha_1 x_1 + \alpha_2 x_2)}{x_2} \in E_{00}.$$

Now consider a point of coordinate k in E_{0p}. This point is seen as $\lim_{t\to 0}[t : 1 : -\gamma_0 t + kt^2]$. Then for $t \sim 0$,

$$F[t : 1 : -\gamma_0 t + kt^2] \sim [kt : \alpha_1 k : \beta_0] \to_{t\to 0} [0 : \alpha_1 k : \beta_0] \in S_0.$$

Hence (13) can be completed and we get the cycle:

$$\hat{S}_0 \longrightarrow E_{00} \longrightarrow E_{01} \longrightarrow E_{02} \longrightarrow \cdots \longrightarrow E_{0p} \longrightarrow \hat{S}_0.$$

Now since $S_1 \twoheadrightarrow A_1 \in S_0$ and $\tilde{F}^{p+2}$ sends $\hat{S}_0$ to itself, it could happen that for some $n \in \mathbb{N}$, $\tilde{F}^{n(p+2)}(A_1) = O_0$, which still is an indeterminacy point of $\tilde{F}$.

If it is not the case, these $\tilde{F}$ is AS. Let us to compute the matrix of $\tilde{F}^*$. The Picard group of X is $Pic(X) =< \hat{L}, E_{00}, E_{01}, \ldots, E_{0p}, E_0 >$. To write $\hat{S}_0$ and $\hat{S}_1$ as a linear combination of basis elements, we are going to use the identity (6). For instance $\pi^*(F^{-1}(L)) = F^{-\hat{1}}(L) + \sum_{j=1}^{p} m_j E_{0j}$ where the multiplicities m_j are the order of vanishing of $F^{-1}(L)$ at generic points of E_{0j}. If $\delta_0 x_0 + \delta_1 x_1 + \delta_2 x_2 = 0$ is the equation of a generic straight line L, then a calculation gives $\delta_0 F[t : 1 : wt + kt^2][1] + \delta_1 F[t : 1 : wt + kt^2][2] + \delta_2 F[t : 1 : wt + kt^2][3] = \delta_1 \alpha_1(\gamma_0 + w)t + o(t^2)$ which let us to write $\pi^*(F^{-1}(L)) = F^{-\hat{1}}(L) + \sum_{j=1}^{p-1} E_{0j} + 2E_{0p}$. Now from $\pi^*(F^{-1}(L)) = 2\hat{L}$ we get that $\tilde{F}^*(\hat{L}) = 2\hat{L} - \sum_{j=1}^{p-1} E_{0j} - 2E_{0p}$. Proceeding in this way we find that the matrix of $\tilde{F}^*$ is:

$$\begin{pmatrix} 2 & 1 & 0 & 0 & \ldots & 0 & 0 \\ -1 & -1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & 0 & 0 & \ldots & 1 & 0 \\ -2 & -1 & 0 & 0 & \ldots & 0 & 0 \\ -1 & -1 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}.$$

It is not hard to see that the characteristic polynomial of such a matrix is $(-z)^p\,(z-1)^2$. Hence the sequence of degrees satisfies $d_{n+p+2} = 2d_{n+p+1} - d_{n+p}$ and its behaviour depends on the initial conditions, i.e., on the first terms $d_1, d_2, \ldots, d_{p+2}$. So, if $h(y)$ is $k-$periodic, then $k = p + 2$ and $f^k(x, y)[2] = y$. It implies that $d_k = d_{k-1}$. Since the first degrees are $2, 3, 4, \ldots, k, k$, from $d_{n+k} = 2d_{n+k-1} - d_{n+k-2}$ we get that $d_n = k$ for all $n \geq k$. It remains to prove that the condition $\tilde{F}^{n(p+2)}(A_1) = O_0$ is equivalent to $1 + \alpha_1^k + \alpha_1^{2k} + \cdots + \alpha_1^{nk} = 0$

for $k = p + 2$. To this end, taking into account the terms of maximum degree of $f^k(x, y)$, (see (15) below) we get that:

$$\tilde{F}^k[0 : x_1 : x_2] = [0 : \alpha_1^{k-1}(\alpha_1 x_1 + x_2) : x_2]$$

and hence

$$\tilde{F}^{nk}[0 : 0 : 1] = [0 : \alpha_1^{k-1}(1 + \alpha_1^k + \alpha_1^{2k} + \cdots + \alpha_1^{(n-1)k} : 1].$$

Therefore $\tilde{F}^{nk}[0 : 0 : 1] = O_1 = [0 : 1 : -\alpha_1]$ if and only if

$$1 + \alpha_1^k + \alpha_1^{2k} + \cdots + \alpha_1^{nk} = 0. \tag{14}$$

Statement (b) is now proved. To see (c) we just compute $f^{(n+1)k}$. In this case since $\alpha_1 \neq 1$ we can consider (doing a translation if necessary) that $\alpha_0 = 0$. Now the expression of f^k is

$$f^k(x, y) = \Big(\alpha_1^k x + \alpha_1^{k-1} y + \alpha_1^{k-2} h(y) + \alpha_1^{k-3} h^2(y) + \cdots + \alpha_1 h^{k-2}(y) + h^{k-1}(y), y\Big) \tag{15}$$

Hence:

$$f^{(n+1)k}(x, y) = \Big(\alpha_1^{(n+1)k} x + (1 + \alpha_1^k + \alpha_1^{2k} + \cdots + \alpha_1^{nk})(\alpha_1^{k-1} y + \alpha_1^{k-2} h(y) + h^{k-1}(y)), y\Big).$$

Then since condition (14) implies that $\alpha_1^{(n+1)k} = 1$ we have that when condition (14) is satisfied, f is a $(n+1)k-$periodic map, and hence also the sequence of degrees is $(n+1)k-$periodic.

We are going to prove (ii). First of all, $\gamma_1 = 0$ implies $\gamma_2 \neq 0$, and from $\gamma_1\beta_2 - \gamma_2\beta_1 = 0$ we get $\beta_1 = 0$. Doing a translation on y and renaiming the coefficients we get equation (11). Since this map is very simple we are going to prove the result on the behaviour of d_n using simple arguments. We observe that the first component of $f^k(x, y)$ is $a_k x + b_k$ for certains a_k, b_k. And the second components are just the iterates of $h(y) = \frac{\beta_0}{\gamma_0 + y}$, a one-dimensional Möbius map. We claim that if $h(y)$ is not a periodic map then $h^k(y)$ is a Möbius map with non-constant denominator for all $k \in \mathbb{N}$ and also that the denominators of $h^i(y)$ and $h^j(y)$ are different for $i \neq j$. From the claim we can deduce that when $h(y)$ is not a periodic map then $d_n = 2$ for all $n \in \mathbb{N}$. And when $h(y)$ is a $k-$periodic map, then the sequence of degrees is $d_n = 2$ for all n which is not a multiple of k and $d_n = 1$ when n is a multiple of k.

To prove the claim we consider N_k and D_k with $h^k(z) = \frac{N_k}{D_k}$ and we see that, if we don't perform simplifications, $N_{k+1} = \beta_0\, D_k$ and $D_{k+1} = \gamma_0\, D_k + N_k$. Let $p_k, q_k \in \mathbb{C}$ such that $D_k = p_k + q_k\, z$. Then, $D_{k+2} - \gamma_0\, D_{k+1} - \beta_0\, D_k = 0$, which implies that $q_{k+2} - \gamma_0\, q_{k+1} - \beta_0\, q_k = 0$. Analyzing this linear recurrence with constant coefficients and taking into account that this sequence is $k-$periodic if and only if $\left(\frac{\lambda_2}{\lambda_1}\right)^k = 1$, where λ_1, λ_2 are the two different roots of $\lambda^2 - \gamma_0\, \lambda - \beta_0 = 0$ (see [11]), the claim follows.

∎

Proposition 9. *Consider the birational mappings*

$$f(x,y)=\left(\alpha_0+\alpha_1x+y,\frac{\beta_0}{\gamma_0+y}\right)\ ,\ \alpha_1\neq 0\neq\beta_0. \tag{16}$$

Then the following hold:

(a) If the onedimensional mapping $h(y):=\frac{\beta_0}{\gamma_0+y}$ *is not a periodic map, then* $f(x,y)$ *has the unique invariant fibration* $V_1(x,y)=y$.

(b) If $h(y)$ *is a* k*- periodic map and* $1+\alpha_1^k+\alpha_1^{2k}+\cdots+\alpha_1^{nk}\neq 0$ *for all* $n\in\mathbb{N}$*, then* $f(x,y)$ *is integrable being*

$$H_1(x,y)=y+h(y)+h(h(y))+\cdots+h^{k-1}(y)$$

a first integral and also has a second invariant fibration $V_2(x,y)$ *:*

(b_1) If $\alpha_1^k\neq 1$ *we can assume that* $\alpha_0=0$ *and then* $V_2(x,y)=$

$$(\alpha_1^k-1)x+\alpha_1^{k-1}y+\alpha_1^{k-2}h(y)+\alpha_1^{k-3}h^2(y)+\cdots+\alpha_1h^{k-2}(y)+h^{k-1}(y) \tag{17}$$

satisfies $V_2(f(x,y))=\alpha_1V_2(x,y)$.

(b_2) If $\alpha_1^k=1$ *but* $\alpha_1\neq 1$ *we can assume that* $\alpha_0=0$ *and then* $V_2(x,y)=$

$$\frac{kx+(k-1)\alpha_1^{k-1}y+(k-2)\alpha_1^{k-2}h(y)+(k-3)\alpha_1^{k-3}h(h(y))+\cdots+2\alpha_1^2h^{k-3}(y)+\alpha_1h^{k-2}(y)}{\alpha_1^{k-1}y+\alpha_1^{k-2}h(y)+\alpha_1^{k-3}h(h(y))+\cdots+\alpha_1h^{k-2}(y)+h^{k-1}(y)}$$

satisfies $V_2(f(x,y))=V_2(x,y)+1$.

(b_3) If $\alpha_1=1$ *then*

$$V_2(x,y)=\frac{kx+(k-1)y+(k-2)h(y)+(k-3)h(h(y))+\cdots+2h^{k-3}(y)+(y)}{k\alpha_0+\alpha_1^{k-1}y+\alpha_1^{k-2}h(y)+\alpha_1^{k-3}h(h(y))+\cdots+h^{k-2}(y)+h^{k-1}(y)}$$

satisfies $V_2(f(x,y))=V_2(x,y)+1$.

(c) If $h(y)$ *is a* k*-periodic map and* $1+\alpha_1^k+\alpha_1^{2k}+\cdots+\alpha_1^{nk}=0$ *for some* $n\in\mathbb{N}$*, then* $f(x,y)$ *has a second first integral* $H_2(x,y)$ *which can be given by* $H_2(x,y))=V_2^{(n+1)k}(x,y)$ *being* $V_2(x,y)$ *be defined by (17).*

The proofs are straightforward. Only say that to find the fibrations we have considered combinations of $x,y,h(y),h(h(y)),\ldots,h^{k-1}(y)$ or quotients of them.

Remark 10. *Assuming the hypothesis* (b)*, since* d_n *is a bounded sequence and* $f(x,y)$ *is not a periodic map, from [6] we know that* $f(x,y)$ *is birationally equivalent to either* $(x,y)\to(ax,by)$ *where* a *is a root of unity and* b *it is not or to* $(x,y)\to(ax,y+1)$*. The fibrations encountered in* (b) *let us to construct such a conjugations. In fact, when* $V_2(f(x,y))=\alpha_1V_2(x,y)$ *we are in the first case while when* $V_2(f(x,y))=V_2(x,y)+1$ *we are in the second one.*

The invariant fibrations and first integrals corresponding to the mappings satisfying (ii) of Theorem 8 are very easy after the adequate affine change of coordinates. Next Proposition gives this information.

Proposition 11. *Consider the birational mappings*

$$f(x,y) = \left(\alpha_0 + \alpha_1 x, \frac{\beta_0}{\gamma_0 + y}\right) \ , \ \alpha_1 \neq 0 \neq \beta_0. \tag{18}$$

These mappings preserve the two generically transverse invariant foliations $V_1(x,y) = x$ and $V_2(x,y) = y$. Furthermore,

(a) If $h(y) = \frac{\beta_0}{\gamma_0+y}$ is periodic of period k then

$$H_1(x,y) = y + h(y) + h(h(y)) + \cdots + h^{k-1}(y)$$

is a first integral of $f(x,y)$.

(b) If $m(x) := \alpha_0 + \alpha_1 x$ is periodic of period p then

$$H_2(x,y) = x + m(x) + m(m(x)) + \cdots + m^{p-1}(x)$$

is a first integral of $f(x,y)$.

(c) If $h(y)$ and $m(x)$ are $k-$periodic then $f(x,y)$ is a $k-$periodic mapping having two independent first integrals $H_1(x,y)$ and $H_2(x,y)$ with $p = k$.

4.3 $(\beta\gamma)_{12} = 0$ with $\gamma_2 = 0$

If $\gamma_2 = 0$ we know that $\gamma_1 \neq 0$ and from $(\beta\gamma)_{12} = 0$ we get $\beta_2 = 0$. Also $\alpha_2 \neq 0$, if not $f(x,y)$ only depens on x.

Theorem 12. *Consider birational mappings*

$$f(x_1,x_2) = \left(\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2, \frac{\beta_0 + \beta_1 x_1}{\gamma_0 + \gamma_1 x_1}\right), (\gamma_1, \alpha_2) \neq (0,0). \tag{19}$$

(a) Assume that $\alpha_1 \neq 0$. Then the dynamical degree of F is $\delta(F) = \delta^*$ and $d_{n+2} = d_n + d_{n+1}$.

(b) Assume that $\alpha_1 = 0$. Then after an affine change of coordinates $f(x,y)$ takes the form:

$$f(x_1,x_2) = \left(x_2, \frac{\beta_0}{\gamma_0 + x_1}\right) \tag{20}$$

and the dynamical degree of F is $\delta(F) = 1$. Furthermore:

(b_1) If $h(z) := \frac{\beta_0}{\gamma_0+z}$ is not a periodic map then $d_n = 2$ for all $n \in \mathbb{N}$.

(b_1) If $h(z)$ is a $k-$periodic map then d_n is a $2k-$periodic sequence.

Proof. To prove (a) we observe that now we have $S_1 \twoheadrightarrow A_1 = O_0 = [0:0:1]$ and $F(A_0) = [0:\alpha_1\gamma_1:0] = A_0 \notin \mathcal{I}(F)$. So we have to blow-up $A_1 = [0:0:1]$ getting E_1. Then $\tilde{F}$ sends $S_1 \to E_1 \twoheadrightarrow [0:1:0] = A_0$. Since $A_1 \in S_1$, $\pi^*(S_1) = \hat{S}_1 + E_1$ and the matrix of $\widetilde{F}^*$ is:

$$\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}. \tag{21}$$

Then the characteristic polynomial associated to F is $z^2 - z - 1$. Hence the dynamical degree is $\delta(F) = \delta^*$ and $d_{n+2} = d_{n+1} + d_n$ for all $n \in \mathbb{N}$.

We are going to prove (b). When $\alpha_1 = 0$ (19) can be transformed in (20) via the conjugation

$$\psi(x,y) = \left(\frac{1}{\gamma_1}x + \frac{\alpha_0\gamma_1 + \alpha_2\beta_1}{\gamma_1}, \frac{1}{\gamma_1\alpha_2}y + \frac{\beta_1}{\gamma_1}\right).$$

From (20) we have that $f(f(x,y)) = (h(x), h(y))$ and generally:

$$f^{2n}(x,y) = (h^n(x), h^n(y)) \ , \ f^{2n+1}(x,y) = \left(h^n(y), h^{n+1}(x)\right). \tag{22}$$

From the same arguments as before if h is not periodic $d_n = 2$ for all $n \in \mathbb{N}$. If h is $k-$periodic then $f^{2k}(x,y) = (x,y)$ and from (22) we get that $d_n = 2$ for all $n \in \mathbb{N}$ such that it is not a multiple of $2k$ and $d_n = 1$ for all $n \in \mathbb{N}$ such that it is a multiple of $2k$. In any case case the dynamical degree of F is $\delta(F) = 1$.

∎

Proposition 13. *Consider the family of mappings:*

$$f(x,y) = \left(y, \frac{\beta_0}{\gamma_0 + x}\right).$$

Then

(a) If $\gamma_0^2 + 4\,\beta_0 \neq 0$, let p and q be the two different roots of $z^2 - \gamma_0\, z - \beta_0 = 0$, and let m such that $m^2 = q/p$. Then $f(x,y)$ preserves the generically transverse fibrations

$$H_1(x,y) = \frac{m^2\,p^2 + mpx + p(m^2 - m + 1)y + xy}{(x+p)\,(y+p)},$$

$$H_2(x,y) = \frac{m^2\,p^2 - mpx + p(m^2 + m + 1)y + xy}{(x+p)\,(y+p)}$$

with $H_1(f(x,y)) = mH_1(x,y)$, $H_2(f(x,y)) = -mH_2(x,y)$. Furthermore $f(x,y)$ is $2k-$ periodic if and only if $m^{2k} = 1$ and in this case $H_1^{2k}(x,y)$ and $H_2^{2k}(x,y)$ are two independent first integrals of $f(x,y)$.

(b) If $\gamma_0^2 + 4\,\beta_0 = 0$ *then it preserves the two generically transverse fibrations*

$$K_1(x,y) = \frac{\gamma_0^2 - 2\gamma_0\,x + 6\gamma_0\,y + 4\,x\,y}{(2\,x+\gamma_0)\,(2\,y+\gamma_0)}\,,\; K_2(x,y) = \frac{2\;\gamma_0\,(x+y+\gamma_0)}{(2\,x+\gamma_0)\,(2\,y+\gamma_0)},$$

with $K_1(f(x,y)) = -K_1(x,y)\,,\; K_2(f(x,y)) = K_2(x,y)+1$. *Furthermore* $f(x,y)$ *is integrable being* $W(x,y) = (K_1(x,y))^2$ *a first integral.*

Proof. When $\gamma_0^2 + 4\,\beta_0 \neq 0$ some calculations give that in fact $H_1(f(x,y)) = mH_1(x,y)$ and $H_2(f(x,y)) = -mH_2(x,y)$. Furthermore $H_1(x,y), H_2(x,y)$ are generically transverse because the determinant of the Jacobian of $H_1(x,y), H_2(x,y)$ is

$$-\frac{2p^2m(m^2-1)}{(p+x)^2(p+y)^2}$$

which is different from zero (if not $m^2 = 1$ and it happens if and only if $p = q$, which is in contradiction with $\gamma_0^2 + 4\,\beta_0 \neq 0$).

Also when $\gamma_0^2+4\,\beta_0 = 0$ the determinant of the Jacobian of $K_1(x,y), K_2(x,y)$ is different from zero because it is equal to:

$$\frac{16c^2}{(2y+c)^2(2x+c)^2}.$$

Finally $W(x,y) = (K_1(x,y))^2$ is a first integral integral of $f(x,y)$ because $W(f(x,y)) = (K_1(f(x,y)))^2 = (-K_1(x,y))^2 = W(x,y)$. ∎

Remark 14. *Simple computations give that when* $\gamma_0^2+4\,\beta_0 \neq 0$, f *is birationally conjugated to* $(mx,-my)$ *via the conjugation* $\varphi(x,y) = (H_1(x,y),H_2(x,y))$ *and that when* $\gamma_0^2+4\,\beta_0 = 0$, f *is birationally conjugated to* $(-x,y+1)$ *via* $\psi(x,y) = (K_1(x,y),K_2(x,y))$.

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