NEW SYMMETRIC PERIODIC SOLUTIONS FOR THE MAXWELL-BLOCH DIFFERENTIAL SYSTEM

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Abstract. We provide sufficient conditions for the existence of a pair of symmetric periodic solutions in the Maxwell-Bloch differential equations modeling laser systems. These periodic solutions come from a zero-Hopf bifurcation studied using recent results in averaging theory.

1. Introduction and statement of the main result

In nonlinear optics the Maxwell–Bloch equations are used to describe laser systems. These equations were obtained by coupling the Maxwell equations with the Bloch equation (a linear Schrödinger like equation which describes the evolution of atoms resonantly coupled to the laser field), see [1]. Now in MathSciNet appear 265 articles related with these equations, see for instance [4, 5, 6, 9, 10, 11].

Recently in [7] it was studied the weak foci and centers of the Maxwell-Bloch system

$$\begin{aligned} \dot{u} &= -au + v, \\ \dot{v} &= -bv + uw, \\ \dot{w} &= -c(w-\delta) - 4uv. \end{aligned} \tag{1}$$

For $c = 0$ the differential system (1) has a singular line $\{(u, v, w)|u = 0, v = 0\}$; for $c \neq 0$ and $ac(\delta - ab) < 0$ the differential system (1) has one equilibrium $\mathbf{p}_0 = (0, 0, \delta)$; and for $c \neq 0$ and $ac(\delta - ab) > 0$ the differential system (1) has three equilibria $\mathbf{p}_+ = (u^*, v^*, w^*)$, $\mathbf{p}_- = (-u^*, -v^*, w^*)$ and $\mathbf{p}_0$, where

$$u^* = \sqrt{\frac{c(\delta - ab)}{4a}},\ v^* = a\sqrt{\frac{c(\delta - ab)}{4a}},\ w^* = ab.$$

For $a = \delta = 0$ the differential system has the singular line $L = \{(u, v, w)|v = 0, w = 0\}$. The periodic orbits bifurcating from the equilibrium $\mathbf{p}_0$ was studied in [2]. Here we complete this study analyzing the periodic orbits which bifurcate form the other two singularities.

We define a *zero-Hopf equilibrium* of a 3-dimensional autonomous differential system as an equilibrium point having two purely conjugate imaginary eigenvalues and a zero eigenvalue. The next result characterizes the zero-Hopf equilibria of system (1) that lies over the singular line L.

Proposition 1. *Consider $a = \delta = 0$, $c = -b$, $\omega \in (0, \infty)$ and*

$$\mathbf{q}_\pm = \left(\pm\frac{1}{2}\sqrt{b^2 + \omega^2}, 0, 0\right) \in L.$$

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The only zero-Hopf equilibria of system (1) *in the singular line* L *are* $\mathbf{q}_\pm$.

Proposition (1) is proved in Section 2. Perturbing the condition $a = \delta = 0$ the line of singularity L disappears. However the next result shows that there are two equilibrium points in L which produce an isolated periodic solution due to a zero-Hopf bifurcation.

Theorem 2. *Let* $\omega \in (0, \infty)$,

$$\begin{aligned} a =& \varepsilon^3 a_3, \\ b =& b_0 - \varepsilon c_1 - \varepsilon^2 c_2 + \varepsilon^3 b_3, \\ c =& -b_0 + \varepsilon c_1 + \varepsilon^2 c_2 + \varepsilon^3 \left(a_3 - b_3 + \frac{2a_3 b_0^2}{\omega^2} \right), \\ \delta =& -\varepsilon^3 \frac{a_3 \omega^2}{b_0}, \end{aligned}$$

with $(a_3, b_0, b_3, c_1, c_2) \in \mathbb{R}^5$ *and* ε *a small parameter. Then for* $|\varepsilon| \neq 0$ *sufficiently small the Maxwell-Bloch differential system* (1) *has two symmetric isolated periodic solutions bifurcating from the equilibrium points* $\mathbf{q}_\pm \in L$ *when* $\varepsilon = 0$ *and* $2a_3 c_1 (1 + 6b_0^2/(\omega^2 - 5b_0^2)) > 0$.

Theorem 2 is proved in Section 2.

2. The proofs

Proof of Proposition 1. Consider $\mathbf{q} = (\bar{u}, 0, 0) \in L$. In the following we discuss the conditions for $\mathbf{q}$ being a zero-Hopf equilibrium point of system (1). The characteristic equation at $\mathbf{q}$ is given as

$$-\lambda^3 + \lambda^2(-b - c) + \lambda \left(-bc - 4\bar{u}^2 \right) = 0. \tag{2}$$

It is easy to check that equation (2) has the pair of pure imaginary roots $\pm i\omega$ $(\omega > 0)$ if and only if $\bar{u} = \pm \frac{1}{2}\sqrt{b^2 + \omega^2}$ and $c = -b$. □

Since system (1) is invariant under the transformation $(x, y, z) \to (-x, -y, z)$ we proceed the proof only for the point $\mathbf{q}_-$.

Proof of Theorem 2. Assuming the conditions of Theorem 2 and translating $\mathbf{q}_-$ to the origin of coordinates, the differential system (1) writes

$$\begin{aligned} \dot{u} =& v + \varepsilon^3 \frac{a_3}{2} \left(\sqrt{b_0^2 + \omega^2} - 2u \right), \\ \dot{v} =& -\frac{w}{2}\sqrt{b_0^2 + \omega^2} - b_0 v + uw + \varepsilon c_1 v + \varepsilon^2 c_2 v - \varepsilon^3 b_3 v, \\ \dot{w} =& 2v\sqrt{b_0^2 + \omega^2} + b_0 w - 4uv - \varepsilon c_1 w - \varepsilon^2 c_2 w \\ &+ \varepsilon^3 \left(a_3 \left(w \left(-\frac{2b_0^2}{\omega^2} - 1 \right) + \omega^2 \right) + b_3 w \right) - \varepsilon^4 \frac{a_3 c_1 \omega^2}{b_0} - \varepsilon^5 \frac{a_3 c_2 \omega^2}{b_0} \\ &+ \varepsilon^6 \frac{a_3 \left(\omega^2 (b_3 - a_3) - 2a_3 b_0^2 \right)}{b_0}. \end{aligned} \tag{3}$$

In order to write the linear part of system (3) into its Jordan normal form, we do the linear change of variables $(u,v,w)\to(x,y,z)$ where

$$(u,v,w)=\left(z-\frac{x}{\omega},y,-\frac{2(b_0y+x\omega)}{\sqrt{b_0^2+\omega^2}}\right).$$

The differential system (3) becomes

$$\begin{aligned}
\dot{x}=&-\omega y+\frac{2(\omega z-x)\left(2b_0^2y+b_0x\omega+y\omega^2\right)}{\omega^2\sqrt{b_0^2+\omega^2}}-\frac{c_1\varepsilon(2b_0y+x\omega)}{\omega}-\frac{c_2\varepsilon^2(2b_0y+x\omega)}{\omega}\\
&+\varepsilon^3\left(-\frac{a_3\left(2b_0^2+\omega^2\right)(b_0y+x\omega)}{\omega^3}-\frac{1}{2}a_3\omega\sqrt{b_0^2+\omega^2}+b_3\left(\frac{2b_0y}{\omega}+x\right)\right)\\
&+\frac{a_3c_1\omega\varepsilon^4\sqrt{b_0^2+\omega^2}}{2b_0}+\frac{a_3c_2\omega\varepsilon^5\sqrt{b_0^2+\omega^2}}{2b_0}+\frac{a_3\varepsilon^6\sqrt{b_0^2+\omega^2}\left(2a_3b_0^2+\omega^2(a_3-b_3)\right)}{2b_0\omega},\\
\dot{y}=&\omega x+\frac{2(x-\omega z)(b_0y+x\omega)}{\omega\sqrt{b_0^2+\omega^2}}+c_1y\varepsilon+c_2y\varepsilon^2-b_3y\varepsilon^3,\\
\dot{z}=&\frac{2(\omega z-x)\left(2b_0^2y+b_0x\omega+y\omega^2\right)}{\omega^3\sqrt{b_0^2+\omega^2}}-\frac{c_1\varepsilon(2b_0y+x\omega)}{\omega^2}-\frac{c_2\varepsilon^2(2b_0y+x\omega)}{\omega^2}\\
&+\frac{\varepsilon^3\left(b_3\omega^2(2b_0y+x\omega)-a_3\left(2b_0^3y+2b_0^2x\omega+b_0y\omega^2+\omega^4z\right)\right)}{\omega^4}\\
&+\frac{a_3c_1\varepsilon^4\sqrt{b_0^2+\omega^2}}{2b_0}+\frac{a_3c_2\varepsilon^5\sqrt{b_0^2+\omega^2}}{2b_0}+\frac{a_3\varepsilon^6\sqrt{b_0^2+\omega^2}\left(2a_3b_0^2+\omega^2(a_3-b_3)\right)}{2b_0\omega^2}.
\end{aligned}\tag{4}$$

To study the periodic orbits of system (4) when $0<|\varepsilon|\ll 1$, we introduce the cylindrical coordinates $x=R\cos\theta$, $y=R\sin\theta$ and $z=Z$. Doing this transformation system (4) becomes

$$\begin{aligned}
\frac{dR}{dt}=&-\frac{2b_0R^2\cos(\theta)(b_0\sin(2\theta)+\omega\cos(2\theta))}{\omega^2\sqrt{b_0^2+\omega^2}}\\
&+\frac{2b_0RZ(b_0\sin(2\theta)+\omega\cos(2\theta))}{\omega\sqrt{b_0^2+\omega^2}}+\mathcal{O}(\varepsilon)\\
=&\mathbf{R}(\theta,R,Z),\\
\frac{d\theta}{dt}=&\omega+\frac{2R\cos(\theta)\left(b_0^2(-\cos(2\theta))+b_0^2+b_0\omega\sin(2\theta)+\omega^2\right)}{\omega^2\sqrt{b_0^2+\omega^2}}\\
&-\frac{2Z\left(b_0^2(-\cos(2\theta))+b_0^2+b_0\omega\sin(2\theta)+\omega^2\right)}{\omega\sqrt{b_0^2+\omega^2}}+\mathcal{O}(\varepsilon)\\
=&\mathbf{O}(\theta,R,Z),\\
\frac{dZ}{dt}=&-\frac{2R^2\cos(\theta)\left(\left(2b_0^2+\omega^2\right)\sin(\theta)+b_0\omega\cos(\theta)\right)}{\omega^3\sqrt{b_0^2+\omega^2}}\\
&+\frac{2RZ\left(\left(2b_0^2+\omega^2\right)\sin(\theta)+b_0\omega\cos(\theta)\right)}{\omega^2\sqrt{b_0^2+\omega^2}}+\mathcal{O}(\varepsilon)\\
=&\mathbf{Z}(\theta,R,Z).
\end{aligned}\tag{5}$$

Rescaling the variables (R,Z) of system (5) as $R=\varepsilon^2r$, $Z=\varepsilon z$, and taking θ as the new independent we obtain the equivalent differential system

$$\left(\frac{dr}{d\theta},\frac{dz}{d\theta}\right)=\varepsilon\mathbf{F}_1(r,z,\theta)+\varepsilon^2\mathbf{F}_2(r,z,\theta)+\varepsilon^3\mathbf{F}_3(r,z,\theta)+\varepsilon^4\mathbf{F}_4(r,z,\theta)+\mathcal{O}(\varepsilon^5),\tag{6}$$

where $\mathbf{F}_i(r,z,\theta)$ for $i=1,\dots,4$ are given in Appendix B.

Applying Theorem 3 from Appendix A to system (6) we calculate the correspondent averaging functions

$$\mathbf{g}_1(r,z)=(0,0),$$
$$\mathbf{g}_2(r,z)=(0,0),$$
$$\mathbf{g}_3(r,z)=\left(0,\frac{\pi\left(a_3\left(b_0^2+\omega^2\right)\left(2b_0^2c_1-4b_0z\sqrt{b_0^2+\omega^2}+c_1\omega^2\right)-2b_0^2r^2\right)}{b_0\omega^3\sqrt{b_0^2+\omega^2}}\right),$$
$$\begin{aligned}\mathbf{g}_4(r,z)=\Bigg(&\frac{4\pi a_3rz\sqrt{b_0^2+\omega^2}\left(5b_0^2+\omega^2\right)}{\omega^5}-\frac{\pi r}{b_0\omega^5\left(b_0^2+\omega^2\right)}\Big(a_3c_1\left(b_0^2+\omega^2\right)\\&\left(10b_0^4+9b_0^2\omega^2+\omega^4\right)-b_0^2r^2\left(5b_0^2+3\omega^2\right)\Big),\frac{2\pi z}{b_0\omega^5\left(b_0^2+\omega^2\right)}\\&\Big(a_3c_1\left(b_0^2+\omega^2\right)\left(12b_0^4+13b_0^2\omega^2+2\omega^4\right)-2b_0^2r^2\left(5b_0^2+4\omega^2\right)\Big)\\&+\frac{\pi}{b_0\omega^5\sqrt{b_0^2+\omega^2}}\Big(2b_0c_1r^2\left(5b_0^2+\omega^2\right)-a_3\left(b_0^2+\omega^2\right)\\&\left(6b_0^3\left(c_1^2+4z^2\right)+2b_0\omega^2\left(-b_0c_2+2c_1^2+6z^2\right)-c_2\omega^4\right)\Big)\Bigg).\end{aligned}$$

Note that the averaged equation $\mathbf{g}_3(r,z)$ vanishes over the graph

$$\mathcal{Z}=\{\mathbf{z}_\alpha=(\alpha,\beta(\alpha)):\alpha\in\mathbb{R}^+\}\subset U,$$

where $\beta(\alpha)=\dfrac{a_3c_1\left(b_0^2+\omega^2\right)\left(2b_0^2+\omega^2\right)-2b_0^2\alpha^2}{4a_3b_0\left(b_0^2+\omega^2\right)^{3/2}}$. Furthermore the Jacobian matrix of $\mathbf{g}_3$ at $\mathbf{z}_\alpha$ is

$$D\mathbf{g}_3(\mathbf{z}_\alpha)=\begin{pmatrix}0 & 0\\ -\frac{4b_0\pi\alpha}{\omega^3\sqrt{b_0^2+\omega^2}} & -\frac{4a_3\pi\left(b_0^2+\omega^2\right)}{\omega^3}\end{pmatrix}. \tag{7}$$

From (7) we have that $\Delta_\alpha=-4a_3\pi\left(b_0^2+\omega^2\right)/\omega^3\neq 0$. This verifies the conditions (i) and (ii) of Theorem 3. Thus we calculate the function $f(x)$ and we get

$$f(\alpha)=\frac{\pi b_0\alpha\left(\omega^2\left(\alpha^2-2a_3b_0^2c_1\right)-2a_3c_1\omega^4-5b_0^2\alpha^2\right)}{\omega^5\left(b_0^2+\omega^2\right)}.$$

It is easy to check that $f(\alpha)$ has the positive zero

$$\alpha^*=\omega\sqrt{2a_3c_1\left(\frac{6b_0^2}{\omega^2-5b_0^2}+1\right)}\quad\text{if}\quad 2a_3c_1\left(\frac{6b_0^2}{\omega^2-5b_0^2}+1\right)>0.$$

Moreover, α^* is a simple zero because $f'(\alpha^*)=4\pi a_3b_0c_1/\omega^3\neq 0$. Thus the result follows from applying Theorem 3. This conclude the proof. □

Appendix A: Averaging theory

We consider differential systems of the form

$$\dot{\mathbf{x}}=\mathbf{F}_0(t,\mathbf{x})+\varepsilon\mathbf{F}_1(t,\mathbf{x})+\varepsilon^2\mathbf{F}_2(t,\mathbf{x})+\varepsilon^3\mathbf{F}_3(t,\mathbf{x})+\varepsilon^4\mathbf{F}_4(t,\mathbf{x})+\varepsilon^5\widetilde{\mathbf{F}}(t,\mathbf{x},\varepsilon), \tag{8}$$

with $\mathbf{x}$ in some open subset Ω of $\mathbb{R}^n$, $t\in[0,\infty)$, $\varepsilon\in[-\varepsilon_0,\varepsilon_0]$. We assume $\mathbf{F}_i$ and $\widetilde{\mathbf{F}}$ for all $i=1,2,3,4$ are T–periodic in the variable t. Let $\mathbf{x}(t,\mathbf{z},0)$ be the solution of the unperturbed system

$$\dot{\mathbf{x}}=\mathbf{F}_0(t,\mathbf{x}),$$

such that $\mathbf{x}(0,\mathbf{z},0)=\mathbf{z}$. We define $M(t,\mathbf{z})$ the fundamental matrix of the linear differential system

$$\dot{\mathbf{y}}=\frac{\partial \mathbf{F}_0(t,\mathbf{x}(t,\mathbf{z},0))}{\partial \mathbf{x}}\mathbf{y},$$

such that $M(0,\mathbf{z})$ is the $n\times n$ identity matrix. The *displacement map* of system (8) is defined as

$$\mathbf{d}(\mathbf{z},\varepsilon)=\mathbf{x}(T,\mathbf{z},\varepsilon)-\mathbf{z}. \tag{9}$$

In order to have $\mathbf{d}(\mathbf{z},\varepsilon)$ well defined we assume that for $|\varepsilon|\neq 0$ sufficiently small the following hypothesis holds:

(H) there exists an open set $U\subset\Omega$ such that for all $\mathbf{z}\in U$ the solution $\mathbf{x}(t,\mathbf{z},\varepsilon)$ is defined on the interval $[0,t_{(\mathbf{z},\varepsilon)})$ with $t_{(\mathbf{z},\varepsilon)}>T$.

This hypothesis is always satisfied when the unperturbed system has a manifold of T-periodic solutions. The standard method of averaging for finding periodic solutions consists in write the displacement map (9) in power series of ε as follows

$$\mathbf{d}(\mathbf{z},\varepsilon)=\mathbf{g}_0(\mathbf{z})+\varepsilon\mathbf{g}_1(\mathbf{z})+\varepsilon^2\mathbf{g}_2(\mathbf{z})+\varepsilon^3\mathbf{g}_3(\mathbf{z})+\varepsilon^4\widetilde{\mathbf{g}}(\mathbf{z},\varepsilon),$$

Where for $i=0,1,2,3,4$ we have

$$\mathbf{g}_i(\mathbf{z})=M(T,\mathbf{z})^{-1}\frac{\mathbf{y}_i(T,\mathbf{z})}{i!},$$

being

$$\begin{aligned}
\mathbf{y}_0(t,\mathbf{z}) =&\mathbf{x}(t,\mathbf{z},0)-\mathbf{z},\\
\mathbf{y}_1(t,\mathbf{z}) =&M(t,\mathbf{z})\int_0^t M(\tau,\mathbf{z})^{-1}\mathbf{F}_1(\tau,\mathbf{x}(\tau,\mathbf{z},0))\mathrm{d}\tau,\\
\mathbf{y}_2(t,\mathbf{z}) =&M(t,\mathbf{z})\int_0^t M(\tau,\mathbf{z})^{-1}\Bigg[2\mathbf{F}_2(\tau,\mathbf{x}(\tau,\mathbf{z},0))+2\frac{\partial\mathbf{F}_1}{\partial\mathbf{x}}(\tau,\mathbf{x}(\tau,\mathbf{x},0))\mathbf{y}_1(\tau,\mathbf{z})\\
&+\frac{\partial^2\mathbf{F}_0}{\partial\mathbf{x}^2}(\tau,\mathbf{x}(\tau,\mathbf{z},0))\mathbf{y}_1(\tau,\mathbf{z})^2\Bigg]\mathrm{d}\tau,\\
\mathbf{y}_3(t,\mathbf{z}) =&M(t,\mathbf{z})\int_0^t M(\tau,\mathbf{z})^{-1}\Bigg[6\mathbf{F}_3(\tau,\mathbf{x}(\tau,\mathbf{z},0))+6\frac{\partial\mathbf{F}_2}{\partial\mathbf{x}}(\tau,\mathbf{x}(\tau,\mathbf{z},0))\mathbf{y}_1(\tau,\mathbf{z})\\
&+3\frac{\partial^2\mathbf{F}_1}{\partial\mathbf{x}^2}(\tau,\mathbf{x}(\tau,\mathbf{z},0))\mathbf{y}_1(\tau,\mathbf{z})^2+3\frac{\partial\mathbf{F}_1}{\partial\mathbf{x}}(\tau,\mathbf{x}(\tau,\mathbf{z},0))\mathbf{y}_2(\tau,\mathbf{z})\\
&+3\frac{\partial^2\mathbf{F}_0}{\partial\mathbf{x}^2}(\tau,\mathbf{x}(\tau,\mathbf{z},0))\mathbf{y}_1(\tau,\mathbf{z})\odot\mathbf{y}_2(\tau,\mathbf{z})+\frac{\partial^3\mathbf{F}_0}{\partial\mathbf{x}^3}(\tau,\mathbf{x}(\tau,\mathbf{z},0))\mathbf{y}_1(\tau,\mathbf{z})^3\Bigg]\mathrm{d}\tau,\\
\mathbf{y}_4(t,\mathbf{z}) =&M(t,\mathbf{z})\int_0^t M(\tau,\mathbf{z})^{-1}\Bigg[24\mathbf{F}_4(\tau,\mathbf{x}(\tau,\mathbf{z},0))+24\frac{\partial\mathbf{F}_3}{\partial x}(\tau,\mathbf{x}(\tau,x,0))\mathbf{y}_1(\tau,\mathbf{z})\\
&+12\frac{\partial^2\mathbf{F}_2}{\partial x^2}(\tau,\mathbf{x}(\tau,\mathbf{z},0))\mathbf{y}_1(\tau,\mathbf{z})^2+12\frac{\partial\mathbf{F}_2}{\partial x}(\tau,\mathbf{x}(\tau,\mathbf{z},0))\mathbf{y}_2(\tau,\mathbf{z})\\
&+12\frac{\partial^2\mathbf{F}_1}{\partial x^2}(\tau,\mathbf{x}(\tau,\mathbf{z},0))\mathbf{y}_1(\tau,\mathbf{z})\odot\mathbf{y}_2(\tau,\mathbf{z})+4\frac{\partial^3\mathbf{F}_1}{\partial x^3}(\tau,\mathbf{x}(\tau,\mathbf{z},0))\mathbf{y}_1(\tau,\mathbf{z})^3\\
&+4\frac{\partial\mathbf{F}_1}{\partial x}(\tau,\mathbf{x}(\tau,\mathbf{z},0))\mathbf{y}_3(\tau,\mathbf{z})+3\frac{\partial^2\mathbf{F}_0}{\partial x^2}(\tau,\mathbf{x}(\tau,\mathbf{z},0))\mathbf{y}_2(\tau,\mathbf{z})^2\\
&+4\frac{\partial^2\mathbf{F}_0}{\partial x^2}(\tau,\mathbf{x}(\tau,\mathbf{z},0))\mathbf{y}_1(\tau,\mathbf{z})\odot\mathbf{y}_3(\tau,\mathbf{z})
\end{aligned}$$

$$+ 6\frac{\partial^3 \mathbf{F}_0}{\partial x^3}(\tau, \mathbf{x}(\tau, \mathbf{z}, 0))\mathbf{y}_1(\tau, \mathbf{z})^2 \odot \mathbf{y}_2(\tau, \mathbf{z}) + \frac{\partial^4 \mathbf{F}_0}{\partial x^4}(\tau, \mathbf{x}(\tau, \mathbf{z}, 0))\mathbf{y}_1(\tau, \mathbf{z})^4 \Bigg] \mathrm{d}\tau.$$

The functions $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$ and $\mathbf{g}_4$ will be called here the *averaged functions* of order 1, 2, 3 and 4 respectively of system (8).

We say that system (8) has a periodic solution bifurcating from the point $\mathbf{z}_0$ if there exists a branch of solutions $x(t, \mathbf{z}(\varepsilon), \varepsilon)$ such that the displacement function satisfies $\mathbf{d}(\mathbf{z}(\varepsilon), \varepsilon) = \mathbf{0}$ and $\mathbf{z}(0) = \mathbf{z}_0$.

Let $\pi : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$ and $\pi^\perp : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$ denote the projections onto the first m coordinates and onto the last $n-m$ coordinates, respectively. For a point $\mathbf{z} \in U$ we also consider $\mathbf{z} = (a, b) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$. Consider the graph

$$\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta(\alpha)) : \alpha \in \overline{V}\} \subset U$$

such that $m < n$, V is an open set of $\mathbb{R}^m$ and $\beta : \overline{V} \to \mathbb{R}^{n-m}$ is a $\mathcal{C}^4$ function.

The next theorem provides sufficient conditions for the existence of periodic solutions of the differential system (8) when the set $\mathcal{Z}$ is a continuum of zeros to the first non vanishing averaged equation.

Theorem 3. *Let $r \in \{0, 1, 2, 3\}$ such that r is the first subindex such that $\mathbf{g}_r \not\equiv 0$. In addition to hypothesis (H) assume that*

(i) the averaged function $\mathbf{g}_r$ vanishes on $\mathcal{Z}$. That is $\mathbf{g}_r(\mathbf{z}_\alpha) = \mathbf{0}$ for all $\alpha \in \overline{V}$, and

(ii) the Jacobian matrix

$$D\mathbf{g}_r(\mathbf{z}_\alpha) = \begin{pmatrix} \Lambda_\alpha & \Gamma_\alpha \\ B_\alpha & \Delta_\alpha \end{pmatrix},$$

where $\Lambda_\alpha = D_a \pi g_r(z_\alpha)$, $\Gamma_\alpha = D_b \pi g_r(z_\alpha)$, $B_\alpha = D_a \pi^\perp g_r(z_\alpha)$ and $\Delta_\alpha = D_b \pi^\perp g_r(z_\alpha)$, satisfies that $\det(\Delta_\alpha) \neq 0$ for all $\alpha \in \overline{V}$.

We define the function

$$f(\alpha) = -\Gamma_\alpha \Delta_\alpha^{-1} \pi^\perp \mathbf{g}_{r+1}(\mathbf{z}_\alpha) + \pi \mathbf{g}_{r+1}(\mathbf{z}_\alpha),$$

Then the following statements hold.

(a) If there exists $\alpha^* \in V$ such that $f(\alpha^*) = 0$ and $\det(Df(\alpha^*)) \neq 0$, for $|\varepsilon| \neq 0$ sufficiently small, then there is an initial condition $\mathbf{z}(\varepsilon) \in U$ such that $\mathbf{z}(0) = \mathbf{z}_{\alpha^*}$ and the solution $\mathbf{x}(t, \mathbf{z}(\varepsilon), \varepsilon)$ of system (8) is T-periodic.

For a proof of Theorem 3 see [4]. The ideas of the proof were first presented in [8].

Appendix B: The functions $\mathbf{F}_i(r, z, \theta)$ for $i = 1, 2, 3, 4$.

In the following functions we take $S = \sin\theta$, $C = \cos\theta$, $S_2 = \sin(2\theta)$, $C_2 = \cos(2\theta)$ and $C_3 = \cos(3\theta)$.

$$\mathbf{F}_1(r, z, \theta) = \left(\frac{2r}{2\omega^2 \sqrt{b_0^2 + \omega^2}} \left(2b_0 z - c_1 \sqrt{b_0^2 + \omega^2} \right) (b_0 S_2 + \omega C_2) - a_3 \omega^2 \left(b_0^2 + \omega^2 \right) C, 0 \right)$$

$$\mathbf{F}_2(r, z, \theta) = \left(\frac{1}{4\omega^4} \left(\frac{1}{r\left(b_0^2 + \omega^2\right)} \left(\omega S \left(a_3 \omega \left(b_0^2 + \omega^2 \right) + 4rC \left(c_1 \sqrt{b_0^2 + \omega^2} - 2b_0 z \right) \right) \right) \right.$$

$$
\begin{aligned}
&+ 2b_0c_1r\sqrt{b_0^2+\omega^2} +2b_0rC_2\left(2b_0z - c_1\sqrt{b_0^2+\omega^2}\right) - 4rz\left(b_0^2+\omega^2\right)\Big)\\
&\left(a_3\omega^2\left(b_0^2+\omega^2\right)C - 2r\left(2b_0z - c_1\sqrt{b_0^2+\omega^2}\right)(b_0S_2+\omega C_2)\right)\\
&+\frac{2\omega}{b_0\sqrt{b_0^2+\omega^2}}\left(C\left(b_0^2\left(a_3c_1\omega^3-2r^2\omega\right)+a_3c_1\omega^5-4b_0^3r^2S_2\right)\right.\\
&\left.-2b_0r\omega\left(c_2\sqrt{b_0^2+\omega^2}(b_0S_2+\omega C_2)+b_0rC_3\right)\right)\Bigg), \frac{r}{\omega^3\sqrt{b_0^2+\omega^2}}\\
&\left(2S\left(-b_0c_1\sqrt{b_0^2+\omega^2}+2b_0^2z+\omega^2z\right)+\omega C\left(2b_0z-c_1\sqrt{b_0^2+\omega^2}\right)\right)\Bigg),\\
\mathbf{F}_3(r,z,\theta) = &\left(\frac{1}{4\omega^5}\left(\frac{1}{2r^2\omega\sqrt{b_0^2+\omega^2}}\left(2r\left(2b_0z-c_1\sqrt{b_0^2+\omega^2}\right)(b_0S_2\right.\right.\right.\\
&\left.+\omega C_2)-a_3\omega^2\left(b_0^2+\omega^2\right)C\right)\left(\frac{1}{b_0^2+\omega^2}\left(\omega S\left(a_3\omega\left(b_0^2+\omega^2\right)\right.\right.\right.\\
&\left.\left.+4rC\left(c_1\sqrt{b_0^2+\omega^2}-2b_0z\right)\right)+2b_0c_1r\sqrt{b_0^2+\omega^2}+2b_0rC_2\right.\\
&\left.\left(2b_0z-c_1\sqrt{b_0^2+\omega^2}\right)-4rz\left(b_0^2+\omega^2\right)\right)^2-\frac{2r\omega}{b_0\sqrt{b_0^2+\omega^2}}\\
&\left(-\omega S\left(a_3c_1\omega^2\left(b_0^2+\omega^2\right)-4b_0c_2r\sqrt{b_0^2+\omega^2}\right.\right.\\
&\left.\left.(b_0S+\omega C)\right)+4b_0r^2\left(2b_0^2+\omega^2\right)S^2C+8b_0^2r^2\omega SC^2+4b_0r^2\omega^2C^3\right)\right)\\
&+\frac{1}{b_0r\left(b_0^2+\omega^2\right)}\left(\omega C\left(4b_0^2r^2C_2-a_3c_1\omega^2\left(b_0^2+\omega^2\right)\right)+8b_0^3r^2SC^2\right.\\
&\left.+2b_0c_2r\omega\sqrt{b_0^2+\omega^2}(b_0S_2+\omega C_2)\right)\left(\omega S\left(a_3\omega\left(b_0^2+\omega^2\right)\right.\right.\\
&\left.\left.+4rC\left(c_1\sqrt{b_0^2+\omega^2}-2b_0z\right)\right)+2b_0c_1r\sqrt{b_0^2+\omega^2}+2b_0rC_2\right.\\
&\left.\left(2b_0z-c_1\sqrt{b_0^2+\omega^2}\right)-4rz\left(b_0^2+\omega^2\right)\right)+\frac{2\omega}{b_0}\\
&\left(-4a_3b_0^3r\omega C^2+a_3C\left(c_2\omega^4\sqrt{b_0^2+\omega^2}-4b_0^4rS\right)\right.\\
&\left.-b_0r\omega^2((a_3-2b_3)(b_0S_2+\omega C_2)+a_3\omega)\right)\Bigg),\\
&\frac{1}{2\omega^5}\left(\frac{\omega}{b_0\sqrt{b_0^2+\omega^2}}\left(a_3\omega^3\left(b_0^2c_1-2b_0z\sqrt{b_0^2+\omega^2}+c_1\omega^2\right)\right.\right.\\
&\left.-2b_0c_2r\omega^2\sqrt{b_0^2+\omega^2}C-4b_0r\left(b_0c_2\omega\sqrt{b_0^2+\omega^2}S\right.\right.\\
&\left.\left.+r\left(2b_0^2+\omega^2\right)SC+b_0r\omega C^2\right)\right)-\frac{1}{b_0^2+\omega^2}\left(2S\right.\\
&\left.\left(-b_0c_1\sqrt{b_0^2+\omega^2}+2b_0^2z+\omega^2z\right)+\omega C\left(2b_0z-c_1\sqrt{b_0^2+\omega^2}\right)\right)
\end{aligned}
$$

$$\left(\omega S\left(a_3\omega\left(b_0^2+\omega^2\right)+4rC\left(c_1\sqrt{b_0^2+\omega^2}-2b_0z\right)\right)\right.$$
$$\left.+2b_0c_1r\sqrt{b_0^2+\omega^2}+2b_0rC_2\left(2b_0z-c_1\sqrt{b_0^2+\omega^2}\right)-4rz\left(b_0^2+\omega^2\right)\right)\Bigg)\Bigg),$$

$$\mathbf{F}_4(r,z,\theta)=\left(\frac{1}{4\omega^4\sqrt{b_0^2+\omega^2}}\Big(-2c_3\delta_3\left(b_0^2+\omega^2\right)C\omega^2+\frac{1}{r}\left(2c_3r\omega C^2+\left(c_2\delta_3\sqrt{b_0^2+\omega^2}\right.\right.\right.$$
$$\left.-2b_0(b_3-c_3)rS\right)C+2b_3r\omega S^2\Big)\left(-4rz\omega^2C^2-S\left(b_0\delta_3\left(b_0^2+\omega^2\right)\right.\right.$$
$$\left.\left.+2r\omega C\left(4b_0z+(b_1-c_1)\sqrt{b_0^2+\omega^2}\right)+2rS\left(4zb_0^2+(b_1-c_1)\sqrt{b_0^2+\omega^2}b_0+2z\omega^2\right)\right)\right)$$
$$+\frac{1}{2r^2\omega^3}\left(-4b_0r^2\omega C^3-2r\left(4rSb_0^2+c_2\omega^2\sqrt{b_0^2+\omega^2}\right)C^2-\omega\left(-4b_0r^2S^2\right.\right.$$
$$\left.\left.-2b_0(b_2-c_2)r\sqrt{b_0^2+\omega^2}S+c_1\delta_3\left(b_0^2+\omega^2\right)\right)C-2b_2r\omega^2\sqrt{b_0^2+\omega^2}S^2\right)$$
$$\left(\frac{1}{b_0^2+\omega^2}\left(4rz\omega^2C^2+S\left(b_0\delta_3\left(b_0^2+\omega^2\right)+2r\omega C\left(4b_0z+(b_1-c_1)\sqrt{b_0^2+\omega^2}\right)\right.\right.\right.$$
$$\left.\left.\left.+2rS\left(4zb_0^2+(b_1-c_1)\sqrt{b_0^2+\omega^2}b_0+2z\omega^2\right)\right)\right)^2+\frac{2r\omega}{\sqrt{b_0^2+\omega^2}}\left(-4r^2\omega^2C^3\right.$$
$$-8b_0r^2\omega SC^2-S\left(2\left(2b_0^2+\omega^2\right)S_2r^2-2b_0(b_2-c_2)\omega\sqrt{b_0^2+\omega^2}Sr\right.$$
$$\left.\left.+c_1\delta_3\omega\left(b_0^2+\omega^2\right)\right)+(b_2-c_2)r\omega^2S_2\sqrt{b_0^2+\omega^2}\right)\Big)-\frac{1}{4r^3\omega^4}\left(2r\omega\left(2b_0z-c_1\sqrt{b_0^2+\omega^2}\right)\right.$$
$$C^2+b_0\left(\delta_3\left(b_0^2+\omega^2\right)+2rS\left(4b_0z+(b_1-c_1)\sqrt{b_0^2+\omega^2}\right)\right)C-2r\omega\left(2b_0z\right.$$
$$\left.+b_1\sqrt{b_0^2+\omega^2}\right)S^2\Big)\left(-4r^2S\left(2(b_3-c_3)r\omega C+2b_0(b_3-c_3)rS-c_2\delta_3\sqrt{b_0^2+\omega^2}\right)\omega^4\right.$$
$$+\frac{2r\omega}{b_0^2+\omega^2}\left(4rz\omega^2C^2+S\left(b_0\delta_3\left(b_0^2+\omega^2\right)+2rS\left(4zb_0^2+(b_1-c_1)\sqrt{b_0^2+\omega^2}b_0\right.\right.\right.$$
$$\left.\left.+2z\omega^2\right)\right)+r\omega S_2\left(4b_0z+(b_1-c_1)\sqrt{b_0^2+\omega^2}\right)\Big)\left(4r^2\omega^2C^3+8b_0r^2\omega SC^2+\omega S\right.$$
$$\left(c_1\delta_3\left(b_0^2+\omega^2\right)-2b_0(b_2-c_2)r\sqrt{b_0^2+\omega^2}S\right)+r\left((c_2-b_2)\sqrt{b_0^2+\omega^2}\omega^2+2r\left(2b_0^2+\omega^2\right)\right.$$
$$\left.S\Big)S_2\right)+\frac{1}{\sqrt{b_0^2+\omega^2}}\left(-4rz\omega^2C^2-S\left(b_0\delta_3\left(b_0^2+\omega^2\right)+2r\omega C\left(4b_0z\right.\right.\right.$$
$$\left.\left.\left.+(b_1-c_1)\sqrt{b_0^2+\omega^2}\right)+2rS\left(4zb_0^2+(b_1-c_1)\sqrt{b_0^2+\omega^2}b_0+2z\omega^2\right)\right)\right)\left(\frac{1}{b_0^2+\omega^2}\right.$$
$$\left(4rz\omega^2C^2+S\left(b_0\delta_3\left(b_0^2+\omega^2\right)+2r\omega C\left(4b_0z+(b_1-c_1)\sqrt{b_0^2+\omega^2}\right)\right.\right.$$
$$\left.\left.+2rS\left(4zb_0^2+(b_1-c_1)\sqrt{b_0^2+\omega^2}b_0+2z\omega^2\right)\right)\right)^2+\frac{2r\omega}{\sqrt{b_0^2+\omega^2}}\left(-4r^2\omega^2C^3\right.$$
$$-8b_0r^2\omega SC^2-S\left(2\left(2b_0^2+\omega^2\right)S_2r^2-2b_0(b_2-c_2)\omega\sqrt{b_0^2+\omega^2}Sr+\right.$$

$$c_1\delta_3\omega\left(b_0^2+\omega^2\right))+(b_2-c_2)r\omega^2S_2\sqrt{b_0^2+\omega^2}\Big)\Big)\Big)\Big),$$

$$-\frac{1}{2\omega^3}\Big(-2a_3r\omega C+2c_3r\omega C-2b_0b_3rS+2b_0c_3rS+\frac{1}{2r\omega^3\left(b_0^2+\omega^2\right)}$$

$$\left(4b_0r^2\omega C^2+2r\left((c_2-a_2)\sqrt{b_0^2+\omega^2}\omega^2+2r\left(2b_0^2+\omega^2\right)S\right)C\right.$$

$$\left.+\omega\left(c_1\delta_3b_0^2-2(b_2-c_2)r\sqrt{b_0^2+\omega^2}Sb_0+\omega^2\left(c_1\delta_3+2a_3z\sqrt{b_0^2+\omega^2}\right)\right)\right)$$

$$\left(4rz\omega^2C^2+S\left(b_0\delta_3\left(b_0^2+\omega^2\right)+2r\omega C\left(4b_0z+(b_1-c_1)\sqrt{b_0^2+\omega^2}\right)\right.\right.$$

$$\left.\left.+2rS\left(4zb_0^2+(b_1-c_1)\sqrt{b_0^2+\omega^2}b_0+2z\omega^2\right)\right)\right)-\frac{1}{4r^2\omega^4\sqrt{b_0^2+\omega^2}}\left(a_3\omega^4\right.$$

$$\left.+a_3b_0^2\omega^2+b_0\delta_3\omega^2-2a_2z\sqrt{b_0^2+\omega^2}\omega^2+2rC\left(2b_0z-c_1\sqrt{b_0^2+\omega^2}\right)\omega+b_0^3\delta_3\right.$$

$$\left.+2rS\left(4zb_0^2+(b_1-c_1)\sqrt{b_0^2+\omega^2}b_0+2z\omega^2\right)\right)\left(\frac{1}{b_0^2+\omega^2}\left(4rz\omega^2C^2+S\right.\right.$$

$$\left.\left.\left(b_0\delta_3\left(b_0^2+\omega^2\right)+2r\omega C\left(4b_0z+(b_1-c_1)\sqrt{b_0^2+\omega^2}\right)+2rS\left(4zb_0^2\right.\right.\right.\right.$$

$$\left.\left.\left.\left.+(b_1-c_1)\sqrt{b_0^2+\omega^2}b_0+2z\omega^2\right)\right)\right)^2+\frac{2r\omega}{\sqrt{b_0^2+\omega^2}}\left(-4r^2\omega^2C^3-8b_0r^2\omega S\right.$$

$$C^2-S\left(2\left(2b_0^2+\omega^2\right)S_2r^2-2b_0(b_2-c_2)\omega\sqrt{b_0^2+\omega^2}Sr\right.$$

$$\left.+c_1\delta_3\omega\left(b_0^2+\omega^2\right))+(b_2-c_2)r\omega^2S_2\sqrt{b_0^2+\omega^2}\right)\Big)+\frac{a_2\sqrt{b_0^2+\omega^2}}{8r^3\omega^4}$$

$$\left(-4r^2S\left(2(b_3-c_3)r\omega C+2b_0(b_3-c_3)rS-c_2\delta_3\sqrt{b_0^2+\omega^2}\right)\omega^4\right.$$

$$+\frac{2r\omega}{b_0^2+\omega^2}\left(4rz\omega^2C^2+S\left(b_0\delta_3\left(b_0^2+\omega^2\right)+2rS\left(4zb_0^2\right.\right.\right.$$

$$\left.\left.\left.+(b_1-c_1)\sqrt{b_0^2+\omega^2}b_0+2z\omega^2\right)\right)+r\omega S_2\left(4b_0z+(b_1-c_1)\sqrt{b_0^2+\omega^2}\right)\right)$$

$$\left(4r^2\omega^2C^3+8b_0r^2\omega SC^2+\omega S\left(c_1\delta_3\left(b_0^2+\omega^2\right)-2b_0(b_2-c_2)\right.\right.$$

$$\left.\left.r\sqrt{b_0^2+\omega^2}S\right)+r\left((c_2-b_2)\sqrt{b_0^2+\omega^2}\omega^2+2r\left(2b_0^2+\omega^2\right)S\right)S_2\right)$$

$$+\frac{1}{\sqrt{b_0^2+\omega^2}}\left(-4rz\omega^2C^2-S\left(b_0\delta_3\left(b_0^2+\omega^2\right)+2r\omega C\Big(4b_0z\right.\right.$$

$$\left.\left.+(b_1-c_1)\sqrt{b_0^2+\omega^2}\right)+2rS\left(4zb_0^2+(b_1-c_1)\sqrt{b_0^2+\omega^2}b_0+2z\omega^2\right)\right)\right)$$

$$\left(\frac{1}{b_0^2+\omega^2}\left(4rz\omega^2C^2+S\left(b_0\delta_3\left(b_0^2+\omega^2\right)+2r\omega C\Big(4b_0z\right.\right.\right.$$

$$\left.\left.\left.+(b_1-c_1)\sqrt{b_0^2+\omega^2}\right)+2rS\left(4zb_0^2+(b_1-c_1)\sqrt{b_0^2+\omega^2}b_0+2z\omega^2\right)\right)\right)^2$$

$$+\frac{2r\omega}{\sqrt{b_0^2+\omega^2}}\left(-4r^2\omega^2C^3-8b_0r^2\omega SC^2-S\left(2\left(2b_0^2+\omega^2\right)S_2r^2\right.\right.$$

$$-2b_0(b_2-c_2)\omega\sqrt{b_0^2+\omega^2}Sr+c_1\delta_3\omega\left(b_0^2+\omega^2\right)\Big)+(b_2-c_2)r\omega^2S_2\sqrt{b_0^2+\omega^2}\Big)\Big)\Big)$$

$$+c_2\delta_3\sqrt{b_0^2+\omega^2}\Big)\Big).$$

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References

[1] F. T. Arecchi, *Chaos and generalized multistability in quantum optics*, Physica Scripta **9** (1985), 85–92.

[2] M. R. Cândido, J. Llibre and D. D. Novaes, *Persistence of periodic solutions for higher order perturbed differential systems via Lyapunov-Schmidt reduction*, Nonlinearity **30** (2017) 35–60.

[3] M. R. Cândido and J. Llibre, *Stability of periodic orbits in the averaging theory: Applications to Lorenz and Thomas' differential systems*, Int. J. Bifurcation and Chaos **28** (2018), 1830007–pp 14.

[4] I. Casu and C. Lazureanu, *Stability and integrability aspects for the Maxwell-Bloch equations with the rotating wave approximation*, Regul. Chaotic Dyn. **22** (2017), no. 2, 109–121.

[5] C. Lazureanu, *The real-valued Maxwell-Bloch equations with controls: from a Hamilton-Poisson system to a chaotic one*, Int. J. Bifur. Chaos Appl. Sci. Engrg. **27** (2017), no. 9, 1750143, 17 pp.

[6] C. Lazureanu, *On a Hamilton-Poisson approach of the Maxwell-Bloch equations with a control*, Math. Phys. Anal. Geom. **20** (2017), no. 3, Art. 20, 22 pp.

[7] L. Liu, O. O. Aybar, V. G. Romanovski and W. Zhang, *Identifying weak foci and centers in Maxwell–Bloch system*, J. Math. Anal. Appl. **430** (2015), 549–571.

[8] J. Llibre and D. D. Novaes, *Improving the averaging theory for computing periodic solutions of the differential equations*, Zeitschrift für angewandte Mathematik und Physik (2015) 1401–1412.

[9] D.W. Zuo, *Modulation instability and breathers synchronization of the nonlinear Schrödinger Maxwell–Bloch equation*, Appl. Math. Lett. **79** (2018), 182–186.

[10] L. Wang, Z.Q. Wang, W.R. Sun, Y.Y. Shi, M. Li and M. Xu, *Dynamics of Peregrine combs and Peregrine walls in an inhomogeneous Hirota and Maxwell-Bloch system*, Commun. Nonlinear Sci. Numer. Simul. **47** (2017), 190–199.

[11] J. Wei, X. Wang and X. Geng, *Periodic and rational solutions of the reduced Maxwell-Bloch equations*, Commun. Nonlinear Sci. Numer. Simul. **59** (2018), 1–14.

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