THE PHASE PORTRAIT OF THE HAMILTONIAN SYSTEM ASSOCIATED TO A PINCHUK MAP

JOAN CARLES ARTÉS, FRANCISCO BRAUN, AND JAUME LLIBRE

Abstract. In this paper we describe the global phase-portrait of the Hamiltonian system associated to a Pinchuk map in the Poincaré disc. In particular, we prove that this phase portrait has 15 separatrices, five of them singular points, and 7 canonical regions, six of them of type strip and one annular.

1. Introduction

As far as we know, the simplest class of non-injective polynomial local diffeomorphisms of $\mathbb{R}^2$ are the Pinchuk maps, constructed by Pinchuk in [17]. The existence of these maps disproves the *real Jacobian conjecture*, that a polynomial local diffeomorphism of $\mathbb{R}^2$ is globally injective. One open problem is to know what exactly fails in this conjecture.

One of the most known conditions for a local diffeomorphism to be a global one is that it is proper. The asymptotic variety of a map of $\mathbb{R}^2$ is the set of points where the map is not proper (i.e. points that are limits of the map under sequences tending to infinity). In particular, a local diffeomorphism is a global diffeomorphism if and only if this set is empty. Gwoździewicz in [11] and Campbell in [6, 7] calculated the asymptotic variety of two Pinchuk maps in details. Our aim in this paper is to do a similar work, i.e. to describe a Pinchuk map, but now from a different point of view.

Let $U \subset \mathbb{R}^2$ be an open connected set. Let $F = (p, q) : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a C^2 local diffeomorphism. Let $H_F(x, y) = (p(x, y)^2 + q(x, y)^2)/2$ and consider the Hamiltonian system

$$\dot{x} = -(H_F)_y(x, y), \qquad \dot{y} = (H_F)_x(x, y), \tag{1}$$

where the dot denotes derivative with respect to the time t.

The singular points of system (1) are characterized by the following result, that we shall prove in section 2.

Lemma 1. *The singular points of system* (1) *are the zeros of F, each of them is a center of system* (1)*.*

The following is a generalization of the characterization of global invertibility of polynomial maps given by Sabatini in [18]. This version will appear in [5].

Date: August 20, 2016.
2010 *Mathematics Subject Classification.* Primary: 14R15; Secondary: 34C25.
Key words and phrases. Centers, global injectivity, real Jacobian conjecture.

Theorem 2 ([5, 18])**.** *Let $z_0 \in U$ such that $F(z_0) = (0,0)$. The center z_0 of system* (1) *is global if and only if (i) F is globally injective and (ii) $F(U) = \mathbb{R}^2$ or $F(U)$ is an open disc centered at the origin.*

In case $U = \mathbb{R}^2$ and F is a polynomial map, it follows that $F(\mathbb{R}^2) = \mathbb{R}^2$ provided F is injective, see [3]. Hence in this case z_0 is a global center of (1) if and only if F is globally injective. See an application of this result in [4].

Thus from Theorem 2 to understand why a local injective polynomial map F is not globally injective is equivalent to knowing that the center of the differential system (1) is not global.

Since the phase portrait on the Poincaré sphere of a Hamiltonian polynomial vector field having a global center is simple, i.e. at the infinite either it does not have singular points, or the infinite singular points are formed by two degenerate hyperbolic sectors (for Hamiltonian vector fields, the infinity contains only isolated singular points), it is interesting to know how complex can be the phase portrait of a non-global center of a Hamiltonian system (1).

In this paper we provide the qualitative global phase portrait of the Hamiltonian system (1) when F is given by the Pinchuk map considered in [6, 7], after a translation in the target in order to have only a point z_0 such that $F(z_0) = (0,0)$. More precisely, we prove the following result.

Theorem 3. *Let $F = (p,q) : \mathbb{R}^2 \to \mathbb{R}^2$, where $(p, q+208) : \mathbb{R}^2 \to \mathbb{R}^2$ is the Pinchuk map considered in* [6, 7] *(see the definition in section 2). Then the phase portrait of the Hamiltonian system* (1) *in the Poincaré disc is topologically equivalent to the phase portrait given in Fig. 1.*

To prove Theorem 3 we first study the infinite singular points of system (1) in section 3. These infinite singular points are very degenerate, and we apply homogeneous and quasi-homogeneous blow ups to study them. In section 4 we complete the proof of Theorem 3 by proving that the separatrix configuration of system (1) is qualitatively the one presented in Fig. 1.

We think that a good understanding of what fails in the real Jacobian conjecture is important to investigate a related problem, the *Jacobian conjecture* in $\mathbb{R}^2$, that a polynomial local diffeomorphism whose Jacobian determinant is constant is globally injective. This conjecture remains unsolved until now. For the Jacobian conjecture we address the reader to [2] or [9].

2. Injectivity, centers and a Pinchuk map

We begin with the proof of Lemma 1.

Proof of Lemma 1. Let z_0 be a singular point of the Hamiltonian system (1). We have

$$\begin{pmatrix} -p_y(z_0) & -q_y(z_0) \\ p_x(z_0) & q_x(z_0) \end{pmatrix} \begin{pmatrix} p(z_0) \\ q(z_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which is true if and only in $F(z_0) = (p,q)(z_0) = (0,0)$ because the Jacobian determinant of F is nowhere zero.

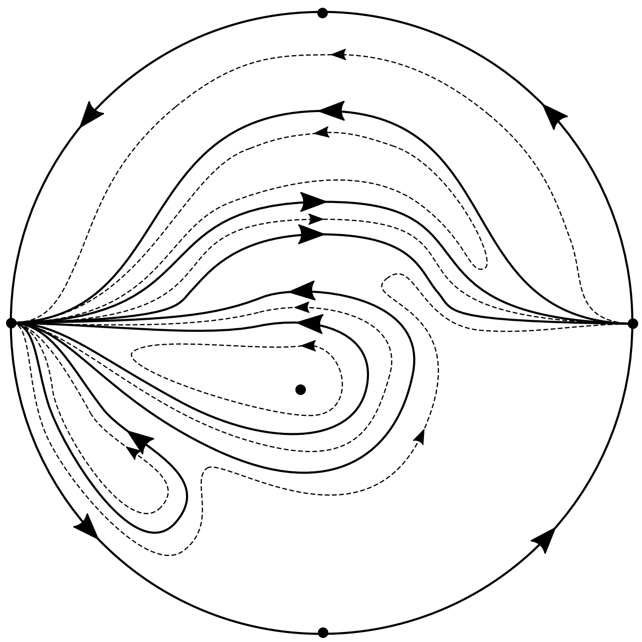


FIGURE 1. The qualitative global phase portrait of system (1) in the Poincaré disc.

The point z_0 is a center of the Hamiltonian system (1) because it is an isolated minimum of H_F. $\square$

Now we select the map F that we are going to work in this paper. Let $t = xy-1$, $h = t(xt+1)$ and $f = (xt+1)^2(t^2+y)$. A *Pinchuk map* is a non-injective polynomial map with nowhere zero Jacobian determinant of the form $(P, Q) : \mathbb{R}^2 \to \mathbb{R}^2$ such that $P = h + f$ and $Q = -t^2 - 6th(h + 1) - u(h, f)$, where u is chosen so that $\det D(P, Q)(x, y) = t^2 + (t + f(13 + 15h))^2 + f^2$. The following is the Pinchuk map studied by Campbell in [7]:

$$\overline{p} = h + f,$$

$$\overline{q} = -t^2 - 6th(h + 1) - 170fh - 91h^2 - 195fh^2 - 69h^3 - 75fh^3 - \frac{75h^4}{4}.$$

According to [7], the points $(-1, -163/4)$ and $(0, 0) \in \mathbb{R}^2$ have no inverse image under $(\overline{p}, \overline{q})$, all the other points of the curve

$$\gamma(s) = \left(s^2 - 1, -75s^5 + \frac{345s^4}{4} - 29s^3 + \frac{117s^2}{2} - \frac{163}{4}\right),\ s \in \mathbb{R},$$

which is a parametrization of the asymptotic variety of $(\overline{p}, \overline{q})$, have exactly one inverse image under this map, and the points of $\mathbb{R}^2 \setminus \gamma(\mathbb{R})$ have two inverse images. Hence, in particular, the point $(0, 208)$ has precisely one inverse image under $(\overline{p}, \overline{q})$.

We consider the map $F = (p, q) : \mathbb{R}^2 \to \mathbb{R}^2$ given by the translation

$$p(x, y) = \overline{p}(x, y), \qquad q(x, y) = \overline{q}(x, y) - 208. \tag{2}$$

Observe that F is a Pinchuk map according to our above-definition. Moreover, now there exists exactly one point $z_0 \in \mathbb{R}^2$ such that $F(z_0) = (0, 0)$. From Lemma 1 the point z_0 is the only finite singular point of system (1), corresponding to a non-global center of this system according to Theorem 2. Further, the curve

$$\beta(s) = \gamma(s) - (0, 208) = \left(s^2 - 1, -75s^5 + \frac{345s^4}{4} - 29s^3 + \frac{117s^2}{2} - \frac{995}{4}\right), \tag{3}$$

$s \in \mathbb{R}$, is the asymptotic variety of F, whose points have exactly one inverse image over F, but the points $(-1, -995/4)$ and $(0, 208)$, which have none.

From now on, we restrict our attention to the specific Pinchuk map (2).

We first calculate the coordinates of the point z_0. Observe that $xt+1 = x^2y-x+1$ is a factor of p. If this factor annihilates, then $h = 0$ and $q = -t^2 - 208 < 0$.

The other factor of p is

$$g(x, y) = -x + (1 - 2x + 3x^2)y - x^2(-2 + 3x)y^2 + x^4y^3.$$

We observe that $g(0, y) = y$ and $q(0, y) = 50y - 799/4$ do not annihilate at the same time, thus the first coordinate of the point z_0 is not 0. Moreover, since the leading coefficient of $q(x, y)$ as a polynomial in y is $-75x^{15}$, it follows that the first coordinate of z_0 will be a point where the resultant in y between $g(x, y)$ and $q(x, y)$ is zero. This resultant is the cubic $c(x) = 31008391 - 11757152x - 155580672x^2 + 2239078400x^3$ multiplied by $-x^{36}/64$. The discriminant of $c(x)$ is negative, so it has only one real root, which will be the first coordinate of the point z_0.

Repeating a similar reasoning now looking g and q as polynomials in x, we calculate their resultant and obtain that its zero is the only real root of the cubic $c(y) = 1789023641600 + 100675956992y + 26252413280y^2 + 1506138481y^3$, which will be the second coordinate of the point z_0.

Hence $z_0 = (-0, 22568337..., -17, 491214...)$ approximately. Since z_0 is a center, the only finite singular point of system (1), near z_0 the phase portrait of this system is simple. Indeed, since z_0 is the minimum point of H_F, it follows that the gradient of H_F points outward of each closed orbit of the center, and so each closed orbit of the center rotates in counterclockwise around z_0.

In the following section we shall investigate the infinite of system (1).

3. The infinite of system (1)

In this section we will use results and notations on the *Poincaré compactification* of the polynomial vector fields of $\mathbb{R}^2$. For details on this technique we refer the reader to [8, Chapter 5] or [10].

We call a singular point of a vector field *linearly zero* when the linear part of the vector field at this point is identically zero.

We begin by proving a general fact about the infinite singular points of Hamiltonian systems of the form (1). Writing $H = H_0 + H_1 + \cdots + H_{d+1}$, where H_i is the homogeneous part of degree i of the polynomial H, it is simple to conclude that the infinite singular points $(u, 0)$ of system (1) in the local charts U_1 and U_2 are the points satisfying $H_{d+1}(1, u) = 0$ and $H_{d+1}(u, 1) = 0$, respectively. Let $(u, 0)$ be an infinite singular point of system (1) and assume it is in the chart U_1. The linear part of the vector field at $(u,0)$ is

$$\begin{pmatrix} (d+1)(H_{d+1})_y(1,u) & dH_d(1,u) \\ 0 & (H_{d+1})_y(1,u) \end{pmatrix}.$$

Assuming $m = \deg p \geq \deg q$, we have $d = 2m-1$ and $H_d = p_m p_{m-1} + q_m q_{m-1}$ and $H_{d+1} = p_m^2 + q_m^2$. Since $H_{d+1}(1,u) = 0$, it follows that $p_m(1,u) = q_m(1,u) = 0$, and hence $(H_{d+1})_y(1,u) = H_d(1,u) = 0$. Therefore, $(u,0)$ is a linearly zero singular point. This proves the following result.

Lemma 4. *The infinite singular points of the Hamiltonian system* (1) *are linearly zero.*

Now we return to the Pinchuk map F defined by (2). Observe that the highest homogeneous part of $H_F(x,y)$ is $5625x^{30}y^{20}/2$. Thus the origins of the charts U_1, V_1 and U_2, V_2 are the infinite singular points of the Hamiltonian system (1), each of them linearly zero from Lemma 4.

We will use the quasi-homogeneous directional blow up technique to desingularize each of these infinite singular points. An exposition about blow-ups can be found in [1], see also [8, Chapter 3].

We now recall the directional blow up transformations.

By the *quasi-homogeneous blow up in the positive* (resp. *negative*) *x-direction with weights α and β*, or simply (α,β)*-blow up in the positive* (resp. *negative*) *x-direction*, we mean the transformation which carries the variables (x_1,y_1) to the variables (x_2,y_2) according to the formulas

$$(x_1,y_1) = (x_2^\alpha, x_2^\beta y_2), \qquad (x_1,y_1) = (-x_2^\alpha, x_2^\beta y_2),$$

respectively. Similarly, by the *quasi-homogeneous blow up in the positive* (resp. *negative*) *y-direction with weights α and β*, or simply (α,β)*-blow up in the positive* (resp. *negative*) *y-direction*, we mean the transformations

$$(x_1,y_1) = (x_2 y_2^\alpha, y_2^\beta), \qquad (x_1,y_1) = (x_2 y_2^\alpha, -y_2^\beta),$$

respectively.

Clearly if α (resp. β) is odd, then the blow up in the positive x-direction (respec. y-direction) provides the information of the respectively negative blow ups. Also, if β is odd, the x-directional blow ups swap the second and third quadrants, while the y-directional blow ups swap the third and the fourth quadrants if α is odd. After the (α,β)-blow up in the x-direction, a system $\dot{x_1} = P(x_1,y_1)$, $\dot{y_1} = Q(x_1,y_1)$ is transformed into

$$\dot{x_2} = \frac{\pm P}{\alpha x_2^{\alpha-1}}, \qquad \dot{y_2} = \frac{\alpha x_2^{\alpha-1} Q \mp \beta x_2^{\beta-1} y_2 P}{\alpha x_2^{\alpha+\beta-1}},$$

with $P = P(\pm x_2^\alpha, x_2^\beta y_2)$ and $Q = Q(\pm x_2^\alpha, x_2^\beta y_2)$, in the positive and negative directions according to $\pm$. Similarly, the (α,β)-blow up in the y-direction transforms $\dot{x_1} = P(x_1,y_1)$, $\dot{y_1} = Q(x_1,y_1)$ into

$$\dot{x_2} = \frac{\beta y_2^{\beta-1} P \mp \alpha x_2 y_2^{\alpha-1} Q}{\beta y_2^{\alpha+\beta-1}}, \qquad \dot{y_2} = \frac{\pm Q}{\beta y_2^{\beta-1}},$$

with $P = P(x_2 y_2^\alpha, \pm y_2^\beta)$ and $Q = Q(x_2 y_2^\alpha, \pm y_2^\beta)$, in the positive and in the negative directions according to $\pm$.

After the blow up in the x-direction (resp. y-direction) we cancel a common appearing factor x_2^k (y_2^k) for a suitable k. So if k is odd, the direction of the orbits are reversed in $x_2 < 0$ ($y_2 < 0$).

The weights α and β are chosen analyzing the *Newton polygon* of (P, Q), see the construction in [1].

The application of (α, β)-blow ups with $\alpha\beta \neq 1$ usually reduces the number of blow ups necessary for studying the local phase portrait of a linearly zero singular point.

To make the exposition clearer, we shall apply the most part of the blow ups in the x-direction. So, sometimes we will first apply a *xy-change*, $(x_1, y_1) \mapsto (y_1, x_1) = (x_2, y_2)$, before making the blow-up.

In the next two subsections we will desingularize the origin of the charts U_1 and U_2, respectively. We will denote the coordinates of the system in the step i of the algorithm as the variables (w_i, z_i), so that after either a wz-change, a translation or a blow up, the new obtained system will be written in the variables (w_{i+1}, z_{i+1}). In each step, we will denote the system $\dot{w}_i = P_i(w_i, z_i)$, $\dot{z}_i = Q_i(w_i, z_i)$ simply as (P_i, Q_i).

Since the Hamiltonian system (1) with the polynomials p and q given by (2) has degree 49, it follows that for the calculations in each step of the algorithm we have to deal with polynomials of very high degree. So we persuade these calculations with the algebraic manipulator *Mathematica*. We do not show in each step the whole expressions of the systems (P_i, Q_i) because this would be impractical.

3.1. **The origin of the chart U_1.** We write the compactification of system (1) in the chart U_1 in the variables (w_0, z_0), as (P_0, Q_0). From Lemma 4, the singular point $(0, 0)$ is linearly zero.

We first apply a wz-change and write the new system in the variables (w_1, z_1) as (P_1, Q_1).

The Newton polygon of system (P_1, Q_1) has only one compact edge contained in the straight line $x + 2y = 38$. We apply $(1, 2)$-blow ups in the positive w-direction and in the positive and negative z-directions obtaining systems (P_2, Q_2) and $(P_2^\pm, Q_2^\pm)$, in the variables (w_2, z_2) and $(w_2^\pm, z_2^\pm)$, after canceling the common factors w_2^{38} and $(w_2^\pm)^{38}$, respectively. The first terms of these systems have the following expressions:

$$
\begin{aligned}
P_2 &= w_2 \left(-56250 + \frac{1125}{2}(447 w_2 + 1900 z_2) + \cdots \right), \\
Q_2 &= 28125 - \frac{1125}{4}(387 w_2 + 2000 z_2) + \frac{75}{4}(1967 w_2^2 + 24138 w_2 z_2 + 57000 z_2^2) + \cdots,
\end{aligned}
$$

and

$$P_2^{\pm} = w_2^{\pm}\left(\mp\frac{28125}{2} + 281250(w_2^{\pm})^2 + \cdots\right),$$
$$Q_2^{\pm} = z_2^{\pm}\left(\pm\frac{140625}{2} - 1350000(w_2^{\pm})^2 + \cdots\right).$$

The only singular point of (P_2, Q_2) over the line $w_2 = 0$ is the linearly zero singular point $(0, 1)$. The origin of the systems $(P_2^{\pm}, Q_2^{\pm})$ are saddles as depicted in the planes $w_2^+ z_2^+$ and $w_2^- z_2^-$ of Fig. 2.

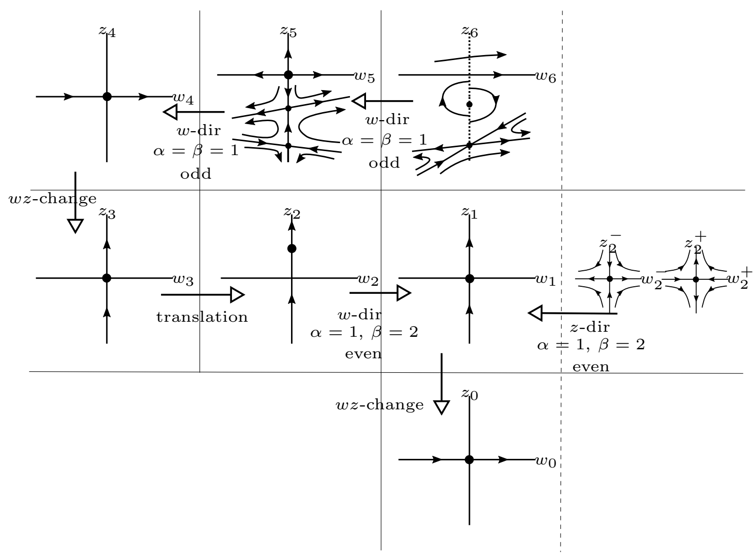


FIGURE 2. The sequence of blow downs in the study of the origin of the chart U_1.

The reader can follow a schema of each step of the calculations in Fig. 2. We just need to analyze the origin of the systems $(P_2^{\pm}, Q_2^{\pm})$, because the other singularities over the lines $z_2^{\pm} = 0$ will correspond to the singularity $(0, 1)$ of (P_2, Q_2).

We now analyze this linearly zero singularity. We first do a translation bringing this point to the origin, obtaining the new system (P_3, Q_3) in the variables (w_3, z_3). We also apply a wz-change obtaining the system (P_4, Q_4) in the variables (w_4, z_4). The Newton polygon of this system has two compact edges. We choose the one contained in the straight line $x+y = 11$. This compact edge has the point of negative abscissa $(-1, 12)$, thus concerning $(1, 1)$-blow ups, it follows from [1, Proposition 3.2] that w_4 is not a characteristic direction, and so we only need to apply a w-directional $(1, 1)$-blow up, obtaining the system (P_5, Q_5) in the variables (w_5, z_5), after canceling the common factor w_5^{11}. The first terms of (P_5, Q_5) are:

$$
\begin{aligned}
P_5 &= w_5\Big(\frac{1}{4}(w_5+z_5)(w_5+2z_5)\big(442125w_5^5z_5+824250w_5^4z_5^2+699990w_5^3z_5^3\\
&\quad +320532w_5^2z_5^4+215904w_5z_5^5+112500w_5^6+217160z_5^6\big)+\cdots\Big),\\
Q_5 &= z_5\Big(\frac{1}{4}(w_5+z_5)(w_5+2z_5)\big(442125w_5^5z_5+824250w_5^4z_5^2+699990w_5^3z_5^3\\
&\quad +320532w_5^2z_5^4+215904w_5z_5^5+112500w_5^6+217160z_5^6\big)+\cdots\Big).
\end{aligned}
\tag{4}
$$

Over the line $w_5 = 0$, the singular points of (P_5, Q_5) are $(0, 0)$ and two points of the form $(0, z_5)$, with z_5 the two real solutions of

$$0 = z_5^4 + 70726z_5^3 + 252941z_5^2 + 290380z_5 + 108580.$$

The discriminant of this quartic equation is negative, thus it has two real solutions. Those are approximately $z_5 = -70722.424...$ and $z_5 = -1.6611121....$ The singular point $(0, 0)$ is linearly zero and the other two singular points are saddles, as represented in the w_5z_5-plane of Fig. 2.

Now we study the linearly zero point $(0, 0)$ of (P_5, Q_5). It is clear from (4) that the characteristic equation of (P_5, Q_5) is identically zero, so $(0, 0)$ is a dicritical singular point. We apply $(1, 1)$-blow ups in both the w- and z-directions obtaining systems (P_6, Q_6) and (P_6^y, Q_6^y) in the variables (w_6, z_6) and (w_6^y, z_6^y), after canceling the factors w_6^9 and $(z_6^y)^9$, respectively. System (P_6^y, Q_6^y) does not have $(0, 0)$ as a singular point, so we just need to consider system (P_6, Q_6) over the line $w_6 = 0$. We have

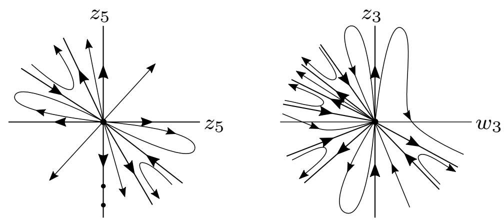
$$
\begin{aligned}
P_6(0,z_6) &= \frac{1}{4}(z_6+1)(2z_6+1)\big(217160z_6^6+215904z_6^5+320532z_6^4+699990z_6^3\\
&\quad +824250z_6^2+442125z_6+112500\big),\\
Q_6(0,z_6) &= \frac{1}{4}z_6^2(z_6+1)(2z_6+1)\big(290380z_6^6+260416z_6^5+421348z_6^4+904140z_6^3\\
&\quad +1032225z_6^2+542250z_6+135000\big).
\end{aligned}
$$

By using Sturm's theorem (see for instance [12]; in the software Mathematica, the Sturm theorem is programed by the instruction *CountRoots*) we see that the polynomial of degree 6 multiplying $(z_6+1)(2z_6+1)/4$ in $P_6(0, z_6)$ has no real roots, so the only singular points are $(0, -1/2)$ and $(0, -1)$. The first one is a weak focus and the second one is a saddle, as depicted in the plane w_6z_6 of Fig. 2. Since the origin of (P_5, Q_5) is dicritical, it follows that each orbit crossing the line $w_6 = 0$ will correspond to two orbits tending to $(0, 0)$ in positive or negative directions.

We now begin the process of blowing down.

It is simple to conclude that the phase portrait of the system (P_5, Q_5) close to the origin is qualitatively the one depicted in (a) of Fig. 3. Consequently, by considering also the information close to the other two singular points in the line $w_5 = 0$ (see the plane w_5z_5 of Fig. 2), we can understand the behavior near the origin of system (P_4, Q_4). We then apply a wz-change and conclude that the behavior of system (P_3, Q_3) near the origin is the one presented qualitatively in (b) of Fig. 3.

By translating $(0, 0)$ to $(0, 1)$ and by using the information provided by the saddles of planes $w_2^{\pm}z_2^{\pm}$, we make the blow downs with $\alpha = 1$ and $\beta = 2$, obtaining



(a) Origin of (P_5, Q_5). (b) Origin of (P_3, Q_3).

FIGURE 3. The origins of systems (P_5, Q_5) and (P_3, Q_3).

the origin of system (P_1, Q_1). We then finally apply a wz-change and conclude that the origin of system (P_0, Q_0) is qualitatively as drawn in Fig. 4.

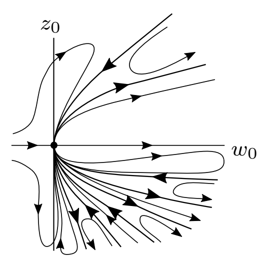


FIGURE 4. The origin of the chart U_1.

3.2. **The origin of chart U_2.** As in the calculations made above, we write the compactified vector field in the chart U_2 as $(\dot{w}_0, \dot{z}_0) = (P_0, Q_0)$. The Newton polygon of (P_0, Q_0) has two compact edges: one of them contained in the straight line $3x + 2y = 87$. We apply a $(3,2)$-blow up in the w-direction, obtaining the system $(\dot{w}_1, \dot{z}_1) = (P_1, Q_1)$ after canceling the factor w_1^{87}. The first terms of P_1 and Q_1 are:

$$P_1 = w_1 \left(-46875 + 11250 z_1^2 (80 w_1 - 47 z_1) + \cdots\right),$$
$$Q_1 = z_1 \left(9375 + 56250 z_1^2 (2 z_1 - 3 w_1) + \cdots\right).$$

The polynomials P_1 and Q_1 have degree 61.

It is clear that at $(0,0)$ we have a saddle. The other singular point of (P_1, Q_1) in the line $w_1 = 0$ is $(0, -1)$, and it is a linearly zero point. See the $w_1 z_1$-plane of Fig. 5. The reader can follow the steps of the calculations in the schema shown in this figure. We just warn that, differently of Fig. 2, we already draw the final phase portrait of each step, including the behavior close to the linearly zero points (information that we will know only after persuading all the blow ups).

We also apply $(3,2)$-blow ups in the positive and negative z-directions, obtaining the systems $\dot{w}_1^{\pm} = P_1^{\pm}$, $\dot{z}_1^{\pm} = Q_1^{\pm}$, respectively, with linearly zero singular points at $(w_1^{\pm}, z_1^{\pm}) = (0,0)$. The polynomials $P_1^{\pm}$ and $Q_1^{\pm}$ have degree 30 and $Q_1^{\pm}$ is a factor of $z_1^{\pm}$.

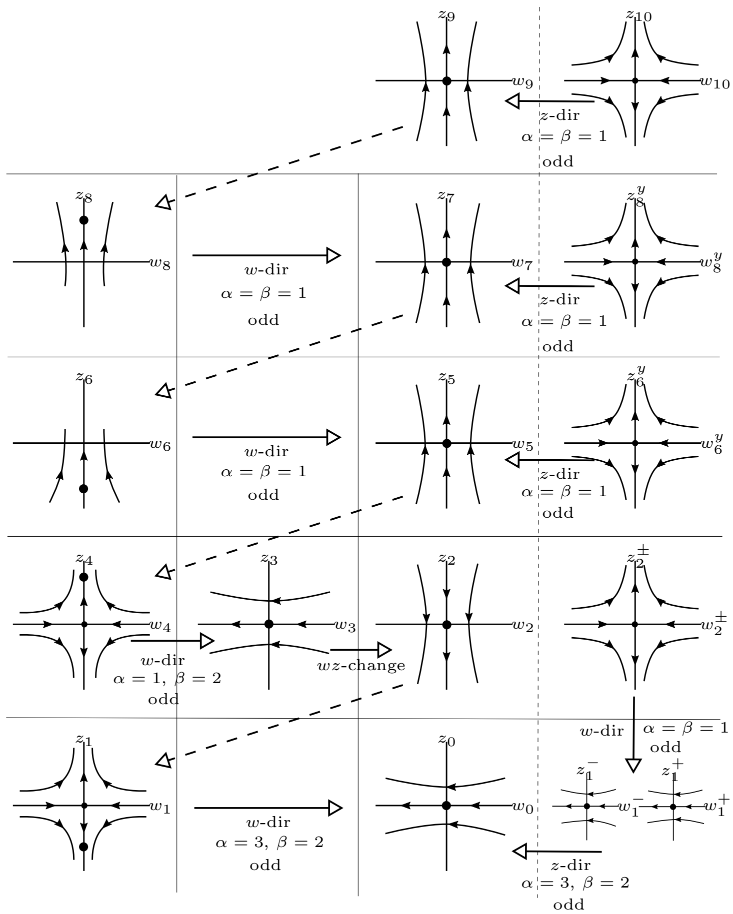


FIGURE 5. The sequence of blow downs in the study of the origin of the chart U_2.

We do not need to analyze the other singular points over the lines $z_1^\pm = 0$, as the information provided by them is already contained in the w-directional blow up. We desingularize these points applying $(1,1)$-blow ups in the w-direction. Here we do not need to apply blow ups in the z-directions because the characteristic equations of the systems are

$$\begin{aligned} 0 = z_1^\pm \big(& -4500(w_1^\pm)^5 z_1^\pm + 1650(w_1^\pm)^4 (z_1^\pm)^2 - 7800(w_1^\pm)^3 (z_1^\pm)^3 \\ & +3025(w_1^\pm)^2 (z_1^\pm)^4 - 500 w_1^\pm (z_1^\pm)^5 + 5625(w_1^\pm)^6 + 2501(z_1^\pm)^6 \big), \end{aligned}$$

and so $w_1^\pm = 0$ are not characteristic directions. We obtain the systems $(P_2^\pm, Q_2^\pm)$ after canceling a factor $(w_2^\pm)^5$. The polynomials $P_2^\pm$ and $Q_2^\pm$ have degree 45, and up to order 2 they have the same expressions:

$$\begin{aligned} P_2^\pm &= w_2^\pm \left(-\frac{28125}{2} + 10125 z_2^\pm + \cdots \right), \\ Q_2^\pm &= z_2^\pm \left(5625 - 4500 z_2^\pm + \cdots \right). \end{aligned}$$

Thus at $(0,0)$ the systems have a saddle, as depicted in the planes $w_2^{\pm} z_2^{\pm}$ of Fig. 5. Moreover, any other singularity of the form $(0, z_2^{\pm})$ must satisfy

$$2501(z_2^{\pm})^6 - 500(z_2^{\pm})^5 + 3025(z_2^{\pm})^4 - 7800(z_2^{\pm})^3 + 1650(z_2^{\pm})^2 - 4500z_2^{\pm} + 5625 = 0.$$

By using Sturm's theorem, we conclude that this equation has no real solution.

Now we desingularize the point $(0,-1)$ of system (P_1, Q_1). First we apply a translation to bring this point to the origin, obtaining the system (P_2, Q_2) in the variables (w_2, z_2). We also apply a wz-change obtaining the system (P_3, Q_3) in the variables (w_3, z_3). The Newton polygon of this system has only one compact edge contained in the line $x + 2y = 11$, and this edge has points of negative abscissa, so concerning $(1,2)$-blow ups we just need to apply them in the w-direction, according to [1, Proposition 3.2]. Hence we apply a $(1,2)$-blow up in the positive w-direction, obtaining the system (P_4, Q_4) in the variables (w_4, z_4), after canceling a factor of w_4^{11}. These polynomials have degree 90, and their first terms are:

$$\begin{aligned} P_4 &= w_4 \left(-4982259375 + \frac{996451875}{4}(260w_4 + z_4) + \cdots \right), \\ Q_4 &= z_4 \left(3985807500 - 110716875(639w_4 + 2z_4) + \cdots \right). \end{aligned}$$

Clearly $(0,0)$ is a singularity corresponding to a saddle. The other singular point in the line $w_4 = 0$ is $(0, \xi)$, where ξ is the only real root of the cubic

$$c(x) = 4x^3 + 216x^2 + 6075x - 218700,$$

which is approximately $\xi = 18.8848.....$ This cubic has only one real root because its discriminant is negative. A calculation shows that $(0,\xi)$ is linearly zero. See the plane $w_4 z_4$ in Fig. 5.

Now we apply a translation to bring the point $(0,\xi)$ to the origin, obtaining system (P_5, Q_5) written in the variables (w_5, z_5). Since ξ is not a rational number, we do this translation with a parameter x, and thus P_5 and Q_5 are polynomials in w_5, z_5 and x. We simplify these polynomials substituting them by the remainder of the division of each of them by $c(x)$, obtaining so polynomials of degree 2 in x, and hence when we substitute x by ξ, we obtain the same expressions. We keep the notation (P_5, Q_5).

The Newton polygon of this system has just one compact edge contained in the line $x + y = 1$. So the blow ups here will be homogeneous ones. The characteristic equation of system (P_5, Q_5) is a multiple of

$$\begin{aligned} 0 = w_5 \big(729 &\left(23090824x^2 + 532204875x - 18375684300\right) w_5^2 \\ &-216 \left(149x^2 + 1828125x - 32221800\right) w_5 z_5 - 4 \left(404x^2 + 8325x - 54675\right) z_5^2 \big), \end{aligned}$$

with $x = \xi$. It would thus be enough to apply a $(1,1)$-blow up in the z-direction and to study the singularities of the new system in $z_6 = 0$ (this could evidently also be concluded by observing that the compact edge of the Newton polygon of (P_5, Q_5) has a point of negative ordinate). We prefer though to apply $(1,1)$-blow ups in the w and z-directions and to study the singularities of the new systems either in the line $w_6 = 0$ and in the origin, respectively. The reason why we do this is that the singularities other than the origin are linearly zero and we have to apply new blow ups after persuading a translation. The matter here is that the blow up in the z-direction produces a vector field of degree 158, while the blow up in the w-direction produces a vector field of degree 109. Thus it is simpler to do a translation and after to apply the polynomial remainder in the vector field with smaller degree.

Then after applying $(1,1)$-blow ups in either the positive w- and z-directions, we obtain the systems (P_6, Q_6) and (P_6^y, Q_6^y) in the variables (w_6, z_6) and (w_6^y, z_6^y), after canceling factors w_6 and z_6^y, respectively. The first terms of these systems are:

$$\begin{aligned} P_6 &= w_6 \left(\frac{59049}{4} \left(149x^2 + 1828125x - 32221800\right) + \cdots \right), \\ Q_6 &= -\frac{4782969}{16} \left(23090824x^2 + 532204875x - 18375684300\right) \\ &\quad + \frac{177147}{64} \left(5928191012x^2 - 9644385686625x + 179165168144100\right) w_6 \\ &\quad + \frac{177147}{2} \left(149x^2 + 1828125x - 32221800\right) z_6 + \cdots, \end{aligned}$$

with $x = \xi$ and

$$\begin{aligned} P_6^y &= w_6^y \left(-\frac{6561}{4}(404x^2 + 8325x - 54675) + \cdots \right), \\ Q_6^y &= z_6^y \left(2187(404x^2 + 8325x - 54675) + \cdots \right), \end{aligned}$$

with $x = \xi$.

The origin of system (P_6^y, Q_6^y) is a saddle (see the plane $w_6^y z_6^y$ in Fig 5). On the other hand, the singularities of (P_6, Q_6) over the line $w_6 = 0$ are the points $(0, z_6)$, with z_6 the real solutions of

$$\begin{aligned} 0 &= 4\left(404x^2 + 8325x - 54675\right) z_6^2 + 216\left(149x^2 + 1828125x - 32221800\right) z_6 \\ &\quad - 729\left(23090824x^2 + 532204875x - 18375684300\right), \end{aligned} \tag{5}$$

with $x = \xi$. The discriminant of this quadratic equation is a polynomial in x whose division by $c(x)$ has remainder equal to 0. This means that the only real solution of (5) is $r_1 = -b/(2a)$, where a and b are the coefficients of z_6^2 and z_6 in (5), respectively. Substituting x by ξ after applying the polynomial remainder again we have

$$r_1 = \frac{3\left(2380\xi^2 + 21\xi - 334440\right)}{5989}$$

The point $(0, r_1)$ is linearly zero, so we translate it to the origin obtaining the system (P_7, Q_7) in the variables (w_7, z_7). We again persuade this translation considering $r_1 = r_1(x)$ as a polynomial of x. Again P_7 and Q_7 will be polynomials in w_7, z_7 and x. As before we substitute these polynomials by the remainder of the division of them by $c(x)$, obtaining polynomials of degree 2 in x. We keep the notation P_7 and Q_7 for them.

The Newton polygon of this system has only one compact edge contained in the straight line $x + y = 1$. The characteristic equation of this system has $w_7 = 0$ as a solution.

As above, we apply $(1,1)$-blow ups in either the positive w- and z-directions, obtaining the systems (P_8, Q_8) and (P_8^y, Q_8^y) in the variables (w_8, z_8) and (w_8^y, z_8^y), respectively. We then study the origin of (P_8^y, Q_8^y) and the singularities of (P_8, Q_8)

over the line $w_8 = 0$. The reason is again computational, as the degree of (P_8^y, Q_8^y) is 196 and the degree of (P_8, Q_8) is 128. The first terms of these systems are:

$$P_8 = w_8 \left(-\frac{6561\left(610023097091x^2 - 7154910819000x - 72219849901200\right)}{95824} + \cdots \right),$$

$$Q_8 = \frac{59049}{4591119488}\big(866106385697199684752x^2 - 63678825997496319079125x \\ + 894244583851567110026100\big) + \cdots,$$

with $x = \xi$, and

$$P_8^y = w_8 \left(-\frac{2187}{2}(404x^2 + 8325x - 54675) + \cdots \right),$$

$$Q_8^y = z_8 \left(\frac{6561}{4}(404x^2 + 8325x - 54675) + \cdots \right),$$

with $x = \xi$.

System (P_8^y, Q_8^y) has a saddle at the origin (see the plane $w_8^y z_8^y$ in Fig 5), while the singular points of (P_8, Q_8) over the line $w_8 = 0$ are the points $(0, z_8)$, with z_8 the real roots of

$$0 = 2295559744\left(404x^2 + 8325x - 54675\right) z_8^2 - 574944\big(610023097091x^2 \\ - 7154910819000x - 72219849901200\big)z_8 + 27\big(866106385697199684752x^2 \\ - 63678825997496319079125x + 894244583851567110026100\big),$$

with $x = \xi$. The discriminant of this equation is a polynomial in x whose division by $c(x)$ has remainder 0. Thus the only real solution is $r_2 = -b/(2a)$, where a and b are the coefficients of z_8^2 and z_8 of the equation, respectively. After applying the polynomial remainder, we substitute x by ξ obtaining

$$r_2 = \frac{-38570325688\xi^2 - 1361034154573\xi + 41691943772820}{430417452}.$$

The point $(0, r_2)$ is linearly zero, so we translate it to the origin obtaining the system (P_9, Q_9) in the variables (w_9, z_9). As before, we make this translation with the parameter x, so that P_9 and Q_9 are polynomials in x. Keeping the notation we substitute these polynomials by the remainder of the division of them by $c(x)$.

As before the Newton polygon of this system has only one compact edge contained in the straight line $x + y = 1$. Moreover, the characteristic equation does not have $z_9 = 0$ as a solution. Here we just apply a $(1,1)$-blow up in the positive z-direction, obtaining system (P_{10}, Q_{10}) in the variables (w_{10}, z_{10}), after canceling the factor z_{10} (here we do not use the superscript y as this is the only system in this step). The degree of this new system is 234, but as we are going to see, just the origin is a singular point in the line $z_{10} = 0$. The first terms of P_{10} and Q_{10} are:

$$P_{10} = w_{10} \left(-\frac{2187}{4}(404x^2 + 8325x - 54675) + \cdots \right),$$

$$Q_{10} = z_{10} \left(\frac{2187}{2}(404x^2 + 8325x - 54675) + \cdots \right),$$

with $x = \xi$.

Now over the line $z_{10} = 0$, the singular points of (P_{10}, Q_{10}) are $(0,0)$ and the points $(w_{10}, 0)$, with w_{10} the real roots of

$$\begin{aligned} 0 = {} & 27(224799605593831132981000196646508x^2 \\ & + 1106076318319862271941876917379662 5x \\ & - 2890483990749333378769851609264081 00)w_{10}^2 \\ & - 1721669808(375535867201456283x^2 + 1078577653550389425 0x \\ & - 338993606077717260600)w_{10} + 41168707330260512(404x^2 + 8325x - 54675), \end{aligned}$$

with $x = \xi$. The discriminant of this equation after applying the polynomial remainder is

$$\begin{aligned} \Delta(x) = {} & 427951390803211906507575958648677316602784861 58784x^2 \\ & + 2546081344010178238089386481604981090589087283168000x \\ & - 633456291588531648452267836321422243593406331826688 00. \end{aligned}$$

It is simple to conclude that $\Delta(\xi) < 0$, thus only $(0,0)$ is a singular point of (P_{10}, Q_{10}) in $z_{10} = 0$. This singular point is the saddle depicted in in the plane $w_{10}z_{10}$ of Fig. 5.

Since the behavior near each appearing singular points in each step above is very simple, the blow down of each step is also very simple: following the arrays in Fig. 5, it is easy to conclude that the origin of U_2 has a degenerate hyperbolic sector as shown in the w_0z_0-plane of Fig 5.

4. The global phase portrait

We begin with a background on separatrices and canonical regions of the Poincaré compactification $p(\mathcal{X})$ in the Poincaré disc $\mathbb{D}$ of a polynomial system $\dot{x} = \mathcal{X}(x)$. Let φ be the flow of $p(\mathcal{X})$ defined in $\mathbb{D}$. As usual we denote by (U, φ) the flow of $p(\mathcal{X})$ on an invariant subset $U \subset \mathbb{D}$. Two flows (U, φ) and (V, ψ) are said to be *topologically equivalent* if there exists a homeomorphism $h : U \to V$ sending orbits of (U, φ) onto orbits of (V, ψ) preserving or reversing the orientation of all the orbits.

Following Markus [13], we say that the flow (U, φ) is *parallel* if it is topologically equivalent to one of the following flows: (i) the flow defined in $\mathbb{R}^2$ by the system $\dot{x} = 1$, $\dot{y} = 0$; (ii) the flow defined in $\mathbb{R}^2 \setminus \{(0,0)\}$ by the system in polar coordinates $\dot{r} = 0$, $\dot{\theta} = 1$; and (iii) the flow defined in $\mathbb{R}^2 \setminus \{(0,0)\}$ by the system in polar coordinates $\dot{r} = r$, $\dot{\theta} = 0$. Parallel flows topologically equivalent to (i), (ii) and (iii) are called *strip*, *annular* and *spiral* (or *radial*), respectively.

We denote by γ_x the orbit of $p(\mathcal{X})$ passing through x when $t = 0$ with maximal interval I_x, and the positive (resp. negative) orbit of γ_x by $\gamma_x^+ = \{\gamma_x(t) \mid t \in I_x \text{ and } t \geq 0\}$ (resp. $\gamma_x^- = \{\gamma_x(t) \mid t \in I_x \text{ and } t \leq 0\}$). Then we set $a^{\pm}(x) = \overline{\gamma_x^{\pm}} \setminus \gamma_x^{\pm}$, here as usual $\overline{\gamma_x^{\pm}}$ denotes the closure of $\gamma_x^{\pm}$. Observe that $a^-(x)$ differs from $\alpha(x)$ in the case of periodic orbits and singular points: indeed, $a^-(x) = \emptyset$ and $\alpha(x) = \gamma_x$ in this case (similarly for $a^-(x)$ and $\omega(x)$). An orbit γ_x of $p(\mathcal{X})$ is called a *separatrix* of $p(\mathcal{X})$ if it is not contained in an open neighborhood U such that (U, φ) is parallel and such that both $a^{\pm}(x) = a^{\pm}(y)$ for all $y \in U$ and $\overline{U} \setminus U$ consists of $a^{+}(x)$, $a^{-}(x)$ and exactly two orbits γ_y and γ_z such that $a^{\pm}(x) = a^{\pm}(y) = a^{\pm}(z)$.

If $\mathcal{X}$ is a polynomial vector field it is known that the separatrices of $p(\mathcal{X})$ are (i) the finite and infinite singular points of $p(\mathcal{X})$; (ii) the orbits of $p(\mathcal{X})$ contained in the boundary $\mathbb{S}^1$ of $\mathbb{D}$; (iii) the limit cycles of $p(\mathcal{X})$; and (iv) the separatrices of the hyperbolic sectors of the finite and infinite singular points of $p(\mathcal{X})$. Moreover, if $p(\mathcal{X})$ has finitely many finite and infinite singular points and finitely many limit cycles, then $p(\mathcal{X})$ has finitely many separatrices. We call each connected component of the complement of the union of separatrices a *canonical region* of $p(\mathcal{X})$. Neumann [14] proved that each canonical region of a vector field $p(\mathcal{X})$ is parallel.

To the union of the separatrices of $p(\mathcal{X})$ together with an orbit belonging to each canonical region of $p(\mathcal{X})$ we call a *separatrix configuration* of $p(\mathcal{X})$. We say that the separatrix configurations S_1 and S_2 of $p(\mathcal{X}_1)$ and $p(\mathcal{X}_2)$ are *topologically equivalent* if there exists an orientation preserving homeomorphism from $\mathbb{D}$ to $\mathbb{D}$ which transforms orbits of S_1 onto orbits of S_2. The following is the Markus-Neumann-Peixoto classification theorem, see [13, 14, 15, 16] or [8] for more details, for the Poincaré compactification in the Poincaré disc of polynomial systems.

Theorem 5 (Markus-Neumann-Peixoto)**.** *Let $p(\mathcal{X}_1)$ and $p(\mathcal{X}_2)$ be the Poincaré compactification of two polynomial systems $\dot{x} = \mathcal{X}_1(x)$ and $\dot{x} = \mathcal{X}_2(x)$, respectively. The flows of $p(\mathcal{X}_1)$ and $p(\mathcal{X}_2)$ on the Poincaré disc are topological equivalent if and only if the separatrix configurations of $p(\mathcal{X}_1)$ and $p(\mathcal{X}_2)$ are topological equivalent.*

Hence in order to qualitatively describe the phase portrait on the Poincaré disc of system (1) it is enough to qualitatively describe its separatrix configuration. This was done in Fig. 1, where we have drawn the separatrices other than singular points with bold lines. The other lines are orbits contained in its respective canonical regions. We observe from Fig. 1 that system (1) has 15 separatrices, five of them singular points, and 7 canonical regions, six of them of type strip and the one formed by the closed orbits surrounding z_0, annular.

Below we prove Theorem 3 by proving that Fig. 1 is a separatrix configuration of system (1).

From the previous sections we conclude that close enough to the singular points, the phase portrait of system (1) is qualitatively the one presented in Fig. 6. For further references we label the hyperbolic, parabolic and elliptic sectors presenting in the origins of the charts U_1 and V_1 in Fig. 6 as h_1, h_2, h_3, h_4, p_1, p_2 and e_1, e_2, respectively.

From the definition of system (1), each of its orbits is a connected component of a level set of $H_F = (p^2 + q^2)/2$ (because the ony singular point of this system is the center z_0), which in turn is the inverse image under $F = (p, q)$ of circles surrounding the point $(0, 0)$. Since F preserves orientation (because the Jacobian determinant of F is positive), each orbit of (1) is carried onto a curve contained in a circle with counterclockwise orientation. As we have seen in section 2, the curve $\beta(s)$ defined in (3) is the asymptotic variety of F. Moreover, the points $\beta(0) = (-1, -995/4)$ and $\beta(1) = (0, 208)$ of this curve have no inverse image under F, all the other points of this curve have exactly one inverse image and the other points of $\mathbb{R}^2$ have precisely two inverse images. Acting as in [6], we delete from the curve $\beta(s)$ the

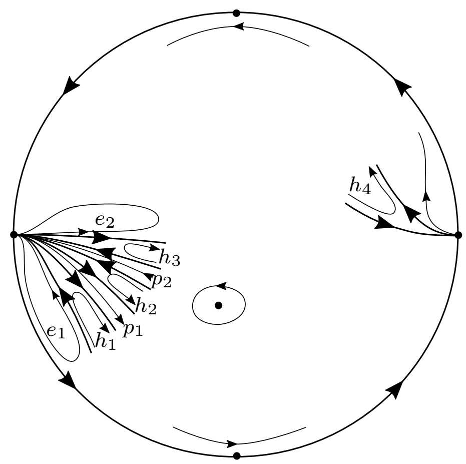
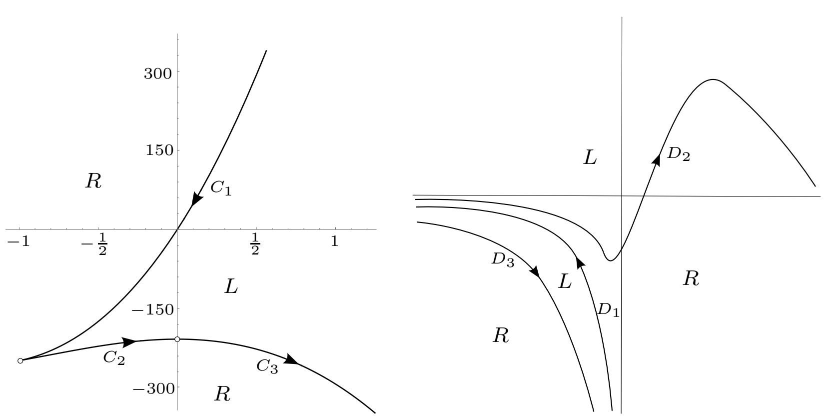


FIGURE 6. The phase portrait of system (1) near the singular points.

points $\beta(0)$ and $\beta(1)$, obtaining three curves: $C_1 = \beta(-\infty, 0)$, $C_2 = \beta(0, 1)$ and $C_3 = \beta(1, \infty)$. According to [6], the inverse image under F of each C_i is a curve that divides the plane into two connected components. We call D_i the inverse image of C_i, $i = 1, 2, 3$. The set $D_1 \cup D_2 \cup D_3$ is called in [6] the *asymptotic flower* of F. It follows that $\mathbb{R}^2 \setminus (D_1 \cup D_2 \cup D_3)$ is formed by 4 connected components, each of them mapped twice onto each of the two connected components of $\mathbb{R}^2 \setminus \{\beta(s)\}$. Each curve C_i has a natural orientation, given by its parametrization (it is the opposite orientation used in [6]). So each curve D_i also has a natural orientation (recall that F preserves orientation). The graphics of C_i and D_i, $i = 1, 2, 3$, are given in (a) and (b) of Fig. 7, respectively. As in [6, 7] the axes in (a) have different scales. Following [6], we label the regions as R (right) and L (left) of the curves C_i and D_i.



(a) Asymtotic variety (b) Asymptotic flower

FIGURE 7. The asymptotic variety and flower of F.

Since for each $s \in \mathbb{R}$

$$\beta'(s) \cdot \beta(s) = \frac{1}{4}(1-s)s(s+1)\big(112500s^6 - 232875s^5 + 301125s^4 - 425760s^3 + 432312s^2 - 86565s + 116423\big),$$

and this polynomial of degree 6 multiplying $(1-s)s(s+1)$ has no real zeros by Sturm's theorem, it follows that the curves C_1, C_2 and C_3 are transversal to the circles centered at $(0,0)$. As a consequence the curves D_1, D_2 and D_3 are transversal to the non-singular orbits of system (1). In particular, the image of a non-periodic orbit of system (1) has α- and ω-limits contained in the curve $\beta(s)$. Below we will say that the image of an orbit *starts* or *finishes* at $\beta(s_0)$ meaning that its α- or ω-limit is $\beta(s_0)$, respectively. Moreover, through each point in the intersection of $C_1 \cup C_2 \cup C_3$ with a circle, it crosses exactly one image of an orbit of system (1).

We call S_1 and S_2 the circles centered at $(0,0)$ and containing the points $\beta(1)$ and $\beta(0)$, respectively.

The point z_0, being the inverse image under F of $(0,0) = \beta(-1)$, is contained in the curve D_1. The images under F of the closed orbits surrounding z_0 are circles surrounding $(0,0)$ contained in the bounded region defined by S_1. Thus the boundary of the period annulus of the center z_0 corresponds to the arc of circle contained in S_1, starting and finishing at the point $\beta(1)$. This means that the boundary of the period annulus is an orbit that goes to infinity through the region labeled by L in (b) of Fig. 7. In particular, in the Poincaré disc, this orbit tends to the origin of the chart V_1. Then analyzing the possibilities in Fig. 6, we see that this orbit contains the two separatrices of the hyperbolic sector h_2. This period annulus is an annular canonical region.

Now we analyze the parabolic sectors p_1 and p_2.

Close to the two points of D_1 cut by the orbit giving the boundary of the period annulus of the center (i.e. the orbit connecting the two separatrices of the hyperbolic sector h_2), and outside the period annulus, there must exist orbits cutting D_1. Analyzing the images of these orbits, they are contained in circles surrounding the circle S_1. So there are two possibilities for the images of these orbits: either they are arcs starting and finishing at a point of the curve C_2, or they are arcs starting at the curve C_3 and finishing at the curve C_2 or C_3. At a first glance both of these possibilities are compatible with the parabolic sectors p_1 and p_2 in Fig. 6. We *claim that the correct possibility is the first one.* Indeed, we can increase the radii of these circles containing the images of the orbits of p_1 and p_2 until we achieve the circle S_2. If we are in the second possibility, the orbit whose image is contained in S_2 and starts at a point of C_3 will contain the separatrice of the end of the parabolic sector p_2. But this orbit will not contain the separatrice of the end of the parabolic sector p_1, because we can continue drawing arcs starting at C_3 with radii bigger than the radius of S_2. Thus the parabolic sector p_1 will not finish, a contradiction with the nature of the vector field at the origin of the chart V_1, as shown in Fig 6. This proves the claim.

So the image of the orbits of the parabolic sectors p_1 and p_2 are arcs starting and finishing at a point of the curve C_2. And since we can continue drawing these arcs until we arrive at circle S_2, this means that the parabolic sectors p_1 and p_2 are connected, and the image of the orbit containing the separatrices that separate p_1 from h_1 and p_2 from h_3 is contained in the arc of S_2 starting and finishing at the point $\beta(0)$. The region connecting p_1 to p_2 is a strip canonical region, see Fig. 1.

Now since the image of this last orbit cuts the curves C_3 and C_1, there must exist orbits near it whose images cross C_3 and C_1. The only possibility is that those images are arcs of circles starting at the curve C_1, rotating a complete turn crossing C_3 and C_1 and continue up to finishing in the curve C_3. We call these orbits the *big orbits*. A big orbit whose image is contained in a circle close enough to S_2 enters both the hyperbolic sectors h_1 and h_3. We have to see where the big orbits start and finish.

The orbits whose images are arcs of the circles with radii smaller than the radius of S_2, starting and finishing at C_1 and contained in the region R correspond to an elliptic sector with boundary formed by an orbit having image contained in the arc of S_2 starting at C_1 an finishing at $\beta(0)$. Close to this boundary and out of the elliptic sector there must exist orbits whose images start at C_1. These orbits are the big orbits. Hence it follows that this elliptic sector is e_1 and that the big orbits start at the origin of V_1, see Fig. 1.

The orbits whose images are arcs starting and finishing at C_1 and contained in the region L form the elliptic sector e_2. Clearly its boundary is formed by the two orbits containing the separatrices of the hyperbolic sector h_4. The image of these orbits are the arcs starting at C_1 and finishing at $\beta(1)$ and starting at $\beta(1)$ and finishing at C_1, respectively.

In particular this means that the big orbits must finish at the origin of the chart U_1, below the hyperbolic sector h_4. Since their images are contained in the circles bigger that S_2, there exist orbits whose images are arcs contained in the circles between S_1 and S_2, starting at C_1, crossing C_2 and finishing at C_3. These orbits produce a parabolic sector between h_3 and e_2, and give rise to a strip canonical region as presented in Fig. 1.

The big orbits also produce a strip canonical region.

The orbits of the strip canonical region formed above the hyperbolic sector h_4 have their images contained in arcs of circles with radii bigger than the radius of S_1, starting at C_3 and finishing at C_1.

The elliptic sectors e_1 and e_2 form another two strip canonical regions.

Hence we have 7 canonical regions, six of them are strip and one is annular. Analyzing Fig. 1, we see there are 6 finite orbits that are separatrices. The infinite has another 4 orbits. Hence, since there are 5 singular points, we have 15 separatrices in the separatrix configuration of system (1) in the Poincaré disc.

Acknowledgements

The second author is partially supported by a BPE-FAPESP grant number 2014/ 26149-3. The first and third authors are partially supported by a MINECO grant number MTM2013-40998-P, an AGAUR grant number 2014SGR 568 and two FP7-PEOPLE-2012-IRSES grants numbers 316338 and 318999. The last two authors are also partially supported by a CAPES CSF–PVE grant 88881. 030454/ 2013-01 from the program CSF-PVE.

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Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: `artes@mat.uab.cat`

Departamento de Matemática, Universidade Federal de São Carlos, 13565–905 São Carlos, São Paulo, Brazil

E-mail address: `franciscobraun@dm.ufscar.br`

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: `jllibre@mat.uab.cat`