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The center problem for $\mathbb{Z}_2$-symmetric nilpotent vector fields

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ABSTRACT

We say that a polynomial differential system $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$ having the origin as a singular point is $\mathbb{Z}_2$-symmetric if $P(-x,-y) = -P(x,y)$ and $Q(-x,-y) = -Q(x,y)$. It is known that there are nilpotent centers having a local analytic first integral, and others which only have a C^∞ first integral. However these two kinds of nilpotent centers are not characterized for different families of differential systems. Here we prove that the origin of any $\mathbb{Z}_2$-symmetric system is a nilpotent center if, and only if, there is a local analytic first integral of the form $H(x,y) = y^2 + \cdots$, where the dots denote terms of degree higher than two.

1. Introduction and statement of the main results

We consider a planar differential system of the form

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y), \tag{1.1}$$

with P and Q defined and analytic in a neighborhood of the origin where the origin is an isolated singular point. The local phase portrait near an isolated singular point can be determined by the Hartman–Grobman theorem except for the case of a monodromic singularity. We recall that a singular point is monodromic when nearby orbits rotate around it. For analytic differential systems it is known that the unique monodromic singularities are centers and foci. We recall that a center is a singular point for which there exists a punctured neighborhood filled of periodic orbits, and a focus has a punctured neighborhood filled of spiraling orbits. The *center problem* consists in distinguishing between a center or a focus at a monodromic singular point. If the linear part has pure imaginary eigenvalues or has zero eigenvalues but the linear part is not identically

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