PERIOD FUNCTION FOR A FAMILY OF PIECEWISE PLANAR HAMILTONIAN SYSTEMS

ALEX C. REZENDE, MIRIANNE A. S. SANTOS, AND JOAN TORREGROSA

Abstract. In this paper, we analyze the monotonicity and the number of critical periods of the period function associated to a family of piecewise planar continuous potential systems. In particular, we describe the bifurcation diagram of the period function, determining the regions of the parameter space where it is monotonous increasing or decreasing and where it has at most one simple critical period. In order to prove the main result, we use that such a function can be written in terms of the two period functions of the uncoupled planar Hamiltonian systems matched by the separation line.

1. Introduction and statement of the main results

There exist several papers dealing with properties of the period function, here denoted by T, in smooth Hamiltonian vector fields with respect to the monotonicity and the number of critical periods, i.e. the number of zeros of the derivative of T. In [13] the authors analyzed the period function of the center singularity at the origin of the planar potential differential system associated to the Hamiltonian function $H(x,y) = y^2/2+V(x)$, where $V(x) = x^2/2 + ax^4/4 + bx^6/6$, with $a, b \in \mathbb{R}$ and $b \neq 0$, as a continuation of the study carried out in [3, 10], where $V(x) = x^2/2 + ax^3/3 + bx^4/4$. Such studies were made thanks to the qualitative properties of the corresponding Picard–Fuchs equations.

There also exist several contributions for piecewise differential systems in the sense of the conditions for the origin to be either a center or an isochronous center as in [1, 5, 9, 11, 12]. Moreover, there exist studies aiming to determine the number of critical periods of piecewise linear systems as in [15]. In [12] it is proved that for any analytical function F at the origin, with $F(0) = F'(0) = 0$ and $|F''(0)| < 1$, the piecewise system

$$(\dot{x}, \dot{y}) = \begin{cases} \big(-y, x + F'(x)\big), & \text{if } y \geq 0, \\ \big(-y, x - F'(x)\big), & \text{if } y < 0, \end{cases}$$

is a center if and only if $F(x)$ is even. Moreover, there is no non-trivial even function $F(x)$ such that the potentials $V^{\pm}(x) = x^2/2 \pm F(x)$ are isochronous and it is not possible to generate a non-trivial isochronous center by matching two centers generated by potentials $V^{\pm}(x)$, with $F(x) = \sum_{n\geq 3} a_n x^n$ an analytic non-linear function. The authors of [12] also consider piecewise planar differential systems

$$(\dot{x}, \dot{y}) = \begin{cases} \big(-y, F'(x)\big), & \text{if } y \geq 0, \\ \big(-y, G'(x)\big), & \text{if } y < 0, \end{cases} \tag{1}$$

where $F(x) = ax^2 + O(x^4)$, with $a > 0$ an analytic even function at 0. They proved that, for every $b > 0$, there exists a unique even function $G(x)$, analytic at the origin, such that $G(x) = bx^2 + O(x^4)$, for which the corresponding system (1) has an isochronous center at the origin.

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