The cyclicity of hyperbolic hemicycles

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Abstract. In this paper we consider families of planar polynomial vector fields of degree n and study the cyclicity of a type of unbounded polycycle Γ called hemicycle. Compactified to the Poincaré disc, Γ consists of an affine straight line together with half of the line at infinity and has two singular points, which are hyperbolic saddles located at infinity. We prove four main results. Theorem A deals with the cyclicity of Γ when perturbed without breaking the saddle connections. For the other results we consider the case $n = 2$. More concretely they are addressed to the quadratic integrable systems belonging to the class Q_3^R and having two hemicycles, Γ_u and Γ_ℓ, surrounding each one a center. Theorem B gives the cyclicity of Γ_u and Γ_ℓ when perturbed inside the whole family of quadratic systems. In Theorem C we study the number of limit cycles bifurcating simultaneously from Γ_u and Γ_ℓ when perturbed as well inside the whole family of quadratic systems. Finally, in Theorem D we show that for three specific cases there exists a simultaneous alien limit cycle bifurcation from Γ_u and Γ_ℓ.

1 Introduction and main results

We begin by recalling the notion of limit periodic set as introduced in [22, Definition 10]. This is the fundamental object that we aim to study and its definition is given in terms of the Hausdorff topology, which for reader's convenience we briefly explain next.

Remark 1.1. Let S be a metrizable space and denote by $\mathcal{C}(S)$ the set of all compact non-empty subsets of S. Given any $K_1, K_2 \in \mathcal{C}(S)$ we define

$$d_H(K_1,K_2) = \sup_{x_1\in K_1, x_2\in K_2} \left\{ \inf_{x_2'\in K_2} d(x_1,x_2')\,,\ \inf_{x_1'\in K_1} d(x_1',x_2) \right\}.$$

One can readily show that d_H is a distance. It defines a topology on $\mathcal{C}(S)$, which is independent of the distance d chosen, that is called the *Hausdorff topology.* Moreover it turns out that

$$d_H(K_1,K_2) = \inf\big\{\varepsilon > 0 : K_1 \subset N_\varepsilon(K_2) \text{ and } K_2 \subset N_\varepsilon(K_1)\big\},$$

where $N_\varepsilon(K)$ is the ε-neighbourhood of K. Finally, if (S,d) is a compact metric space then so is $(\mathcal{C}(S), d_H)$. The interested reader is referred to [20, p. 279] for both assertions. □

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