ON THE DISTRIBUTION OF THE ZEROS OF SOME POLYNOMIAL MAPS $(P,Q)\colon \mathbb{R}^2 \to \mathbb{R}^2$

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ABSTRACT. Consider a map $(P,Q)\colon \mathbb{R}^2 \to \mathbb{R}^2$ of degree (n,m) with $m \geq n$ and assume that it has nm different real zeros. We study the distribution of these nm zeros in the plane $\mathbb{R}^2$ for $n = 1,2,3$ and $m \leq 4$.

1. Introduction and statement of the main results

Consider two polynomials $P(x,y)$ and $Q(x,y)$ with real coefficients of degrees n and m respectively with $m \geq n$. Then we say that the polynomial map $(P,Q)\colon \mathbb{R}^2 \to \mathbb{R}^2$ has degree (n,m). We assume that the map (P,Q) has exactly nm different real zeros. The objective of the paper is to study the distribution of the nm zeros of the map (P,Q) in the plane $\mathbb{R}^2$ when $n = 1,2,3$ and $m \leq 4$.

All the papers in the literature concerning the distribution of the zeroes of a polynomial map are for polynomial maps of one variable, see for instance [1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. As far as the authors are concerned this is the first time in the literature where the distribution of the zeros of a class of polynomial maps $(P,Q)\colon \mathbb{R}^2 \to \mathbb{R}^2$ in two variables is studied.

We introduce some definitions. Given a finite subset B of points of $\mathbb{R}^2$, we denote by $\hat{B}$, $\partial\hat{B}$ and $\#B$ its convex hull, the boundary of the convex hull, and its cardinal, respectively.

We denote by A the set of nm zeros of the map (P,Q). Set $A_0 = A$ and $A_{i+1} = A_i \cap \partial\widehat{A_i}$ for $i \geq 0$. Note that there exists a positive integer q such that $A_q \neq \emptyset$ and $A_{q+1} = \emptyset$.

We say that A has the *distribution of zeros* $(K_0; K_1; K_2; \ldots; K_q)$ if $K_i = \#(A_i \cap \partial\hat{A}_i)$. We say that A has the distribution of zeros $(K_0; K_1; K_2; \ldots; K_r; *)$ if we do not specify the values of K_i for i between $r+1$ and q. We also say that the zeros of (P,Q) belonging to $A_i \cap \partial\hat{A}_i$ are on the *i-th level*.

The main result of this paper is the following.

Theorem 1. *Let $(P,Q)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map of degree (n,m) with exactly nm different zeros with $n = 1,2,3$ and $m \leq 4$. The distribution of the zeros of the map (P,Q) is:*

(a) (m) *if* $(n,m) = (1,m)$.
(b) (4) *and* $(3;1)$ *if* $(n,m) = (2,2)$.
(c) (6), $(4;2)$ *and* $(3;3)$ *if* $(n,m) = (2,3)$.
(d) (8), $(5;3)$, $(4;4)$, $(4;3;1)$ *and* $(3;5)$ *if* $(n,m) = (2,4)$.
(e) (9), $(8;1)$, $(7;2)$, $(6;3)$, $(5;4)$, $(5;3;1)$, $(4;5)$; $(4;4;1)$, $(4;3;2)$, $(3;6)$, $(3;5;1)$, $(3;4;2)$ *and* $(3;3;3)$ *if* $(n,m) = (3,3)$.
(f) (12), $(11;1)$, $(10;2)$, $(9;3)$, $(8;4)$, $(8;3;1)$, $(7;5)$, $(7;4;1)$, $(7;3;2)$, $(6;6)$, $(6;5;1)$, $(6;4;2)$, $(6;3;3)$, $(5;7)$, $(5;6;1)$, $(5;5;2)$, $(5;4;3)$, $(5;3;4)$, $(5;3;3;1)$, $(4;8)$, $(4;7;1)$, $(4;6;2)$, $(4;5;3)$, $(4;4;4)$, $(4;4;3;1)$, $(4;3;5)$, $(4;3;4;1)$, $(4;3;3;2)$, $(3;9)$, $(3;8;1)$, $(3;7;2)$, $(3;6;3)$, $(3;5;4)$, $(3;5;3;1)$, $(3;4;5)$, $(3;4;4;1)$, $(3;4;3;2)$, $(3;3;6)$, $(3;3;5;1)$, $(3;3;4;2)$, $(3;3;3;3)$ *if* $(n,m) = (3,4)$.

Moreover there exist examples of such maps whose zeros have these distributions.

2010 *Mathematics Subject Classification.* Primary 26C10, 26D0, 530C10, 30C15, 42A05.
Key words and phrases. zeros, distribution of zeros, planar polynomial maps.