ON THE INVARIANT STRAIGHT LINES OF THE POLYNOMIAL DIFFERENTIAL SYSTEMS IN $\mathbb{R}^3$

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Abstract. In this paper we deal with the polynomial differential systems $dx/dt = P(x, y, z)$, $dy/dt = Q(x, y, z)$, $dz/dt = R(x, y, z)$ in $\mathbb{R}^3$. Let p, q and r be the degrees of the polynomials P, Q and R, respectively. For such polynomial differential systems having finitely many invariant straight lines we provide an upper bound for the maximum number of invariant straight lines. Additionally, in the case of $p = q = r = 1$, we show that the upper bound is three and it is reached, and when $p = q = r = 2$, we provide an example of a quadratic differential system exhibiting exactly 19 invariant straight lines.

1. Introduction and statement of the main results

Let $\mathbb{R}[x_1, \cdots, x_n]$ denote the ring of the polynomials in the variables $x_1, \cdots, x_n$ with coefficients in $\mathbb{R}$. We say that the *polynomial differential system*

$$\dot{x}_i = P_i(x_1, \cdots, x_n), \quad i = 1, \cdots, n, \tag{1}$$

where $P_i \in \mathbb{R}[x_1, \cdots, x_n]$, has *degree* m if the degree of the polynomial $P_1^2 + \cdots + P_n^2$ is $2m$.

The straight line $l : (\alpha_1, \cdots, \alpha_n) + \lambda(v_1, \cdots, v_n)$, where λ varies in $\mathbb{R}$, is an *invariant straight line* of system (1) if there exists a function $\mu(\lambda)$ such that

$$\big(P_1(\alpha_1+\lambda v_1, \cdots, \alpha_n+\lambda v_n), \cdots, P_n(\alpha_1+\lambda v_1, \cdots, \alpha_n+\lambda v_n)\big) = \mu(\lambda)(v_1, \cdots, v_n), \tag{2}$$

for all $\lambda \in \mathbb{R}$.

Assume that the polynomial differential system (1) of degree m has a finite number of invariant straight lines. We denote by $\alpha(n, m; P_1, \cdots, P_n)$ the number of invariant straight lines of system (1). We define $\alpha(n, m)$ as the maximum of $\alpha(n, m; P_1, \cdots, P_n)$ when $P_1, \cdots, P_n$ vary. An interesting open problem in general can be stated as follows:

For given positive integers n and m, what is the maximum number of invariant straight lines of system (1) *for all possible polynomials $P_1, \cdots, P_n$ for which system (1) has a finite number of invariant straight lines? In other words, what is $\alpha(n, m)$?*

For $n = 2$ consider the polynomial differential systems

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{3}$$

in $\mathbb{R}^2$. Let p be the degree of $P(x, y)$ and q be the degree of $Q(x, y)$. In the middle of the 1980's for the polynomial differential systems (3) with $p = q = n$ Ye Yanqian states the following conjecture, circulating among mathematicians working in polynomial differential equations:

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