THE EASIEST POLYNOMIAL DIFFERENTIAL SYSTEMS IN $\mathbb{R}^4$ HAVING AN INVARIANT SPHERE

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Abstract. In this paper we study the easiest polynomial differential systems in $\mathbb{R}^4$ exhibiting the invariant sphere $x^2 + y^2 + z^2 + w^2 = 1$, and we describe the dynamics of the orbits on this invariant sphere.

1. Introduction and statement of the main result

Consider the polynomial differential system

$$\dot{x} = P(x), \quad x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n, \tag{1}$$

where $P(x) = \big(P_1(x), \cdots, P_n(x)\big)$ is a vectorial polynomial function. The *degree* of the polynomial differential system (1) is the maximum of the degrees of the polynomials $P_1, \cdots, P_n$.

Let $H(x)$ be a C^1 function, defined on a full Lebesgue measure subset U of $\mathbb{R}^n$, locally non-constant in U. The function $H(x)$ is a *first integral* of the differential system (1) if H is constant on all the orbits contained in U, i.e.

$$\frac{dH}{dt} = \frac{\partial H}{\partial x_1} P_1 + \frac{\partial H}{\partial x_2} P_2 + \cdots + \frac{\partial H}{\partial x_n} P_n = 0. \tag{2}$$

Let $\mathbb{R}[x]$ be the ring of polynomials in the variable x with coefficients in $\mathbb{R}$. An algebraic hypersurface $f(x) = 0$, where $f \in \mathbb{R}[x]$, is *invariant* for the polynomial differential system (1) if there exists a $k \in \mathbb{R}[x]$ such that

$$P_1(x)\frac{\partial f}{\partial x_1} + P_2(x)\frac{\partial f}{\partial x_2} + \cdots + P_n(x)\frac{\partial f}{\partial x_n} = kf. \tag{3}$$

The polynomial $k(x)$ is called the *cofactor* of the invariant algebraic surface $f(x) = 0$. Note that, from (3), the degree of the cofactor is at most the degree of the polynomial differential system (1) minus one. Clearly, if $f(x) = 0$ is an invariant algebraic hypersurface of system (1), any orbit of system (1) with an initial point x_0 satisfying $f(x_0) = 0$ will always remain in the invariant algebraic hypersurface $f(x) = 0$. For more information about invariant algebraic curves, surfaces and hypersurfaces see for instance [1, 2].

An interesting invariant algebraic hypersurface in $\mathbb{R}^n$ is the sphere $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Obviously not all polynomial differential systems have an invariant sphere, so a natural question is: *What is the easiest polynomial differential system having an invariant sphere? And what is the dynamics of the polynomial differential system on such invariant sphere?*

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