ON THE ABEL DIFFERENTIAL EQUATIONS AND THE POLYNOMIAL DIFFERENTIAL SYSTEMS

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Abstract. We study the realtionships between the rational trigonometric Abel differential equations and the polynomial differential systems of the form $\dot{x} = -y^m + P_n(x, y)$, $\dot{y} = x^m + Q_n(x, y)$ with P_n and Q_n homogeneous polynomials of degree $n > m \geq 1$ with m odd.

For these differential systems we provide sufficient conditions which control how many limit cycles can be born in a Hopf bifurcation at the singular point localized at the origin of coordinates.

We also provide sufficient conditions for the integrability of such polynomial differential systems.

1. Introduction and statement of the results

Two of the main problems in the qualitative theory of real planar differential systems is to determine the existence or non–existence of limit cycles and of first integrals for a given class of polynomial differential systems.

We recall that a *limit cycle* of a differential system is a periodic orbit of the system isolated in the set of all periodic orbits of the system.

Poincaré [15] started to study seriously the limit cycles of the planar differential systems, and the first works showing their importance in applied problems appeared at the end of the 1920s by van der Pol [16], Liénard [10], Andronov [1], and continue up to nowdays.

The first objective of this paper is to obtain information on the limit cycles which can exhibit the Hopf bifurcation at the origin of coordinates for the polynomial differential systems

$$\dot{x} = -y^m + P_n(x, y), \qquad \dot{y} = x^m + Q_n(x, y), \tag{1}$$

with P_n and Q_n homogeneous polynomials of degree $n > m \geq 1$ with m odd. As usual the dot in (1) denotes derivative with respect the independent variable t.

We write the polynomial differential systems (1) in polar coordinates (r, θ), defined by

$$x = r\cos\theta \quad \textit{and} \quad y = r\sin\theta, \tag{2}$$

and we obtain

$$\begin{aligned} \dot{r} &= r^m \left(\sin\theta\cos^m\theta - \cos\theta\sin^m\theta\right) + r^n f(\theta), \\ \dot{\theta} &= r^{m-1}\left(\sin^{m+1}\theta + \cos^{m+1}\theta\right) + r^{n-1} g(\theta), \end{aligned} \tag{3}$$

where

$$f(\theta) = \cos\theta P_n(\cos\theta, \sin\theta) + \sin\theta Q_n(\cos\theta, \sin\theta),$$

$$g(\theta) = \cos\theta Q_n(\cos\theta, \sin\theta) - \sin\theta P_n(\cos\theta, \sin\theta).$$

We note that f and g are homogeenous trigonometric polynomials in the variables $\cos\theta$ and $\sin\theta$ of degree $n+1$. Rescaling the independent variable t and taking as new independent the variable s

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