THE GLOBAL DYNAMICS OF THE PAINLEVÉ-GAMBIER EQUATIONS XVIII, XXI AND XXII

JIE LI[1] AND JAUME LLIBRE[2]

ABSTRACT. In this paper we describe the global dynamics of the Painlevé-Gambier equations numbered XVIII: $x'' - (x')^2/(2x) - 4x^2 = 0$, XXI: $x'' - 3(x')^2/(4x) - 3x^2$ and XXII: $x'' - 3(x')^2/(4x) + 1 = 0$. We obtain three rational functions as their first integrals and classify their phase portraits in the Poincaré disc. The main reason for considering these three Painlevé-Gambier equations is due to the paper [5] where the authors studied these three differential equations in order to illustrate a method to generate nonlocal constants of motion for a special class of nonlinear differential equations. Here we want to complete their studies describing all the dynamics of these equations.

1. Introduction and statement of the main results

Some of the most studied differential systems are the second-order ordinary differential equations (ODEs) $x'' + F(x, x') = 0$, in particular those equivalent to conservation equations, $x'' + g(x) = 0$, where the prime denotes derivative with respect to the time t. By adding to a linear term of x', $x'' + f(x)x' + g(x) = 0$, we obtain the famous Liénard differential equation. In this paper we consider a class of systems equivalent to second order ordinary differential equations in which the quadratic term of x' is considered,

$$x'' + f(x)(x')^2 + g(x) = 0. \tag{1}$$

Such a study is motivated by a wide range of interests. In [5] the authors studied the following three types of the Painlevé-Gambier equations

$$\textbf{XVIII:} \quad x'' - \frac{1}{2x}(x')^2 - 4x^2 = 0,$$

$$\textbf{XXI:} \quad x'' - \frac{3}{4x}(x')^2 - 3x^2,$$

$$\textbf{XXII:} \quad x'' - \frac{3}{4x}(x')^2 + 1 = 0.$$

They develop a method based in a generalization of the Sundman transformation in order to obtain new nonlocal first integrals of autonomous second-order ordinary differential equations.

The previous three differential equations are inside 50 second-order ordinary differential equations possessing a canonical form whose solutions have the Painlevé property developed by Painlevé, Gambier and their pupils. The results of [4] show that apart of 6 Painlevé-equations, the remaining 44 of the Painlevé-Gambier classification have solutions that can be expressed in terms of elementary functions. Ince studied many aspects of the Painlevé-Gambier equations for instance first integrals and analytical solutions, see the details in [7]. The authors of [6] used the generalized Sundman transformation method in order to construct other first integrals for some of 50 differential equations of classified by Painlevé and Gambier.

The generalized Sundman transformation proposed by Akande et al. [1] is used to study first integrals, symmetries and solutions of equations of Painlevé-Gambier method, such as trigonometric periodic solutions and periodic solutions in [8, 1, 5]. In [8] the authors calculated the explicit and exact general periodic solutions of the cubic Duffing equation and of some

2010 *Mathematics Subject Classification.* 34C05.
Key words and phrases. Painlevé-Gambier equations; Phase portrait; Poincaré disc; first integral.