DYNAMICS OF THE FLOWS WITH CONSTANT SLOPE ON THE CYLINDER AND ON THE MÖBIUS STRIP

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Abstract. The study of the flows with constant slope on the 2-dimensional torus goes back to Denjoy and Siegel. More recently the study of these flows have been extended to the Klein bottle. Here we characterize the dynamics of the flows with constant slope on two non-compact surfaces, the cylinder and the Möbius strip. On the cylinder we consider smooth flows, and on the Möbius strip we consider both smooth and piecewise smooth flows.

1. Introduction and statement of the main results

The study of the flows on surfaces goes back to Poincaré [16] at the end of the 19th century. Many phenomena in our world, from the evolution of the weather to the motion of an electron in a metal, can be described by using the dynamical systems. The motivation of Poincaré for studying the flows on surfaces came from his studies on celestial mechanics, see [17].

Since in general the solutions of the differential equations cannot be found explicitly in function of the time, Poincaré started the qualitative theory of the differential equations. The main objective of this theory is to describe where the orbits of a differential equation born, die, or if they are periodic or not, homoclinic, heteroclinic, ...

We recall that a *flow* on a manifold M is a one-parameter family $\{\varphi_t\}_{t\in\mathbb{R}}$ of homeomorphisms of M which satisfy $\varphi_0 = \text{id}$ and $\varphi_s \circ \varphi_t = \varphi_{s+t}$ for $s, t \in \mathbb{R}$.

The *orbit* of a point $p \in M$ under the flow φ_t is the set $\mathcal{O}(p) = \{\varphi_t(p); t \in I_p\}$ being I_p the maximal interval of definition of the solution $\varphi_t(p)$ such that $\varphi_0(p) = p$.

If $q \in \mathcal{O}(p)$, then $\mathcal{O}(p) = \mathcal{O}(q)$, because there exists $t^* \in \mathbb{R}$ such that $q = \varphi_{t^*}(p)$, and $\varphi_t(q) = \varphi_{t+t^*}(p)$ for $t \in \mathbb{R}$.

The orbit $\mathcal{O}(p)$ is a *periodic orbit* if the orbit $\mathcal{O}(p)$ is a closed solution curve, i.e. if for all $t \in \mathbb{R}$, $\varphi_{t+T}(p) = \varphi_t(p)$ for some $T > 0$. The minimal T such that this last equality holds is the *period* of the periodic orbit $\mathcal{O}(p)$.

Since the periodic motions arise naturally in many fields of the sciences and phenomena of real life, for instance, the rotation of the earth around its axis, the rotation of the earth around the sun, movements of the water waves, the motion of a pendulum, etc. Then the periodic orbits play an important role in understanding the behavior of the dynamical systems, and in the last few decades, considerable attention has been focused in the study of both the existence and number of periodic orbits of a dynamical system. When a periodic orbit of a flow is isolated in the set of all periodic orbits of the flow it is called a *limit cycle*. See for instance [1, 8, 21] and the references therein.

It is known that the existence of periodic orbits depends on the manifold where the dynamical system is defined. For instance, in [5, 10, 14] the authors studied the existence and

2020 *Mathematics Subject Classification.* Primary 34C05, 34C07, 34C25, 34C40.

Key words and phrases. Flow with constant slope, periodic orbits, cylinder, Möbius strip.