CHARACTERIZATION OF THE RICCATI AND ABEL POLYNOMIAL DIFFERENTIAL SYSTEMS HAVING INVARIANT ALGEBRAIC CURVES

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Abstract. The Riccati polynomial differential systems are the differential systems of the form $x' = c_0(x)$, $y' = b_0(x) + b_1(x)y + b_2(x)y^2$, where c_0 and b_i for $i = 0, 1, 2$ are polynomial functions.

We characterize all the Riccati polynomial differential systems having an invariant algebraic curve. We show that the first four higher coefficients of the polynomial in the variable y defining the invariant algebraic curve determine completely the Riccati differential system. A similar result is obtained for any Abel polynomial differential systems.

1. Introduction and statement of the main results

In this work we study the Riccati differential equation of the form

$$\frac{dy}{dx} = B_0(x) + B_1(x)y + B_2(x)y^2, \tag{1}$$

where $B_i(x)$ are rational functions. Indeed this differential equation can be transformed into the polynomial differential system

$$x' = c_0(x), \qquad y' = b_0(x) + b_1(x)y + b_2(x)y^2, \tag{2}$$

where $B_i(x) = b_i(x)/c_0(x)$ for i=1,2,3. The maximum degree of the polynomials $c_0(c)$ and $= b_0(x) + b_1(x)y + b_2(x)y^2$ is the *degree* of the polynomial differential system (2).

Already Euler [3] proved that if we know one particular solution, for instance $y_1(x)$, of the Riccati equation (1), then the general solution of (1) is $y(x) = y_1(x) + 1/v(x)$ where $v(x)$ is the solution of the first-order linear differential equation

$$\frac{dv}{dx} = -(B_1(x) + 2B_2(x)y_1(x))v - B_2(x).$$

The Ricatti differential (1) is the standard example of a nonlinear first order differential equation with a fundamental set of solutions whose general solution is

$$H(y, g_1(x), g_2(x), g_3(x)) = \frac{(y - g_1(x))(g_3(x) - g_2(x))}{(y - g_2(x))(g_3(x) - g_1(x))} = C, \tag{3}$$

the so-called *cross–ratio* of three arbitrary particular solutions $y = g_1(x)$, $y = g_2(x)$, and $y = g_3(x)$, where C is an arbitrary constant. Indeed, other nonlinear equations

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