3–DIMENSIONAL PIECEWISE LINEAR AND QUADRATIC VECTOR FIELDS WITH INVARIANT SPHERES

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Abstract. We consider the class $\mathcal{X}$ of 3-dimensional piecewise smooth vector fields that admit a first integral which leaves invariant any sphere centered at the origin. In this class, we prove that a linear vector field does not admit isolated invariant cones. Moreover, we provide the existence of at least ten 1-parameter families of crossing closed trajectories for quadratic vector fields in $\mathcal{X}$.

1. Introduction

Differential equations and dynamical systems can be used to model natural phenomena and we can obtain information about it from their solutions. An interesting tool used to understand the behavior of the solutions of a dynamical system is the existence of first integrals because, when they exist, the trajectories of the corresponding vector field remain restricted to the level surfaces of these functions. We say that a n-dimensional differential system is completely integrable when it has $n-1$ independent first integrals and the orbits of it are obtained just intersecting the level sets of the first integrals. Moreover, if it has less than $n-1$ first integrals, it is said to be partially integrable. The $2n$-dimensional Hamiltonian systems are particular cases of partially integrable systems, for which we commonly study their behavior restricted to their invariant level sets. The study of Hamiltonian systems has many applications and it is very important in mechanics, for example, as we can see in [28].

Observe that, if the system restricted to an invariant level set of the first integral has a hyperbolic closed trajectory, then the original system has a 1-parameter family of hyperbolic periodic orbits. As we will work with 3-dimensional piecewise smooth vector fields having a first integral that keeps invariant all the spheres centered at origin, in fact we deal with 1-parameter radial families. For more details about how to consider 3-dimensional smooth vector fields (resp. 3-dimensional piecewise smooth vector fields) with invariant spheres as 1-parameter radial family see for instance Section 5 of [4] (resp. [5]). In [6] it was proved that the behavior of a homogeneous vector field restricted to an invariant sphere of radius $\rho = 1$ is topologically equivalent to the behavior of the same system restricted to any other level. So, when a homogeneous vector field restricted to an invariant sphere has a limit cycle (resp. a center), the 3-dimensional vector field has an isolated (resp. non-isolated) invariant cone fulfilled of closed trajectories. On the other hand, the behavior of non-homogeneous vector fields could be totally different in distinct levels of invariant spheres (see again [6]). In this case, each hyperbolic closed trajectory restricted to an invariant sphere of radius ρ generates a 1-parameter radial family of closed trajectories of the 3-dimensional vector field near the sphere of radius ρ. So, it has locally a topological invariant

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cylinder near the sphere of radius ρ fulfilled of closed trajectories. In general, it is very difficult to classify the invariant surfaces generated by these 1-parameter families, as they can have very different behavior depending on the vector field. Understanding it certainly depends on the knowledge on the behavior of the vector fields restricted to each invariant sphere.

As these invariant surfaces are generated by closed trajectories of the restricted vector field, this problem is strictly related to the Hilbert's 16th problem, presented by D. Hilbert in 1900, at the International Congress of Mathematicians, in Paris. The second part of the Hilbert's 16th problem asks for an estimation of the maximal number of limit cycles that a planar polynomial vector field can have, being one of the most important open problems in Qualitative Theory of Ordinary Differential Equations and Dynamical Systems. For more details we refer the reader to [20].

In the last years, many classes of piecewise smooth dynamical systems have also been studied and a rigorous formulation of their qualitative properties was given by Filippov, in [15]. This theory is very important in many areas of science, see for instance [12]. Note that, the Hilbert's 16th problem has been extended to piecewise polynomial vector fields in a natural way (see for example [16, 24]). In part of this paper we will analyze the existence of (crossing) invariant cones for piecewise linear and quadratic vector fields. This dynamics also appears in 3-dimensional piecewise linear systems as in [7, 8, 9].

In this work, we consider 3-dimensional piecewise differential vector fields with a separation set given by $\Sigma = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$, that is

$$Y(x,y,z) = \begin{cases} X^+(x,y,z), & z \geq 0, \\ X^-(x,y,z), & z \leq 0. \end{cases} \tag{1}$$

As Y can be multi-valued in Σ, we will follow the Filippov's convention on the escaping and sliding regions, see again [15].

In the piecewise smooth case, as in the smooth one, the integrability of the vector fields $X^\pm$ is an important tool used to understand the behavior of the trajectories of $Y = (X^+, X^-)$, in the classification of phase portraits, and also to answer questions related to the existence of crossing limit cycles (i.e. isolated crossing periodic orbits). See [27] for more details. Furthermore, when both $X^\pm$ have the same first integral the dimension of the phase space where the trajectories of the piecewise smooth vector field $Y = (X^+, X^-)$ are defined is reduced by one. This property has motivated us to study 3-dimensional piecewise smooth vector fields partially integrable (that is, having both $X^\pm$ the same first integral $H : \mathbb{R}^3 \to \mathbb{R}$) restricted to invariant level sets of H, as we explain on the following.

Let $\mathfrak{X}$ be the class of smooth vector fields $X : \mathbb{R}^3 \to \mathbb{R}^3$ that admits $H(x, y, z) = x^2 + y^2 + z^2$ as a first integral. This class was previously studied in [6]. Note that all the spheres centered at the origin with radius ρ, $\mathbb{S}^2_\rho = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = \rho^2\}$, are invariant by the flow of $X \in \mathfrak{X}$. We denote by $\mathfrak{X}_n$ (resp. $\mathfrak{X}_n^H$) the class of polynomials (resp. homogeneous polynomials) vector fields of degree n in $\mathfrak{X}$. In this work, we consider the class of piecewise differential vector fields given by (1) such that $X^\pm \in \mathfrak{X}$. We denote this class by $\mathcal{X}$ and by $\mathcal{X}_n$ (resp. $\mathcal{X}_n^H$) when $X^\pm \in \mathfrak{X}_n$ (resp. $X^\pm \in \mathfrak{X}_n^H$). Hence, if $Y \in \mathcal{X}$, then any sphere centered at the origin is invariant by the flow of the piecewise differential system Y. Observe that we can consider invariant ellipsoids instead of invariant spheres. Although all the results can be easily generalized to this case, we have preferred not to do it here, to avoid repetitions. In the sequel, we describe the results that we have obtained for piecewise linear and quadratic (homogeneous and nonhomogeneous) differential systems in $\mathcal{X}$. We remark that, in general, the 3-dimensional homogeneous vector fields will not be when we consider them projected to a 2-dimensional space.

Before introducing our main results, we recall some properties about homogeneous vector fields $X \in \mathfrak{X}^H$. $X \in \mathfrak{X}_1$ is homogeneous and it writes in the form

$$X(\mathrm{x}; a_1, a_2, a_3) = (-a_1 y - a_2 z, a_1 x - a_3 z, a_2 x + a_3 y), \tag{2}$$

where, $\mathrm{x} = (x, y, z)$. In [6] it was proved that (2) has (generically) only a line of equilibrium points passing through the origin. Further, when we consider the restriction of (2) to the invariant spheres $\mathbb{S}^2_\rho$, we conclude that (2) has only two equilibrium points on each sphere which are centers and antipodals of each other (see Lemma 3, for more details). It means that the 3-dimensional smooth vector field (2) has a continuous of invariant cones fulfilled of non-isolated closed trajectories. In Proposition 14 we show that a quadratic homogeneous vector field $X \in \mathfrak{X}^H_2$ can present an isolated invariant cone, fulfilled of closed trajectories, showing an important difference between linear and quadratic homogeneous vector fields in the class $\mathfrak{X}$.

Using (2) we can see that each 3-dimensional piecewise linear system $Y = (X^+, X^-) \in \mathcal{X}_1$ is of the form

$$Y(x, y, z) = \begin{cases} X^+(\mathrm{x}; a_1^+, a_2^+, a_3^+), & z \geq 0, \\ X^-(\mathrm{x}; a_1^-, a_2^-, a_3^-), & z \leq 0, \end{cases} \tag{3}$$

with

$$X^\pm(\mathrm{x}; a_1^\pm, a_2^\pm, a_3^\pm) = (-a_1^\pm y - a_2^\pm z, a_1^\pm x - a_3^\pm z, a_2^\pm x + a_3^\pm y). \tag{4}$$

As explained in Section 2.3, we use the stereographic projection to study the local behavior of $Y \in \mathcal{X}$ restricted to the invariant spheres. Moreover, the projection of a linear (resp. quadratic) vector field defined on an invariant sphere is a quadratic (resp. cubic) planar vector field. We observe that they lose the property of homogeneity once projected. Usually, the behavior of piecewise smooth vector fields is richer than the behavior of the smooth ones. This property made us to look for isolated invariant cones in $\mathcal{X}_1$. However, the next result proves that they do not exist.

Theorem 1. *No piecewise differential system $Y \in \mathcal{X}_1$, given by (3), admits an isolated invariant cone.*

We prove it in Section 3, where we also show the possible phase portraits of (3), restricted to the invariant sphere $\mathbb{S}^2_\rho$, with respect to the admissibility of its equilibria (see Figures 3 and 4). We point out that the existence of crossing invariant cones for piecewise linear vector fields which are continuous in the separation set Σ was studied in [8, 9]. But the results cannot be applied to our study because the continuity condition is not satisfied.

Inspired by the homogeneity property of the linear vector fields in $\mathcal{X}$, we study some families in $\mathcal{X}^H_2$ in Section 4, where we prove that they can present isolated and non-isolated crossing invariant cones, showing an important difference between piecewise linear and quadratic homogeneous vector fields in $\mathcal{X}$. We prove it considering the restriction of a piecewise smooth vector field $Y \in \mathcal{X}^H_2$ to the sphere of radius $\rho = 1$ and showing that they can present centers for some specific values of the coefficients and crossing limit cycles for others. Another difference can be observed when we compare the piecewise quadratic homogeneous vector fields defined on $\mathbb{S}^2_1$ and on $\mathbb{R}^2$. To see it, we recall briefly the concept of reversible vector field defined in open regions of $\mathbb{R}^m$. Let $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ be a $\mathcal{C}^r$-involution. It means that $\varphi \circ \varphi(\mathrm{x}) = \mathrm{x}$, where

$\mathrm{x} \in \mathbb{R}^m$. Let $\mathrm{Fix}(\varphi) = \{\mathrm{x}; \varphi(\mathrm{x}) = \mathrm{x}\}$. We say that a differential vector field X, defined in $\mathbb{R}^m$, is φ-reversible if $D\varphi \circ X = -X \circ \varphi$, where $D\varphi$ denotes the Jacobian matrix of φ. We say that X is reversible with respect to a line (resp. a point) when $\mathrm{Fix}(\varphi)$ is a line (resp. a point). We refer [23], for an interesting survey about reversible differential systems. We recall that any quadratic homogeneous vector field defined in $\mathbb{R}^2$ is reversible with respect to the origin and, because of that, it does not have an equilibrium point of center type (see [2]). Thus, we cannot consider the center-focus problem for piecewise quadratic homogeneous vector fields on the plane. Note that the concept of reversibility can also be considered for piecewise smooth vector fields. For more details see Section 2.2.

Finally, we have also analyzed the local behavior of a piecewise quadratic vector field $\mathcal{X}_2$, proving the following result.

Theorem 2. *There exist at least ten* 1*-parameter radial families of invariant crossing closed trajectories in the quadratic family* $\mathcal{X}_2$*, near the radius* $\rho = 1$.

For proving Theorem 2, see Section 5, we consider the restriction of a piecewise smooth vector field $Y \in \mathcal{X}_2$ to the invariant sphere of radius $\rho = 1$ and we show that it has 10 hyperbolic crossing limit cycles on the sphere $\mathbb{S}^2_1$. Since these crossing limit cycles are hyperbolic on $\mathbb{S}^2_1$, they are normally hyperbolic with respect to the radial direction. This implies that $Y \in \mathcal{X}_2$ has at least ten 1-parameter radial families of crossing periodic orbits which cross the sphere of radius ρ in isolated closed trajectories, with $1 - \varepsilon < \rho < 1 + \varepsilon$ for ε sufficiently small. So, the 3-dimensional vector field $Y \in \mathcal{X}_2$ has invariant surfaces, foliated by crossing closed trajectories, which are locally topologically equivalent to cylinders. The global structure of each invariant surface is due to the birth or death of limit cycles. For example, this surface is topologically equivalent to a sphere when we have exactly two Hopf points in $\mathbb{S}^2_{\rho_*}$ and $\mathbb{S}^2_{\rho^*}$, being $\rho_* < 1 < \rho^*$.

This paper is structured as follows. Section 2 is devoted to recalling the tools used to prove our main results. In Section 3 we study piecewise linear vector fields with invariant spheres and we also prove Theorem 1. In Section 4 we give some families of centers for piecewise continuous quadratic homogeneous vector fields, in the sphere $\mathbb{S}^2_1$. Finally, in Section 5 we prove Theorem 2.

2. Preliminary results

This section is dedicated to recall some concepts and bifurcation techniques for piecewise smooth vector fields, that we use in the proofs of the results of this paper. Firstly, we recall the integrability concept and the Filippov's convention for piecewise smooth vector fields. After that, we consider a smooth vector field $X : \mathbb{R}^3 \to \mathbb{R}^3$ having $H(x, y, z) = x^2 + y^2 + z^2$ as a first integral and define a piecewise smooth vector field with the same property. Considering that the center-focus problem and local cyclicity will be studied projecting each 3-dimensional piecewise smooth vector field defined on the invariant sphere to a planar one, we also recall some definitions and the computation algorithm of the center conditions (or Lyapunov constants) for planar piecewise smooth vector fields.

2.1. **Integrability.** Let $P : \mathbb{R}^m \to \mathbb{R}^m$ such that $P(\mathrm{x}) = (P_1(\mathrm{x}), \ldots, P_m(\mathrm{x}))$, where $\mathrm{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and P_i, $i = 1, \ldots, m$, are polynomials in the variables x_i with real coefficients. Let n be the maximum between the degrees of P_i, $i = 1, \ldots, m$ and consider an m-dimensional differential system

$$\dot{\mathrm{x}} = P(\mathrm{x}). \tag{5}$$

Let $U \subset \mathbb{R}^m$ be an open subset. If there exists a non-constant analytic function $H : U \to \mathbb{R}$ such that

$$\langle P(\mathrm{x}), \nabla H(\mathrm{x})\rangle = \sum_{i=1}^{m} P_i(\mathrm{x}) \frac{\partial H}{\partial x_i}(\mathrm{x}) = 0, \text{ for } \mathrm{x} \in U,$$

then (5) is *partially integrable* on U and H is a *first integral* of (5) on U. Moreover, if P has $m-1$ independent first integrals then P is called a *completely integrable* system. In [14], it was proved that any m-dimensional linear system has $m-1$ independent first integrals and then this is an example of a class of completely integrable systems.

It is worth to say that when a system P is completely integrable its trajectories are determined by the intersection of the level sets of its first integrals, see [13] for more details about it. Moreover, each $X \in \mathfrak{X}$ has at least one first integral and, in Section 3 we will see that the key point for the proof of Theorem 1 is the existence of a second first integral for $X^{\pm} \in \mathfrak{X}_1$ and to have a good knowledge of how the levels of these first integrals interact with the separation curve of piecewise system (3).

2.2. **Filippov vector fields.** In this subsection we recall the definition of a piecewise smooth vector field under the Filippov's convention (see [15] for more details). We restrict our attention to piecewise smooth vector fields defined in $\mathbb{R}^m$, the same definitions can be extended easily to m-dimensional manifolds.

Let $\mathrm{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and consider $f : \mathbb{R}^m \to \mathbb{R}$ a $\mathcal{C}^r$-class function such that $0 \in \mathbb{R}$ is a regular value of f. Therefore, $\Sigma = f^{-1}(0) = \{\mathrm{x} \in \mathbb{R}^m : f(\mathrm{x}) = 0\}$ is an embedded codimension one submanifold of $\mathbb{R}^m$. Consider $\Sigma^+ = f^{-1}([0, +\infty)) = \{\mathrm{x} \in \mathbb{R}^m : f(\mathrm{x}) \geq 0\}$, $\Sigma^- = f^{-1}((-\infty, 0]) = \{\mathrm{x} \in \mathbb{R}^m : f(\mathrm{x}) \leq 0\}$ and the piecewise smooth vector field with separation set Σ defined by

$$Y(\mathrm{x}) = \begin{cases} X^+(\mathrm{x}), & \mathrm{x} \in \Sigma^+, \\ X^-(\mathrm{x}), & \mathrm{x} \in \Sigma^-, \end{cases} \tag{6}$$

where $X^{\pm}$ are smooth vector fields defined on $\Sigma^{\pm}$. The equilibrium points of X^+ and X^- located in Σ^+ and Σ^-, respectively, are called admissible (or visible) equilibrium points or simply equilibrium points of (6). On the other hand, the equilibrium points of X^+ and X^- located in Σ^- and Σ^+, respectively, are called nonadmissible (or invisible) equilibrium points of (6).

The Lie derivative of f with respect to the vector field $X^{\pm}$ at the point $p \in \Sigma$ is defined by $X^{\pm}f(p) = X^{\pm}(p) \cdot \nabla f(p)$. The successive Lie derivatives are given by $(X^{\pm})^n f(p) = X^{\pm}(p) \cdot \nabla (X^{\pm})^{n-1} f(p)$, $n \geq 2$. When $X^+f(p) = X^-f(p)$, for all $p \in \Sigma$, we say that (6) is a refractive system (on Σ). For more details about refractive systems we refer the reader to [3, 5].

On the following, we recall the definitions of tangency points and tangency sets of (6). We say that $p \in \Sigma$ is a fold point of Y if $X^+f(p) = 0, (X^+)^2 f(p) \neq 0$ and $X^-f(p) \neq 0$ (or $X^-f(p) = 0, (X^-)^2 f(p) \neq 0$ and $X^+f(p) \neq 0$). Hence, p is a fold-fold point when $X^+f(p) = 0$, $X^-f(p) = 0$, $(X^+)^2 f(p) \neq 0$, and $(X^-)^2 f(p) \neq 0$. We also define the tangency set of $X^{\pm}$ with Σ by $S_{X^{\pm}} = \{p \in \Sigma : \; X^{\pm}f(p) = 0\}$ and the tangency set of Y by $S_Y = S_{X^+} \cup S_{X^-}$.

As usual, we consider the crossing region $\Sigma^c = \{p \in \Sigma : X^+f(p)X^-f(p) > 0\}$, the sliding region $\Sigma^s = \{p \in \Sigma : X^+f(p) < 0, \; X^-f(p) > 0\}$ and the escaping region $\Sigma^e = \{p \in \Sigma : X^+f(p) > 0, \; X^-f(p) < 0\}$. So Σ is the disjoint union $\Sigma^c \cup \Sigma^s \cup \Sigma^e \cup S_Y$

and following the Filippov's convention we define the Filippov vector field $F_Y(p)$ on $\Sigma^s \cup \Sigma^e$ by

$$F_Y(p) = \frac{1}{X^- f(p) - X^+ f(p)} \left(X^- f(p) X^+(p) - X^+ f(p) X^-(p) \right).$$

Finally, we say that (6) is time φ-reversible if $\mathrm{Fix}(\varphi) \subset \Sigma^c$ and $D\varphi \circ Y = -Y \circ \varphi$, where φ is an $\mathcal{C}^r$-involution defined in $\mathbb{R}^m$. As in the smooth case, each piecewise reversible vector field presents a certain symmetry. For more details, see [21].

2.3. **Orthogonal change of coordinates and stereographic projection.** We say that a change of coordinates is orthogonal when the matrix of it is orthogonal, in other words, if M is this matrix it must satisfy $M^t = M^{-1}$. This kind of change of coordinates keeps all the spheres invariant and using it we can assume that the equilibrium point of a smooth vector field, that always exists on each invariant sphere $\mathbb{S}^2_\rho$, can be located at any (x_0, y_0, z_0) that we choose. Note that, when we consider piecewise smooth vector fields on invariant spheres, this kind of change of coordinates (on the whole sphere) allows us to assume that some equilibrium point of the Filippov vector field or some fold point can be located at any $(x_0, y_0, 0) \in \Sigma$.

To study local behaviors, we use the stereographic projection with respect to the point $(0, -\rho, 0)$. It allows us to consider planar vector fields instead of 3-dimensional ones restricted to spheres. In the following, we define the piecewise projected vector field. Consider $\mathfrak{p} : \mathbb{S}^2_\rho \setminus \{(0, -\rho, 0)\} \to \mathbb{R}^2$ the stereographic projection on the plane $\{(x, y, z) \in \mathbb{R}^3 : y = \rho\}$ given by $\mathfrak{p}(x, y, z) = 2\rho(x, z)/(y + \rho)$. We define the projected vector field associated to $X \in \mathfrak{X}$ by

$$\mathcal{P}_X(\mathrm{u}) = d\mathfrak{p}_{\mathfrak{p}^{-1}(\mathrm{u})} \circ X \circ \mathfrak{p}^{-1}(\mathrm{u}),$$

where $X = X_{|_{\mathbb{S}^2_\rho}}$, $\mathrm{u} = (u, v)$ and $\mathfrak{p}(\mathrm{x}) = \mathrm{u}$. Note that, this stereographic projection sends the separation set $\Sigma = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ of a piecewise smooth vector field $Y \in \mathcal{X}$ to $\{(u, v) \in \mathbb{R}^2 : v = 0\}$. Thus, the projection $\mathcal{P}_Y : \mathbb{R}^2 \to \mathbb{R}^2$ of (1) is written as

$$\mathcal{P}_Y(\mathrm{u}) = \begin{cases} \mathcal{P}_{X^+}(\mathrm{u}), \ v \geq 0, \\ \mathcal{P}_{X^-}(\mathrm{u}), \ v \leq 0, \end{cases} \tag{7}$$

where $X^\pm = X^\pm_{|_{\mathbb{S}^2_\rho}}$, $\mathrm{u} = (u, v)$. Besides, $\mathfrak{p}$ preserves closed curves and contact between curves contained on its domain of definition, so $p \in \mathbb{S}^2_\rho$ is said to be a monodromic equilibrium point of (1) if $q = \mathfrak{p}(p)$ is a monodromic equilibrium point of (7).

2.4. **Lyapunov constants and local cyclicity for planar piecewise systems.** In this section, we will recall the stability algorithm for planar piecewise smooth vector fields of the form

$$Y(x, y) = \begin{cases} X^+(x, y), \ y \geq 0, \\ X^-(x, y), \ y \leq 0, \end{cases} \tag{8}$$

having both $X^\pm$ an equilibrium point of nondegenerate center-focus type at the origin. That is,

$$X^\pm(x, y) = \left(\alpha^\pm x - \beta^\pm y + \sum_{k=2}^{n} P_k^\pm(x, y), \beta^\pm x + \alpha^\pm y + \sum_{k=2}^{n} Q_k^\pm(x, y) \right),$$

with $P_k^\pm$ and $Q_k^\pm$ homogeneous polynomials of degree k in the variables x and y. We have assumed that both linear parts are in Jordan's normal form. Furthermore, we follow the Filippov's convention to define the trajectories of Y on the separation set $\Sigma = \{(x, y) \in \mathbb{R}^2 : y = 0\}$ and we assume $\beta^\pm \neq 0$ as the non degeneracy condition for each $X^\pm$. Using polar coordinates, $(x, y) = (r\cos\theta, r\sin\theta)$, we write system (8) as

$$\begin{cases} \dot{r} = R^+(r, \theta), & \theta \in [0, \pi], \\ \dot{r} = R^-(r, \theta), & \theta \in [\pi, 2\pi], \end{cases}$$

where the dot represents the derivative with respect to θ.

Consider $r^\pm(\theta, r_0)$ the solution of $\dot{r} = R^\pm(r, \theta)$ with initial condition $r^\pm(0, r_0) = r_0$ and $r_0 > 0$ sufficiently small. The expansion in Taylor's series of the solution $r^\pm(\theta, r_0)$ can be written as

$$r^\pm(\theta, r_0) = r_0 + \sum_{k=1}^{\infty} r_k^\pm(\theta) r_0^k,$$

with $r_k^\pm(0) = 0$, for all $k \geq 1$, and with r^+ defined for $\theta \in [0, \pi]$ and r^- defined for $\theta \in [\pi, 2\pi]$. The *Poincaré half-return maps* are defined by

$$\begin{aligned} \Pi^+(r_0) &= r^+(\pi, r_0), \\ \widetilde{\Pi}^-(r_0) &= r^-(-\pi, r_0), \end{aligned}$$

where $\widetilde{\Pi}^-$ denotes the inverse of Π^- since both $r^\pm$ are defined with initial condition $\theta = 0$ and $r_0 > 0$ sufficiently small. The *displacement function*, which is analytic, is given by

$$\Delta(r_0) = \widetilde{\Pi}^-(r_0) - \Pi^+(r_0) = \sum_{k=1}^{\infty} L_k r_0^k,$$

for r_0 small enough. When $\alpha^+\alpha^- \neq 0$ the origin is a hyperbolic equilibrium point. Otherwise $L_1 = 0$ and, for $k \geq 2$, we can define the k-th *Lyapunov constant* by $L_k \neq 0$, when $L_1 = \cdots = L_{k-1} = 0$. In this case, if there exists $k \geq 2$ so that $L_k \neq 0$, then the origin of system (8) is a weak focus of order k. Otherwise the origin is a center. For more details see for instance [16, 17]. Usually, to simplify computations we take $\alpha^+ = \alpha^- = 0$. Note that on the smooth case the first non-vanishing Lyapunov constant has always odd subscript while in the piecewise class this property does not hold. Recall that, for analytical vector fields, the classical *Hopf bifurcation* occurs when one limit cycle of small amplitude bifurcates from a weak focus of first order (with the above notation it occurs when $L_1 = L_2 = 0$ and $L_3 \neq 0$), while the limit cycles arise from a higher-order weak focus in the degenerate Hopf bifurcation (see [1] for more details). Moreover, for piecewise smooth vector fields, in [11] it is shown that one more limit cycle appears moving the equilibrium points on Σ. Because a sliding or escaping segment is created adding adequately some perturbative parameters. This is known as a pseudo-Hopf type bifurcation. Because in [22], this limit cycle bifurcation was called *pseudo-Hopf* near a fold-fold point and proved previously in [15]. For more details see [10, 19]. We notice that a weak focus of order k, generically, unfolds exactly k limit cycles. Note that when we deal with continuous or refractive perturbations we do have not pseudo-Hopf type bifurcations because, in these cases, we never have sliding or escaping segments on Σ.

As we deal with polynomial perturbations of a piecewise center, we can use the Implicit Function Theorem to obtain hyperbolic crossing limit cycles of small amplitude in a neighborhood of the origin of (8). In this case, like in the analytical one, when we perturb a center under the condition $\alpha^\pm = 0$, the expressions of L_k are polynomials that vanish when the perturbative parameters do. Therefore, we can compute the

Taylor series of $L_2, \dots, L_l$ with respect to the perturbative parameters. We denote by $L_i^{[1]}$, $i = 2, \dots, l$ their linear parts. Consequently, if the matrix $[L_2^{[1]}, \dots, L_l^{[1]}]$, with respect to the perturbative parameters, has rank $l-1$, as we have previously explained, adding the traces and the sliding or escaping segments we can get l small amplitude hyperbolic crossing limit cycles in a neighborhood of the origin. For more details see [17] and references therein.

We can also study bifurcations of small amplitude limit cycles for piecewise smooth vector fields using the Melnikov's method. It is also used to study global bifurcations that occur near one-parameter families of periodic orbits. In particular, the first Melnikov Function and the first-order of the Lyapunov constants are related and we know that if, after perturbing a center, the rank of the matrix defined by the coefficients of $[L_2^{[1]}, \dots, L_m^{[1]}]$, with respect to the parameters, is $l-1$, where $m > l$, then there exist l hyperbolic crossing limit cycles bifurcating from this center, when we also use the trace and the sliding parameters. For more details, see [18].

3. Piecewise linear vector fields on invariant spheres

In this section, we study piecewise linear vector fields defined on invariant spheres. We prove Theorem 1 and we provide all phase portraits of piecewise smooth vector fields $Y \in \mathcal{X}$ when we restrict on a invariant sphere. From the complete analysis of the phase portraits we can have a more complete result, Proposition 5, that shows the nonexistence of other type of limit cycles on the invariant spheres. In fact, Theorem 1 can be also thought as a corollary of it. Of course, the nonexistence of limit cycles in spheres, by the homogeneity, proves immediately the nonexistence of any kind of isolated invariant cones.

At first we summarize some results about smooth vector fields presented in [6] that we use in what follows.

Lemma 3. *Let $X \in \mathfrak{X}_1$. The following statements are true.*

(a) If $p \in \mathbb{S}^2_\rho = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = \rho^2\}$ is an equilibrium point of system (2)*, then p is a center.*

(b) Any $X \in \mathfrak{X}_1$ is completely integrable with the second first integral

$$\widetilde{H}(x,y,z) = a_3x - a_2y + a_1z. \tag{9}$$

(c) $X \in \mathfrak{X}_1$ has only two equilibrium points of center type on each sphere $\mathbb{S}^2_\rho$ which are antipodal of each other.

(d) The equilibrium points of (2) *are $(0,0,\rho)$ if, and only if, $a_2 = a_3 = 0$. In this case, the second first integral of* (2)*, given by* (9)*, is of the form $\widetilde{H}(x,y,z) = a_1z$.*

(e) Suppose that $a_3 \neq 0$. Then the equilibrium points of (2) *are of the form $\{(x,y,z) \in \mathbb{R}^3 : y = -(a_2/a_3)x, z = (a_1/a_3)x\}$.*

(f) Suppose that $a_2 \neq 0$. Then the equilibrium points of (2) *are of the form $\{(x,y,z) \in \mathbb{R}^3 : x = -(a_3/a_2)y, z = -(a_1/a_2)y\}$.*

(g) System (2) *is invariant by the change of coordinates $(x,y,z,t) \mapsto (-x,-y,-z,t)$.*

(h) The phase portrait of any $X \in \mathfrak{X}_1$ on $\mathbb{S}^2_\rho$, with $\rho > 0$, is topologically equivalent to the one on $\mathbb{S}^2_1$.

So, (2) has (generically) only a line of equilibrium points passing through the origin. As we observed in the introduction, by Lemma 3, we conclude that the 3-dimensional smooth vector field (2) has a continuous of invariant cones fulfilled of non-isolated closed trajectories. One of these cones is illustrated in Figure 1.

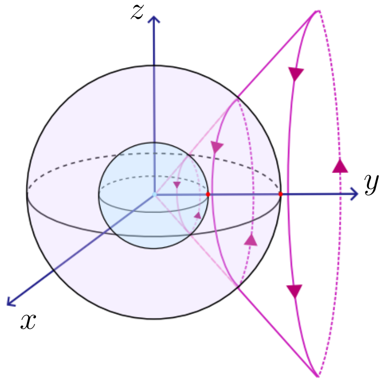

FIGURE 1. Invariant cone of the linear vector field $X(x,y,z) = (z,0,-x) \in \mathfrak{X}_1$.

Now we consider the class of 3-dimensional piecewise smooth vector fields $Y = (X^+, X^-) \in \mathcal{X}_1$ given by (3), with separation set $\Sigma = \{(x,y,z) \in \mathbb{R}^3 : z = 0\}$. Observe that, when we restrict our study to an invariant sphere $\mathbb{S}^2_\rho$ we deal with a piecewise smooth vector field defined on $\mathbb{S}^2_\rho$ with separation set $\{(x,y,z) \in \mathbb{S}^2_\rho : z = 0\}$. Sure that there will be no doubt, to simplify the notation we will continue calling the separation set and the vector fields Y and $X^\pm$ restricted to the sphere $\mathbb{S}^2_\rho$ by Σ, Y, and $X^\pm$.

Firstly, we use Lemma 3 to analyze the possible positions of the equilibrium points of (4) with respect to the separation set Σ. By Lemma 3(d), the equilibria of the linear systems $X^\pm$, defined by (4), are $(0,0,\rho)$ if, and only if, $a_2^\pm = a_3^\pm = 0$. Note that the item (d) of Lemma 3 also implies that Σ is invariant by the flow of $X^\pm$ and that $(0,0,\pm\rho)$ are the unique equilibria of $X^\pm$ on each sphere when $a_2^\pm = a_3^\pm = 0$. So, on the following we assume that $(a_2^\pm)^2 + (a_3^\pm)^2 \neq 0$. We do all the calculations assuming that $a_3^\pm \neq 0$, the case $a_2^\pm \neq 0$ is analogous. Under this condition, Lemma 3(e) implies that the equilibria of $X^\pm$ are of the form $\{(x,y,z) \in \mathbb{R}^3 : y = -(a_2^\pm/a_3^\pm)x, z = (a_1^\pm/a_3^\pm)x\}$. Hence, the equilibria of $X^\pm$ are on the separation set Σ if, and only if, $a_1^\pm = 0$. Moreover, by Lemma 3(c), both $X^\pm$ have two equilibrium points of center type on each sphere. So, if $a_1^\pm \neq 0$ we conclude that the vector field $X^\pm$ has one admissible and one non-admissible equilibrium point.

Following the approach introduced in [27], we use the first integrals $H(x,y,z) = x^2 + y^2 + z^2$ and $\widetilde{H}^\pm(x,y,z)$, given by (9), of the linear vector fields (4) to calculate a difference map, on Σ, defined below. With this map we can analyze and describe the behavior of the levels curves of (9) on $\mathbb{S}^2_\rho$ and how these levels interact with the separation set Σ. It allows us to know the behavior of the trajectories of (4) on each sphere $\mathbb{S}^2_\rho$, and, in particular, see if any system (3) admits crossing limit cycles on $\mathbb{S}^2_\rho$.

Lemma 4. *No piecewise differential system $Y \in \mathcal{X}_1$, given by (3), admits crossing limit cycles restricting the dynamics on each fixed sphere $\mathbb{S}^2_\rho$, with $\rho > 0$.*

Proof. As we saw before, $a_2^\pm = a_3^\pm = 0$ implies that Σ is invariant by the flow of (4). Therefore, in this case we cannot define a difference map using (9). Then, on the following, we assume that $(a_2^\pm)^2 + (a_3^\pm)^2 \neq 0$. We do all the calculations assuming that $a_3^\pm \neq 0$. The case $a_2^\pm \neq 0$ is analogous.

Let $p=(x_0,y_0,0)\in\Sigma\cap\mathbb{S}^2_\rho$. Then, there exist $k^\pm$ such that $\widetilde{H}^\pm(p)=k^\pm$. The half-return maps $\pi^\pm(p)=q^\pm=(x_1^\pm,y_1^\pm,0)$ satisfy

$$\begin{aligned} H(q^\pm)&=\rho^2,\\ \widetilde{H}^+(q^+)&=a_3^+x_1^+-a_2^+y_1^+=k^+,\\ \widetilde{H}^-(q^-)&=a_3^-x_1^--a_2^-y_1^-=k^-. \end{aligned}$$

Solving the systems of equations

$$\{H(q^+)=\rho^2,\widetilde{H}^+(q^+)=k^+\},\{H(q^-)=\rho^2,\ \widetilde{H}^-(q^-)=k^-\}$$

we obtain the solutions

$$q^\pm=\left(-\frac{((a_2^\pm)^2-(a_3^\pm)^2)x_0+2a_2^\pm a_3^\pm y_0}{(a_2^\pm)^2+(a_3^\pm)^2},\frac{((a_2^\pm)^2-(a_3^\pm)^2)y_0-2a_2^\pm a_3^\pm x_0}{(a_2^\pm)^2+(a_3^\pm)^2},0\right).$$

So, the difference map, $d(p)=\pi^+(p)-\pi^-(p):\Sigma\to\mathbb{R}$, is such that

$$\begin{aligned} d(p)=&\big(2(a_2^-a_3^+-a_3^-a_2^+)((a_2^-a_3^++a_2^+a_3^-)x_0-(a_2^-a_2^+-a_3^-a_3^+)y_0),\\ &-2(a_2^-a_3^+-a_3^-a_2^+)((a_2^-a_2^+-a_3^-a_3^+)x_0+(a_2^-a_3^++a_2^+a_3^-)y_0),0\big). \end{aligned}$$

Consequently, it is identically zero if, and only if, $a_2^+a_3^-=a_2^-a_3^+$. Hence, either all the crossing trajectories of (3), on $\mathbb{S}^2_\rho$, are closed or none of them are, which concludes the proof. □

Proof of Theorem 1. It is a direct consequence of the fact that system (3) does not admit crossing periodic orbits, by Lemmas 3(a) and 4. □

The remainder of this section is devoted to describing the behavior of any piecewise smooth vector field (3) restricted to the sphere of radius ρ, that is $\mathbb{S}^2_\rho$. We show also that no piecewise differential system $Y\in\mathcal{X}_1$, given by (3), admits sliding limit cycles on each fixed sphere $\mathbb{S}^2_\rho$, with $\rho>0$. Moreover, we provide the possible phase portraits of $Y\in\mathcal{X}_1$ and, consequently, we will prove the following result.

Proposition 5. *A piecewise differential system $Y\in\mathcal{X}_1$, given by (3), does not admit neither a crossing nor a limit cycle on each fixed invariant sphere $\mathbb{S}^2_\rho$, with $\rho>0$.*

We start studying the behavior of the tangency lines of (4), under the condition $(a_2^\pm)^2+(a_3^\pm)^2\neq0$, assuming that $a_3\neq0$.

Lemma 6. *The tangency lines of (4) are given by*

$$S_{X^\pm}=\left\{(x,y,z)\in\mathbb{R}^3:y=-(a_2^\pm/a_3^\pm)x,z=0\right\}.$$

Moreover, these tangency lines intersect the sphere $\mathbb{S}^2_\rho$ at the points

$$\left\{x=-\rho a_3^\pm/\sqrt{(a_2^\pm)^2+(a_3^\pm)^2},\ y=\ \rho a_2^\pm/\sqrt{(a_2^\pm)^2+(a_3^\pm)^2}\right\}$$

and its antipodal. Then, (3) has two fold points on $\mathbb{S}^2_\rho$, for all $\rho\in\mathbb{R}$, $\rho\neq0$. Besides, one of these tangency points is visible and the other one is invisible unless that $a_1^\pm=0$. Finally, when $S_{X^+}=S_{X^-}$ we have two fold-fold points of $Y=(X^+,X^-)$ on each sphere and it occurs if, and only if, $a_2^+a_3^-=a_2^-a_3^+$.

Proof. Let $f:\mathbb{R}^3\to\mathbb{R}$ given by $f(x,y,z)=z$. So, $\Sigma=\{(x,y,z)\in\mathbb{R}^3;z=0\}=f^{-1}(0)$. Thus, $X^\pm f=X^\pm\cdot\nabla f=a_2^\pm x+a_3^\pm y$ and the first part of the result follows. With straightforward computations we prove the other statements. □

It is important to note that the symmetry of the problem guarantees that if one equilibrium point of X^+ or X^- remains on Σ, then so does the other. Moreover, if an equilibrium point of X^+ coincides with a tangency or an equilibrium point of X^- the other one also coincides. Besides, the change of coordinates $(x, y, z) \mapsto (x, y, -z)$ allows us to change the behavior of the southern and northern hemispheres and then we can fix the behavior in one of them in the next analysis.

As we saw in Section 2, we can define the projected vector field associated to (3), on the sphere $\mathbb{S}^2_\rho$, using (7). It is of the form

$$\mathcal{P}_Y(u, v) = \begin{cases} \mathcal{P}_{X^+}(u, v),\ v \geq 0, \\ \mathcal{P}_{X^-}(u, v),\ v \leq 0, \end{cases}$$

where,

$$\begin{aligned}\mathcal{P}_{X^\pm}(u, v) =& \big(-4\rho^2 a_1^\pm - 4\rho a_2^\pm v - a_1^\pm u^2 + 2a_3^\pm uv + a_1^\pm v^2, \\ & 4\rho^2 a_3^\pm + 4\rho a_2^\pm u - a_3^\pm u^2 - 2a_1^\pm uv + a_3^\pm v^2\big).\end{aligned}$$

The projected Filippov vector field is 1-dimensional and it is well defined at the points $(u, 0)$ such that $(\mathcal{P}_{X^+} f)(\mathcal{P}_{X^-} f)(u) = (4\rho^2 a_3^+ + 4\rho a_2^+ u - a_3^+ u^2)(4\rho^2 a_3^- + 4\rho a_2^- u - a_3^- u^2) < 0$. In this case, we have

$$F_Y(u) = \frac{(4\rho^2 + u^2)\big((a_1^- a_3^+ - a_1^+ a_3^-)u^2 + 4(a_1^+ a_2^- - a_1^- a_2^+)u\rho + 4(a_1^+ a_3^- - a_1^- a_3^+)\rho^2\big)}{(a_3^- - a_3^+)u^2 + 4\rho(a_2^+ - a_2^-)u + 4\rho^2(a_3^+ - a_3^-)}. \tag{10}$$

Now, we summarize the key points of the proof of Proposition 5, providing after the necessary technical lemmas.

Using the two first integrals of $X^\pm$ we prove, in Lemma 7, that there exist only 10 possible behaviors for the levels curves of (3) on each sphere $\mathbb{S}^2_\rho$, concerning the admissibility of equilibrium points of $Y \in \mathcal{X}_1$, which are the ones in Figure 2, where we draw the tangency points as blue dots, the equilibrium points on Σ as red dots and when tangency and equilibrium points coincide we draw a pink dot. After that, we study the behavior of the Filippov vector field, F_Y, given by (10). Note that, (10) is not defined when Σ is a trajectory of X^+ or X^- on $\mathbb{S}^2_\rho$. In Lemma 9, we conclude that if the equilibrium points of both $X^\pm$ stay on Σ ($a_1^+ = a_1^- = 0$) or if there exists $\lambda \in \mathbb{R}$ such that $X^+ = \lambda X^-$, then the Filippov vector field (10) is identically zero. In Lemma 11, we prove that (10) has two symmetric equilibrium points r_1 and r_2 if, and only if, $a_1^+ a_3^- - a_1^- a_3^+ \neq 0$, $a_1^+ a_1^- < 0$ and $a_2^+ a_3^- - a_2^- a_3^+ \neq 0$. In this case, the equilibrium points r_1 and r_2 have the same (1-dimensional) stability and they are stable (resp. unstable) if $(a_2^+ a_3^- - a_2^- a_3^+)(a_1^+ - a_1^-) > 0$ (resp. < 0). In addition, (10) can have isolated equilibrium points only when the sliding and escaping segments are delimited by two tangency points of the same type otherwise both vector fields X^+ and X^- point towards the same direction on Σ. Moreover, (10) does not have isolated equilibrium points when $Y \in \mathcal{X}_1$ has fold-fold points or the equilibrium points of X^+ or X^- stay on Σ.

Now, changing the time orientation of the piecewise smooth vector field (3), if it is necessary, we can fix an orientation for the vector field X^- in $\Sigma^- = \{(x, y, z) \in \mathbb{S}^2_\rho :\ z \leq 0\}$ and choose between two different ones for X^+ in $\Sigma^+ = \{(x, y, z) \in \mathbb{S}^2_\rho :\ z \geq 0\}$. Doing this, in Figure 2 we draw the possible phase portraits for system (3) on the sphere $\mathbb{S}^2_\rho$, with respect to the admissibility of equilibrium points of $Y \in \mathcal{X}_1$, which are the ones in Figures 3 and 4, where we represent Σ by a gray segment when the Filippov's convention does not apply, by a blue segment, in Σ, the escaping region, by a pink one the sliding region, by a black one the crossing region and we use the same colors used in Figure 2 for tangency and equilibrium points on Σ. Note that, in Figures 3 and 4 we do not distinguish the cases in which it is possible to have connections (see Remark 8), because it will not be necessary to conclude the proof of Proposition 5.

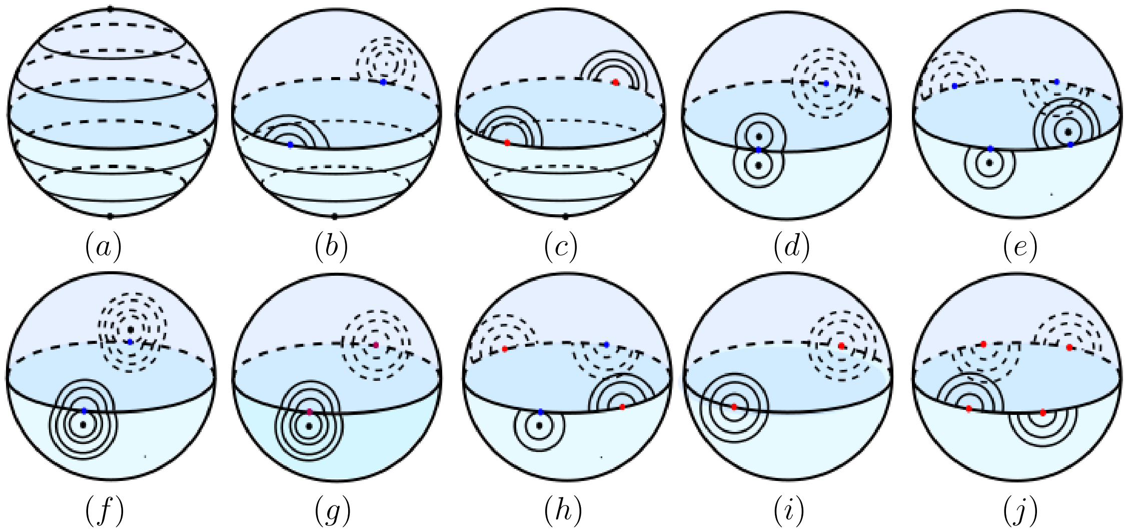

FIGURE 2. Invariant curves of $Y \in \mathcal{X}_1$.

Joining the information about the positions of the equilibrium points on each sphere $\mathbb{S}^2_\rho$ with the property of the difference map detailed on Lemma 4, we can classify the possible behavior of the invariant curves of piecewise smooth vector fields (3) on $\mathbb{S}^2_\rho$, with respect to the admissibility of its equilibrium points.

Lemma 7. *With respect to the admissibility of equilibrium points, the behavior of the level curves of* (3) *on each fixed sphere* $\mathbb{S}^2_\rho$*,* $\rho > 0$*, are shown in Figure 2.*

Proof. Let $Y = (X^+, X^-) \in \mathcal{X}_1$. Denote by $p_i^\pm$ and $q_i^\pm$, with $i = 1, 2$ the equilibrium and the tangency points of $X^\pm$, respectively. On the following we assume that $p_1^\pm = (x_0, y_0, z_0)$ is such that $z_0 \geq 0$ and $p_2^\pm = (x_0, y_0, z_0)$ is such that $z_0 \leq 0$. We show the possible behaviors of the level curves of $Y \in \mathcal{X}_1$ on $\mathbb{S}^2_\rho$ in Figure 2, where we represent the tangency points by blue dots, the equilibrium points on Σ by red dots, and when tangency points and equilibrium points coincide we represent it by a pink dot. We divide the analysis into three cases depending on the position of the equilibrium points of X^- on $\mathbb{S}^2_\rho$.

Firstly, if $p_1^- = (0, 0, \rho)$ and $p_2^- = -p_1^-$ the trajectories of X^- on $\mathbb{S}^2_\rho$ are parallel to Σ. The same property holds when $p_1^+ = (0, 0, \rho)$ and $p_2^+ = -p_1^+$. If $p_i^+ = (x_0, y_0, z_0)$, $i = 1, 2$, are such that $z_0 \neq \pm\rho$ and $z_0 \neq 0$, then we have one admissible and one non-admissible center for X^+ on $\mathbb{S}^2_\rho$ and therefore, two tangency points, $q_i^+ \in \Sigma$, $i = 1, 2$, one visible and one invisible. Finally, if $p_i^+ = (x_0, y_0, 0)$, $i = 1, 2$, both equilibrium points of X^+ are on Σ. We draw the invariant curves of these cases in Figure 2 $(a)-(c)$.

Now we consider the case where $p_i^- = (x_0, y_0, z_0)$, $i = 1, 2$, with $z_0 \neq 0$ and $z_0 \neq \rho$. Then, we have one admissible and one non-admissible center for X^- on $\mathbb{S}^2_\rho$ and therefore, two tangency points, $q_i^- \in \Sigma$, $i = 1, 2$, one visible and one invisible, respectively. Here, as we have already considered the case where the trajectories of X^- are parallel

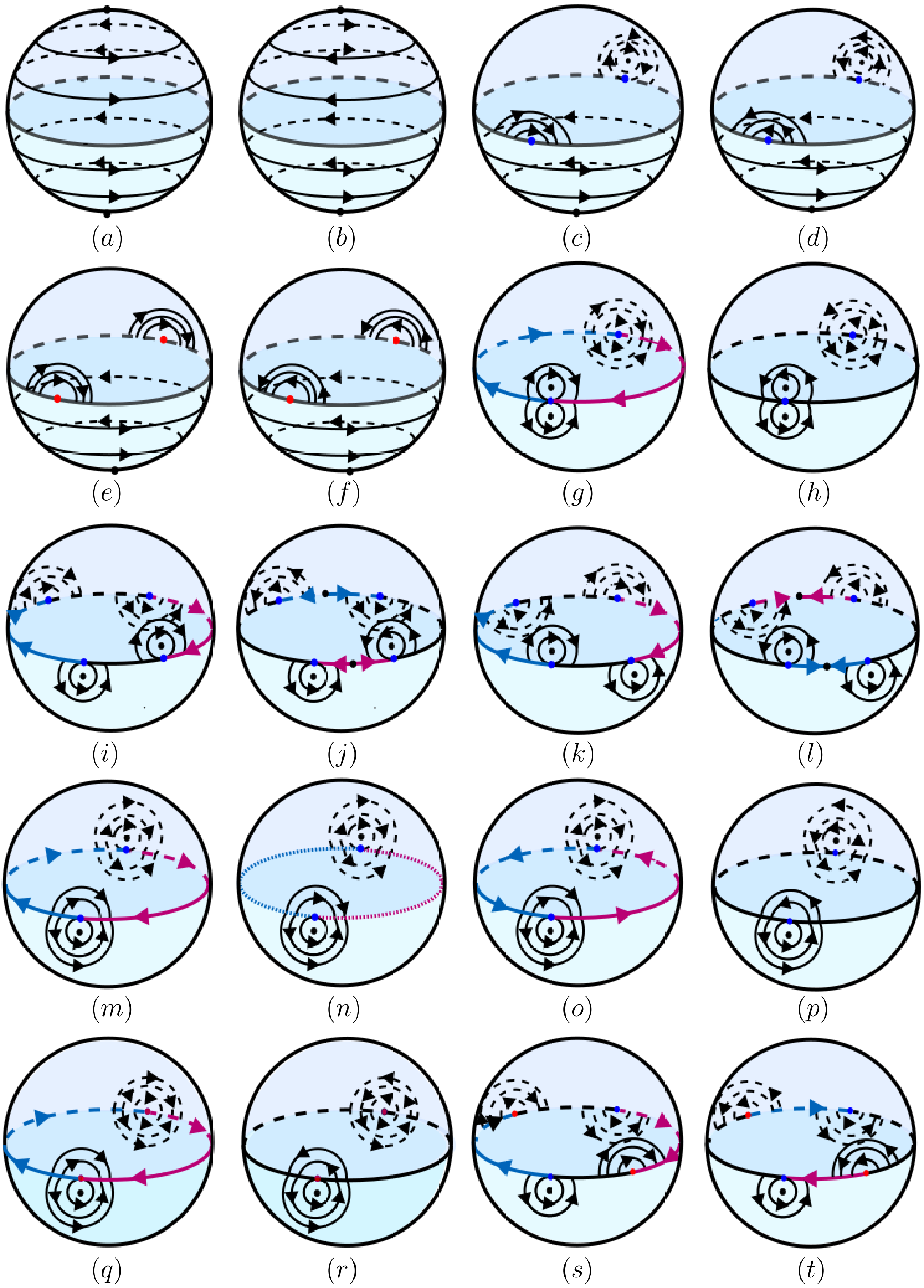

FIGURE 3. Possible phase portraits of $Y \in \mathcal{X}_1$.

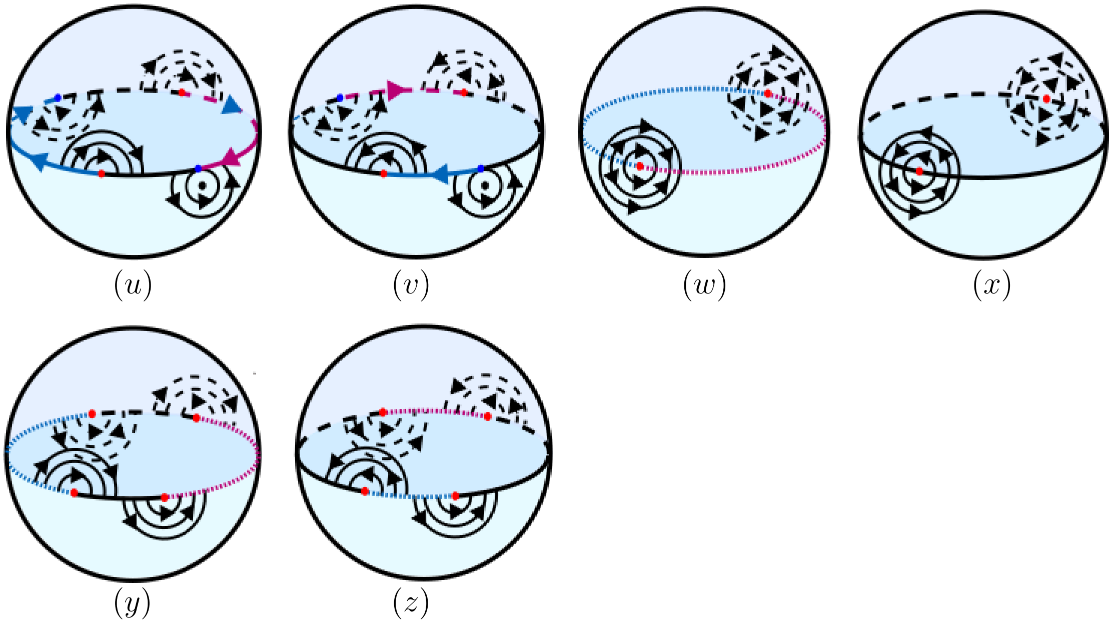

FIGURE 4. Possible phase portraits of $Y \in \mathcal{X}_1$.

to Σ, using the change of coordinates $(x, y, z) \mapsto (x, y, -z)$ explained before, we only need to consider the following two behaviors of X^+ on $\mathbb{S}^2_\rho$. If $p_i^+ = (x_0, y_0, z_0)$, $i = 1, 2$, are such that $z_0 \neq \rho$ and $z_0 \neq 0$, we have one admissible and one non-admissible center for X^+ and therefore, two tangency points $q_i^+ \in \Sigma$, $i = 1, 2$, one visible and one invisible, respectively. Hence, we have three new global behaviors depending on the relative position of $q_i^\pm$, $i = 1, 2$ that occur when $q_1^+ = q_1^-$ and $q_2^+ = q_2^-$, when they do not coincide and when $q_1^+ = q_2^-$ and $q_2^+ = q_1^-$. Finally, if $p_i^+ = (x_0, y_0, 0)$, $i = 1, 2$, the two equilibrium points of X^+ are on Σ and we have two new global behaviors depending on the positions of these equilibrium points, that is $p_i^+ = q_i^-$ or $p_i^+ \neq q_i^-$, $i = 1, 2$. We show the invariant curves of these cases in Figure 2 $(d) - (h)$.

We finish the analysis considering the case where the two centers of X^- on $\mathbb{S}^2_\rho$ are on Σ, it means that $p_i^- = (x_0, y_0, 0)$, $i = 1, 2$. Using the change of coordinates $(x, y, z) \mapsto (x, y, -z)$, we can restrict to the case in which the two equilibrium points of X^+, p_i^+ for $i = 1, 2$, are also on Σ. Here we have two new global behaviors depending on the positions of these equilibrium points: $p_i^+ = p_i^-$ or $p_i^+ \neq p_i^-$. We draw the invariant curves of these cases in Figure 2 $(i) - (j)$. □

Remark 8. *Note that the tangency points of $X^\pm$ are antipodal of each other. Therefore, the tangency lines $S_{X^\pm}$ of $X^\pm$ are contained in the plane $\{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ and pass through the origin. Observe that when these tangency lines are perpendicular $Y \in \mathcal{X}_1$ admits a tangential connection. It occurs because the trajectories of $X^\pm$ are restricted to the level curves of (9), on $\mathbb{S}^2_\rho$. Thus, depending on the relative position of the tangency lines, the behavior illustrated in the cases (e), (h), and (j) of Figure 2 are not unique. But for our purpose we do not need to distinguish the cases in which there are or not separatrix connections.*

As in the above analysis, we only have considered the level curves of $X^\pm$ we have not taken into account the behavior of the Filippov vector field. On the following lemmas we describe the behavior of it using the projected Filippov vector field (10) associated to (3) restricted to the sphere $\mathbb{S}^2_\rho$, because it is 1-dimensional.

Lemma 9. *The Filippov vector field* (10) *is well defined when* $(\mathcal{P}_{X^+}f)(\mathcal{P}_{X^-}f) < 0$. *In this case, it is identically zero if, and only if,* $a_1^- a_3^+ - a_1^+ a_3^- = 0$ *and* $a_1^- a_2^+ - a_1^+ a_2^- = 0$.

Proof. It follows since (10) is a rational function and its numerator is identically zero if, and only if, $(a_1^- a_3^+ - a_1^+ a_3^-)u^2 + 4\rho(a_1^+ a_2^- - a_1^- a_2^+)u + 4\rho^2(a_1^+ a_3^- - a_1^- a_3^+) \equiv 0$. □

Remark 10. *Firstly, we note that* $a_2^+ = a_2^- = 0$ *and* $a_3^+ = a_3^- = 0$ *imply that* $a_1^- a_3^+ - a_1^+ a_3^- = 0$ *and* $a_1^- a_2^+ - a_1^+ a_2^- = 0$. *But, in this case,* $(\mathcal{P}_{X^+}f)(\mathcal{P}_{X^-}f)$ *is identically zero and then* (10) *is not defined for these values of the coefficients. In addition,* $a_1^+ = a_1^- = 0$ *implies that* $a_1^- a_3^+ - a_1^+ a_3^- = 0$ *and* $a_1^- a_2^+ - a_1^+ a_2^- = 0$ *and then* (10) *vanishes identically when the equilibrium points of both* X^+ *and* X^- *are on* Σ. *Finally, when* $(a_1^+)^2 + (a_1^-)^2 \neq 0$, *the conditions* $a_1^- a_3^+ - a_1^+ a_3^- = 0$ *and* $a_1^- a_2^+ - a_1^+ a_2^- = 0$ *imply that* $a_2^- a_3^+ - a_3^- a_2^+ = 0$ *and then* X^+ *and* X^- *are multiple of each other, which means that the equilibrium points and tangency lines of* X^+ *and* X^- *coincide. So, we have that* (10) *is identically zero if, and only if, the equilibrium points of both* $X^\pm$ *are on* Σ *or if* X^+ *and* X^- *are multiple of each other.*

Note that when $a_1^+ a_3^- - a_1^- a_3^+ \neq 0$, the projected Filippov vector field (10) can have at most two real roots given by

$$r_{1,2} = \frac{2\rho\big((a_1^+ a_2^- - a_1^- a_2^+) \pm \sqrt{(a_1^+ a_2^- - a_1^- a_2^+)^2 + (a_1^+ a_3^- - a_1^- a_3^+)^2}\big)}{a_1^+ a_3^- - a_1^- a_3^+} \tag{11}$$

if $(\mathcal{P}_{X^+}f)(\mathcal{P}_{X^-}f)(r_{12}) < 0$. In addition, $(\mathcal{P}_{X^+}f)(\mathcal{P}_{X^-}f)(r_{12}) > 0$ means that X^+ and X^- are parallel at a crossing point and then (10) is not defined at this point. On the other hand, when $a_1^+ a_3^- - a_1^- a_3^+ = 0$ the projected Filippov vector field has a unique possible real root at the origin. The symmetry of the problem ensures that the other root is situated at infinity. We can avoid this and suppose that $(0,0)$ is not an equilibrium point of (10) making the same orthogonal change of coordinates in X^+ and X^- that put $(0,\rho,0)$ in (x_0,y_0,z_0) with $y_0 \neq \rho$, as it was done previously. So, without loss of generality, we only analyze the case $a_1^+ a_3^- - a_1^- a_3^+ \neq 0$ and study the stability of the equilibrium points of the Filippov vector field, when it is well defined.

Lemma 11. *The Filippov vector field* (10) *is well defined when* $(\mathcal{P}_{X^+}f)(\mathcal{P}_{X^-}f) < 0$. *In this case, it has two symmetric equilibrium points* r_1 *and* r_2 *defined in* (11) *if, and only if,* $a_1^+ a_3^- - a_1^- a_3^+ \neq 0$, $a_1^+ a_1^- < 0$ *and* $a_2^+ a_3^- - a_2^- a_3^+ \neq 0$. *The equilibrium points* r_1 *and* r_2 *have the same (*1*-dimensional) stability. Moreover, they are stable (resp. unstable) if* $(a_2^+ a_3^- - a_2^- a_3^+)(a_1^+ - a_1^-) > 0$ *(resp.* < 0*).*

Proof. As we saw before, when $a_1^+ a_3^- - a_1^- a_3^+ \neq 0$, the Filippov vector field can have at most two real roots r_1 and r_2 given in (11). Note that

$$(\mathcal{P}_{X^+}f)(\mathcal{P}_{X^-}f)(r_{1,2}) = \frac{64\rho^4 a_1^+ a_1^- (a_2^+ a_3^- - a_2^- a_3^+)^2}{(a_1^+ a_3^- - a_1^- a_3^+)^4} \left(\sqrt{(a_1^+ a_2^- - a_1^- a_2^+)^2 + (a_1^+ a_3^- - a_1^- a_3^+)^2} \pm (a_1^+ a_2^- - a_1^- a_2^+)\right)^2.$$

As we are assuming $a_1^+ a_3^- - a_1^- a_3^+ \neq 0$, then $\sqrt{(a_1^+ a_2^- - a_1^- a_2^+)^2 + (a_1^+ a_3^- - a_1^- a_3^+)^2} \pm (a_1^+ a_2^- - a_1^- a_2^+)$ are always different from zero. Consequently, the projected vector field (10) is defined at r_{12} if, and only if, $a_1^+ a_1^- < 0$ and $a_2^+ a_3^- - a_2^- a_3^+ \neq 0$. It means that, when the equilibrium points of both X^+ and X^- are on Σ and when the tangential points of X^+ and X^- coincide, the projected Filippov vector field does not have isolated equilibrium points.

When $a_2^+a_3^- - a_2^-a_3^+ \neq 0$ and $a_1^+a_1^- < 0$ we study the stability of these equilibrium points. As $4\rho^2 + u^2$ is a positive factor of (10), we can study the stability of the equilibrium points of $F_Y(u)/(4\rho^2 + u^2)$. In this case, the derivative with respect to u is nonvanishing for all u, because it is

$$-\frac{4\rho(4\rho^2 + u^2)(a_2^+a_3^- - a_2^-a_3^+)(a_1^+ - a_1^-)}{((a_3^- - a_3^+)u^2 + 4\rho(a_2^+ - a_2^-)u + 4\rho^2(a_3^+ - a_3^-))^2}.$$

Thus, r_1 and r_2 have the same stability which depends on the sign of $(a_2^+a_3^- - a_2^-a_3^+)(a_1^+ - a_1^-)$. □

Remark 12. *As we saw above,* (10) *is not defined when* Σ *is a trajectory of* X^+ *or* X^-*. Moreover* (10) *does not have isolated equilibrium points neither when it has fold-fold points nor when the equilibrium points of* X^+ *or* X^- *stay on* Σ*. Besides this, the Filippov vector field* (10) *can have equilibrium points only when the sliding and escaping segments are delimited by two tangency points of the same type, otherwise both vector fields* X^+ *and* X^- *point on the same direction.*

Now, changing the time orientation of the piecewise smooth vector field (3), if it is necessary, we can fix a time orientation for the vector field X^- and choose two different ones for X^+ on $\mathbb{S}^2_\rho$. Hence, when we add a time orientation in Figure 2 we obtain the possible behaviors for system (3), that are depicted in Figures 3 and 4. Note that Figures 3 and 4 do not take into account connections of (3). These elements do not influence in the existence of limit cycles. Moreover, the non-existence of limit cycles in $\mathbb{S}^2_\rho$ is not related to the existence of connections. This is due to the arrangement of tangency points, admissible and non-admissible equilibrium points, and, as (3) is completely integrable, the difference map does not have isolated zeros.

With this analysis we conclude that system (3) has neither limit cycles nor crossing limit cycles on the spheres $\mathbb{S}^2_\rho$, with $\rho > 0$. So, the proof of Proposition 5 follows.

4. Centers and limit cycles for piecewise continuous quadratic homogeneous vector fields

In this section, inspired by the homogeneity property of linear vector fields with invariant spheres, we study the center-focus problem for piecewise quadratic homogeneous vector fields in $\mathcal{X}_2^H$. Because of the difficulty of the problem, we restrict our attention to the class of continuous homogeneous vector fields and give some families of centers in Proposition 15. Even with this restriction, in Proposition 17 we exhibit a system in $\mathcal{X}_2^H$ with a weak focus of third-order at the point $(0, 1, 0)$ from which 2 small amplitude crossing limit cycles bifurcate on $\mathbb{S}^2_1$ with a continuous perturbation in $\mathcal{X}_2^H$. Note that with a continuous perturbation, we cannot produce a sliding segment and then it is natural that we do not reach the maximum upper bound for the number of small amplitude limit cycles that can bifurcate from a generic weak focus of third-order. Moreover, in this section we only consider the perturbation in $\mathcal{X}_2^H$ and, in the next section we deal with a general quadratic perturbation in $\mathcal{X}_2$.

On the following, we recall some assumptions given in [6], for a quadratic homogeneous vector field $X \in \mathfrak{X}_2^H$. Firstly, doing an orthogonal change of coordinates we can assume, without loss of generality, that $(0, \rho, 0) \in \mathbb{S}^2_\rho$ is an equilibrium point of

$X \in \mathfrak{X}_2^H$. With this assumption, we can write X in the form

$$\begin{aligned} \dot{x} &= -a_4xy - a_5xz - (a_6 + a_7)yz - a_8z^2, \\ \dot{y} &= a_4x^2 + a_6xz - a_9z^2, \\ \dot{z} &= a_5x^2 + a_7xy + a_8xz + a_9yz. \end{aligned} \tag{12}$$

Observe that the equilibrium point $(0, \rho, 0)$ of (12) is located at the origin after projection and let J be the Jacobian matrix associated to the projected vector field $\mathcal{P}_X$ at the origin. Therefore, $(0, \rho, 0)$ is of nondegenerate center-focus type if, and only if, the trace of J is zero and its determinant is positive. A straightforward computation shows that it occurs if, and only if, $a_4 - a_9 = 0$ and $a_6a_7 + a_7^2 - a_9^2 > 0$. We also assume $a_7 \neq 0$, otherwise $a_6a_7 + a_7^2 - a_9^2 = -a_9^2 \leq 0$. Hence, with these assumptions $(0, \rho, 0)$ is a weak focus of (12). Doing $w^2 = a_6a_7 + a_7^2 - a_9^2$, and $\varrho = (w^2 + a_4a_9)/a_7$ the projected system $\mathcal{P}_X$ is of the form:

$$\begin{aligned} \dot{u} &= -4a_4u - 4\varrho v - 4a_5uv - 4a_8v^2 - a_4u^3 - (\varrho - 2a_7)u^2v + (a_4 + 2a_9)uv^2 + \varrho v^3, \\ \dot{v} &= 4a_7u + 4a_9v + 4a_5u^2 + 4a_8uv - a_7u^3 - (2a_4 + a_9)u^2v - (2\varrho - a_7)uv^2 + a_9v^3. \end{aligned} \tag{13}$$

Now, the trace and the determinant of J are $-4(a_4 - a_9)$ and $16w^2$, respectively. The next theorem was proved in [6] and gives the conditions to have a center of (12) at the point $(0, 1, 0)$, on the sphere $\mathbb{S}_1^2$.

Theorem 13. [6] *The equilibrium point* $(0, 1, 0)$ *of system* (12) *is a nondegenerate center if, and only if,* $a_7 \neq 0$, $a_4 = a_9$, *and* $a_4a_5a_8a_9 + a_5a_6a_7a_8 + a_5^2a_7a_9 + a_5a_8a_9^2 - a_7a_8^2a_9 = 0$.

Next we will show an important difference between polynomial homogeneous vector fields defined on the sphere $\mathbb{S}_1^2$ and on the plane. Also in our special case that the dynamics is restricted on a invariant sphere. Firstly, we recall that a planar quadratic homogeneous vector field does not have limit cycles. The following example shows a quadratic homogeneous vector field $X \in \mathfrak{X}_2^H$ which has at least one limit cycle on the sphere $\mathbb{S}_1^2$. It occurs because the projected vector field (13) is a planar cubic non-homogeneous vector field. Fore more results about quadratic homogeneous vector fields defined on invariant spheres we refer the reader to [25, 26]. As in the previous section, this limit cycle forces the existence of an invariant cone fulfilled of periodic orbits for (12).

Proposition 14. *The quadratic homogeneous vector field* (12) *has at least one limit cycle bifurcating from* $(0, 1, 0)$ *on the sphere* $\mathbb{S}_1^2$.

Proof. Consider the quadratic homogeneous vector field (12) and its projection (13) with the parameters values $(a_4, a_5, a_7, a_8, a_9, w) = (1 + \varepsilon, 1, 1, 0, 1, 1)$. Note that with these values, (13) writes as the following cubic vector field

$$\begin{aligned} \dot{u} &= (-4 + \varepsilon)u - 4(2 + \varepsilon)v - 4uv - (1 + \varepsilon)u^3 + \varepsilon u^2v + (3 + \varepsilon)uv^2 + (2 + \varepsilon)v^3, \\ \dot{v} &= 4u + 4v + 4u^2 - u^3 - (3 + \varepsilon)u^2v - (3 + \varepsilon)uv^2 + v^3. \end{aligned} \tag{14}$$

As we observe before, the origin is an equilibrium point of (14). Let J be the Jacobian matrix associated to (14) at the origin. As the trace of J is ε and its determinant is $16 + 12\varepsilon$, then the origin is a weak focus for $\varepsilon = 0$. Note that we can use the algorithm explained in Section 2.4 to calculate the Lyapunov constants of analytical vector fields assuming that $\mathcal{P}_{X^+}$ and $\mathcal{P}_{X^-}$ are both defined by (14), because it is a generalization of the algorithm presented in [1]. So, when $\varepsilon = 0$, we calculate the first Lyapunov constant of (14) being $L_3 = 4 \neq 0$. Thus, by the classical Hopf bifurcation, there exist values of ε for which (14) has one limit cycle bifurcating from the origin. $\square$

On the following we will focus our attention on the center-focus problem that appears naturally for the piecewise smooth system

$$Y(x,y,z) = \begin{cases} X^+(x,y,z), \; z \geq 0, \\ X^-(x,y,z), \; z \leq 0, \end{cases} \tag{15}$$

where we obtain $X^\pm$ doing $a_i = a_i^\pm$ in (12) and assuming that $p = (0,1,0) \in \Sigma = \{(x,y,z) \in \mathbb{R}^3 : \; z = 0\}$ is of center type for both X^+ and X^- on $\mathbb{S}^2_\rho$. Here, as we commented above, because of the number of free parameters, we also assume that the system (15) is continuous but not differentiable on the separation set Σ. Note that, system (15), and the projected associated systems $\mathcal{P}_Y = (\mathcal{P}_{X^+}, \mathcal{P}_{X^-})$, where $\mathcal{P}_{X^\pm}$ are obtained doing $a_i = a_i^\pm$ in (13), are continuous on its separation set if, and only if, $a_4^- = a_4^+$, $a_5^- = a_5^+$, and $a_7^- = a_7^+$. Consequently, on the following, we are assuming these conditions.

As we are interested in exhibiting some families of centers for this family of piecewise smooth vector fields we use the method explained in Section 2.4, to calculate the Lyapunov constants for the projected system $\mathcal{P}_Y$. To do that, we need to consider $\mathcal{P}_{X^\pm}$ in its Jordan canonical form.

Note that the change of coordinates $\{\mathrm{u} = v, \mathrm{v} = (cu + dv)/w\}$, where $c = 4a_7$ and $d = 4a_9$ put the linear part of (13) in its Jordan canonical form

$$\begin{aligned} \dot{\mathrm{u}} =& \mathrm{v} + \frac{a_9\,(a_5a_9 - a_7a_8)}{wa_7^2}\mathrm{u}^2 - \frac{(2a_5a_9 - a_7a_8)}{a_7^2}\mathrm{uv} + \frac{a_5 w}{a_7^2}\mathrm{v}^2 + \frac{wa_9}{2a_7^2}\mathrm{u}^3 \\ &+ \frac{(a_7^2 + a_9^2 - 2w^2)}{4a_7^2}\mathrm{u}^2\mathrm{v} - \frac{w^2}{4a_7^2}\mathrm{v}^3, \\ \dot{\mathrm{v}} =& -\mathrm{u} + \frac{(a_5a_9 - a_7a_8)(a_7^2 + a_9^2)}{a_7^2 w^2}\mathrm{u}^2 - \frac{(a_5a_7^2 + 2a_5a_9^2 - a_7a_8a_9)}{wa_7^2}\mathrm{uv} \\ &+ \frac{a_5a_9}{a_7^2}\mathrm{v}^2 + \frac{(a_7^2 + a_9^2)}{4a_7^2}\mathrm{u}^3 + \frac{(2a_7^2 + 2a_9^2 - w^2)}{4a_7^2}\mathrm{uv}^2 - \frac{wa_9}{2a_7^2}\mathrm{v}^3. \end{aligned} \tag{16}$$

Moreover, the change of coordinates $\{\mathrm{u} = v, \mathrm{v} = (cu + dv)/w^\pm\}$ puts $\mathcal{P}_{X^\pm}$ in the canonical form and the separation set $\Sigma = \{(u,v) \in \mathbb{R}^2 : v = 0\}$ becomes $\widetilde{\Sigma} = \{(\mathrm{u},\mathrm{v}); \mathrm{u} = 0, \mathrm{v} = cu/w^\pm\}$. Consequently, after this change of coordinates, we deal with the piecewise smooth system

$$\mathcal{P}_Y(\mathrm{u},\mathrm{v}) = \begin{cases} \mathcal{P}_{X^+}(\mathrm{u},\mathrm{v}), \; \mathrm{u} \geq 0, \\ \mathcal{P}_{X^-}(\mathrm{u},\mathrm{v}), \; \mathrm{u} \leq 0, \end{cases} \tag{17}$$

where $\mathcal{P}_{X^\pm}$ are obtained doing $a_i = a_i^\pm$ in (16) and then, in polar coordinates, it is written as

$$\begin{cases} \dot{r} = R^+(r,\theta), \; \theta \in [-\pi/2, \pi/2], \\ \dot{r} = R^-(r,\theta), \; \theta \in [-\pi/2, -3\pi/2]. \end{cases}$$

Therefore, we use the technique shown in Section 2.4 after a rotation of angle $\pi/2$, to calculate the Lyapunov constants of (17). Note that after the change of coordinates $\{\mathrm{u} = v, \mathrm{v} = (cu + dv)/w^\pm\}$ the separation set of (17), $\widetilde{\Sigma} = \{(\mathrm{u},\mathrm{v}); \mathrm{u} = 0, \mathrm{v} = cu/w^\pm\}$, is parameterized in two different ways when $w^+ \neq w^-$ and then the continuity condition must be considered before doing it and we also take it into account when we compute the Lyapunov constants.

On the following, we give some families of centers for the piecewise smooth vector field (15). Some of these centers appear in a family of reversible vector fields with respect to a line (see the definition in Section 2.2).

Proposition 15. *The piecewise continuous vector field* (15) *has a center at the equilibrium point* $(0,1,0)$*, on* $\mathbb{S}^2_1$*, if* $a_7^{\pm} \neq 0$*,* $a_4^{\pm} = a_9^{\pm}$ *and one of the following conditions is satisfied:*

(a) $a_8^- = -a_8^+$*,* $a_9^- = 0$*, and* $w^+ = w^-$*;*
(b) $a_7^- = \pm w$*,* $a_9^- = 0$*, and* $w^+ = w^-$*;*
(c) $a_8^+ = a_8^-$*,* $-(a_5^-)^2 a_7^- a_9^- + a_5^-(a_7^-)^2 a_8^- - a_5^- a_8^-(a_9^-)^2 - a_5^- a_8^- w^2 + a_7^-(a_8^-)^2 a_9^- = 0$*, and* $w^+ = w^-$*;*
(d) $a_5^- = 0$ *and* $a_9^- = 0$*.*

Proof. In case (a), the piecewise projected continuous vector field $\mathcal{P}_Y = (\mathcal{P}_{X^+}, \mathcal{P}_{X^-})$ is reversible with respect to the separation set. The case (b) follows because in polar coordinates we have $dr/dt = 0$ for both $\mathcal{P}_{X^{\pm}}$, which implies that the difference map defined in Σ, in a neighborhood of the origin, is zero. In case (c), the vector field $\mathcal{P}_Y$ is smooth and it satisfies the condition given on Theorem 13. Finally, case (d) follows because both vector fields $\mathcal{P}_{X^{\pm}}$ are reversible with respect to the u-axis and then the difference map defined in Σ, in a neighborhood of the origin, is zero which concludes the proof. □

If $w^+ = w^-$ the change of coordinates that puts the system (13) on form (16) is the same for X^+ and X^- and then the parametrization of $\widetilde{\Sigma}$ coincides before this change of coordinates. In this case, the only possible center families for (15) are that given on items (a)-(d) of Proposition 15.

Proposition 16. *The piecewise continuous vector field* (15) *with* $w^+ = w^-$ *has a center at the equilibrium point* $(0,1,0)$*, on* $\mathbb{S}^2_1$*, if, and only if,* $a_7^{\pm} \neq 0$*,* $a_4^{\pm} = a_9^{\pm}$ *and one of the conditions (a), (b), (c), or (d) of Proposition 15 is satisfied.*

Proof. To simplify the notation of this proof, we eliminate the superscript $\pm$ when the corresponding coefficients of X^+ and X^- are equal. Hence, we consider $w^+ = w^- = w$, $a_4^+ = a_4^- = a_4$, $a_5^+ = a_5^- = a_5$ $a_7^+ = a_7^- = a_7$, and $a_9^+ = a_9^- = a_9$. According to the proof of Proposition 15, all the families detailed in the statement have a center at the origin. Consequently, we only need to check that these are the only ones when $w^+ = w^- = w$. To do that, we compute four Lyapunov constants using the method explained in Section 2.4 for system (17), with the statement assumptions, and we obtain

$$
\begin{aligned}
L_2 =& \frac{2}{3wa_7} a_9(a_8^+ - a_8^-),\\
L_3 =& \frac{\pi}{8w^3a_7^3}\big(2a_5^2a_7a_9(a_7^2 + a_9^2 + w^2) - a_5(a_7^4a_8^+ - 2a_8^+a_9^4 - 4a_8^+a_9^2w^2\\
&- (a_8^+ + a_8^-)w^4) - 2a_7a_9(a_7^2(a_8^+)^2 + (a_8^-)^2a_9^2 + (a_8^+)^2w^2)\big),\\
L_4 =& \frac{4}{45a_7^4w^5} a_5((a_8^+)^2 - (a_8^-)^2)(a_7^2 - w^2)(4a_7^4 + 3a_7^2w^2 + 2w^4),\\
L_5 =& \frac{\pi}{12w^3a_7^3}(a_5^2 + (a_8^+)^2 - w^2)\big(2a_5^2a_7a_9 - a_5a_7^2a_8^+ - a_5a_7^2a_8^-\\
&+ 2a_5a_8^+a_9^2 + a_5a_8^+w^2 + a_5a_8^-w^2 - 2a_7(a_8^+)^2a_9\big).
\end{aligned}
$$

When we solve the system of equations $\mathcal{S}_L = \{L_2 = \cdots = L_5 = 0\}$ we obtain the real solutions given on statements (a)-(d) of Proposition 15 and more four complex solutions given by $\left\{a_5 = \pm\sqrt{-(a_8^-)^2 + w^2}, a_7 = \pm\,\mathrm{i}\,\sqrt{a_9^2 + w^2}, a_8^+ = a_8^-\right\}$ and $\{a_7 = \pm\,\mathrm{i}\,a_9, a_8^+ = a_8^-, w = 0\}$. As we are interested in real families with $w \neq 0$ we conclude the proof. □

Proposition 17. *Consider system* (15) *with* $a_5^- = 1, a_7^- = 1, a_8^+ = 3, a_8^- = 1, a_9^- = 0$, *and* $w^+ = w^- = 2$. *Then, the equilibrium point* $p = (0, 1, 0)$ *is a weak focus of third-order and there exist* 2 *small amplitude limit cycles, on* $\mathbb{S}_1^2$, *bifurcating from* p *with a continuous perturbation in* $\mathcal{X}_2^H$.

Proof. Using the expressions of L_i, $i = 2, \ldots, 5$ given in the above proposition we conclude that, for these values of parameters, we have $L_2 = 0$ and $L_3 = 15\pi/16 \neq 0$. Hence, adding the trace parameter and using the derivation-division algorithm (see more details in [29]) we obtain 2 small amplitude crossing limit cycles bifurcating from the equilibrium point $(0, 1, 0)$ on $\mathbb{S}_1^2$. □

We emphasize that, when we deal with a continuous perturbation, we do not have the sliding parameter to get the maximum upper bound for the number of small amplitude crossing limit cycles bifurcating from a center or a weak focus, as we have explained in Section 2.4. Because of that, the maximum number of limit cycles that we can obtain bifurcating from the weak focus of third-order, in the last result, is 2.

5. Local cyclicity for quadratic vector fields in $\mathfrak{X}_2$ with piecewise smooth perturbation in $\mathcal{X}$.

In this section, we study the local cyclicity of centers and weak focus families of quadratic smooth vector fields with piecewise quadratic perturbations in $\mathcal{X}_2$. The continuous or refractive perturbation cases are also analyzed. We show the results that we have obtained in Propositions 18 and 19. The proof of Theorem 2 follows directly from Proposition 19(c).

On the following, we summarize some assumptions and results given in [6] for a quadratic vector field $X \in \mathfrak{X}_2$ which will be useful in the sequence. Firstly, the behavior of system X can be totally different in two different levels of invariant spheres. Hence, we restrict our analysis to the unit sphere $\mathbb{S}_1^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. In this case any $X \in \mathfrak{X}_2$ can be written in its canonical form

$$\begin{aligned}
\dot{x} &= -a_1 y - a_2 z - a_4 xy - a_5 xz - a_{10} y^2 - (a_6 + a_7) yz - a_8 z^2,\\
\dot{y} &= a_1 x - a_3 z + a_4 x^2 + a_{10} xy + a_6 xz - a_{11} yz - a_9 z^2,\\
\dot{z} &= a_2 x + a_3 y + a_5 x^2 + a_7 xy + a_8 xz + a_{11} y^2 + a_9 yz.
\end{aligned} \tag{18}$$

Note that $(0, 1, 0)$ is an equilibrium point of (18) if, and only if, $a_1 + a_{10} = 0$, $a_3 + a_{11} = 0$. Consequently, to have the origin as an equilibrium point of the projected vector field $\mathcal{P}_X$ associated to (18), we assume these conditions on the following. Next we will impose the conditions that ensure that $(0, 1, 0)$ is an equilibrium point of nondegenerate center-focus type on the sphere $\mathbb{S}_1^2$. We will do that analyzing the trace and the determinant of the Jacobian matrix J associated to the projected vector field $\mathcal{P}_X$ at the equilibrium point $(0, 0)$. Recall that $\mathcal{P}_X$ has an equilibrium point of nondegenerate center-focus type at origin if, and only if, the trace of J is zero and its determinant is positive. It occurs when $a_4 = a_9$ and $a_2a_6 + a_6a_7 + 2a_2a_7 + a_2^2 + a_7^2 - a_9^2 > 0$. As explained in [6], due to the high number of free parameters, we will restrict our analysis adding two extra conditions: $a_9 = 0$ and $a_2 + a_7 = 1$. With these assumptions, the projected vector field $\mathcal{P}_X$ has a weak focus at the origin if, and only if, $a_4 = 0$ and $a_6 + 1 > 0$. Moreover, with these restrictions the projected vector field has a center-focus point at the origin with a Jacobian matrix in Jordan normal form. Doing $a_4 = 0$ and $w^2 = a_6 + 1$, with $w \neq 0$, the following system is obtained from (18)

$$\begin{aligned}
\dot{x} &= -a_1 y - (1-a_7)z - a_5 xz + a_1 y^2 + (1 - a_7 - w^2)yz - a_8 z^2,\\
\dot{y} &= a_1 x + a_{11} z - a_1 xy + (w^2 - 1)xz - a_{11} yz,\\
\dot{z} &= (1-a_7)x - a_{11} y + a_5 x^2 + a_7 xy + a_8 xz + a_{11} y^2.
\end{aligned} \tag{19}$$

After a time reparameterization and the change of coordinates $u \to wu$, the corresponding projected system obtained from (19) is

$$\begin{aligned}
\dot{u} &= -v - \frac{a_1}{2}u^2 - \frac{a_5}{w}uv - \frac{a_1 + 2a_8}{2w^2}v^2 + \frac{2a_7 - w^2}{4}u^2 v\\
&\quad + \frac{w^2 + 2a_7 - 2}{4w^2}v^3 - \frac{a_1 w^2}{8}u^4 - \frac{a_{11} w}{4}u^3 v - \frac{a_{11}}{4w}uv^3 + \frac{a_1}{8w^2}v^4,\\
\dot{v} &= u + \frac{(2a_5 - a_{11})w}{2}u^2 + a_8 uv - \frac{a_{11}}{2w}v^2 - \frac{(2a_7 - 1)w^2}{4}u^3\\
&\quad - \frac{2w^2 + 2a_7 - 3}{4}uv^2 + \frac{w^3 a_{11}}{8}u^4 - \frac{w^2 a_1}{4}u^3 v - \frac{a_1}{4}uv^3 - \frac{a_{11}}{8w}v^4.
\end{aligned} \tag{20}$$

In items (a) and (b) of Proposition 19, we show that with a continuous (resp. refractive) perturbation in $\mathcal{X}_2$ we obtain 5 (resp. 6) crossing limit cycles bifurcating from a center family of (19). Moreover, when we consider a piecewise quadratic general perturbation in $\mathcal{X}_2$ we obtain 10 limit cycles, as we will see in item (c) of Proposition 19, which proves Theorem 2. We also exhibit a piecewise quadratic perturbation of a weak focus in Proposition 18 for a fixed value of w.

On the following, we will describe the type of piecewise smooth perturbation of $X \in \mathfrak{X}_2$ that we will consider and which are the conditions that will make this perturbation continuous or refractive.

Let $X = X(\mathrm{x}, \mathrm{a}) \in \mathfrak{X}_2$ given by (19) where $\mathrm{x} = (x, y, z)$ and $\mathrm{a} = (a_1, a_5, a_7, a_8, a_{11}, w)$. Denoting $\mathrm{a} + \varepsilon^{\pm} = (a_1 + \varepsilon_1^{\pm}, \ldots, w + \varepsilon_6^{\pm})$ we consider the piecewise smooth perturbation of X defined by

$$Y(\mathrm{x}, \varepsilon) = \begin{cases} X(\mathrm{x}; \mathrm{a} + \varepsilon^+),\ z \geq 0,\\ X(\mathrm{x}; \mathrm{a} + \varepsilon^-),\ z \leq 0, \end{cases} \tag{21}$$

and the projected vector field associated, defined by (7), which is of the form

$$\mathcal{P}_Y(\mathrm{u}, \varepsilon) = \begin{cases} \mathcal{P}_X(\mathrm{u}; \mathrm{a} + \varepsilon^+),\ v \geq 0,\\ \mathcal{P}_X(\mathrm{u}; \mathrm{a} + \varepsilon^-),\ v \leq 0, \end{cases} \tag{22}$$

where $\mathrm{u} = (u, v)$ and $\mathcal{P}_X(\mathrm{u}, 0)$ is given by (20).

So, when $\varepsilon = 0$ we have the unperturbed analytical systems (19) and (20). Following the same idea of the last section, we will say that the perturbation of the vector field Y is continuous (resp. refractive) if (21) is continuous (resp. refractive) in the separation set. With a straightforward computation we see that (21) is continuous (resp. refractive) if, and only if, $\varepsilon_i^+ = \varepsilon_i^-$, for $i = 1, 2, 3, 4$ (resp. $\varepsilon_i^+ = \varepsilon_i^-$, for $i = 2, 3, 4$).

Note that the origin is on the boundary of two crossing segments of the perturbative system (22) as well as of the unperturbed one. This is because we assumed that the origin is an equilibrium point of the center type for the unperturbed system and the perturbative parameters do not change the linear part of it. If we assume $a_3 = -a_{11}+\varepsilon_7$ instead of $a_3 = -a_{11}$ we can create a sliding segment in a neighborhood of the origin, because in this case, the projected system is of the form

$$\Big(-\frac{a_4}{w}u - v + \mathcal{O}_2(u,v), \frac{\varepsilon_7}{w} + u + \mathcal{O}_2(u,v)\Big).$$

Thus, we also use the perturbative parameter ε_7 when we deal with a piecewise perturbation instead of piecewise continuous or piecewise refractive ones, to obtain one more crossing limit cycle of small amplitude creating from a sliding or escaping segment, as it was explained in Section 2.

Now we are able to prove the last results. It is important to note that we confine the dynamics to an invariant sphere of fixed radius, which remains unchanged with the considered perturbations. Conversely, the notion of limit cycle make no sense in this context.

Proposition 18. *Consider the system*

$$\begin{aligned}
\dot{x} &= 2\alpha y + \frac{9}{20}z - xz - 2\alpha y^2 - \frac{89}{20}yz - \alpha z^2,\\
\dot{y} &= -2\alpha x + 2z + 2\alpha xy + 3xz - 2yz,\\
\dot{z} &= -\frac{9}{20}x - 2y + x^2 + \frac{29}{20}xy + \alpha xz + 2y^2.
\end{aligned} \tag{23}$$

Then, for $\alpha = \pm\sqrt{857/488}$ there exists a piecewise quadratic perturbation in $\mathcal{X}$ such that at least 9 hyperbolic crossing limit cycles of small amplitude bifurcate from the equilibrium point $(0,1,0)$ on $\mathbb{S}^2_1$.

Proof. Let $\alpha = \pm\sqrt{857/488}$. Note that system (23) is obtained doing $a_1 = -2\alpha, a_4 = 0, a_5 = 1, a_7 = 29/20, a_8 = \alpha, a_{11} = 2$, and $w = 2$ in (19). It was proved in [6] that the equilibrium point $p = (0,1,0)$ of (23) is a weak focus of fourth-order and that there exist 4 small amplitude limit cycles, on $\mathbb{S}^2_1$, bifurcating from p considering an analytical perturbation of (23) inside family (19). Now we consider a piecewise smooth perturbation $(a_1^\pm, a_5^\pm, a_7^\pm, a_8^\pm, a_{11}^\pm, w^\pm) = (-2\alpha + \varepsilon_1^\pm, 1 + \varepsilon_2^\pm, 29/20 + \varepsilon_3^\pm, \alpha + \varepsilon_4^\pm, 2+\varepsilon_5^\pm, 2+\varepsilon_6^\pm)$ in the piecewise projected system (20). As we saw before, we consider the separation set $\{(u,v) \in \mathbb{R}^2 : v = 0\}$ of the projected system and then we consider the perturbative parameter $\varepsilon^+ = (\varepsilon_1^+, \ldots, \varepsilon_6^+)$ for $v > 0$, $\varepsilon^- = (\varepsilon_1^-, \ldots, \varepsilon_6^-)$ for $v < 0$, and joining all $\varepsilon = (\varepsilon_1^+, \ldots, \varepsilon_6^+, \varepsilon_1^-, \ldots, \varepsilon_6^-)$. Let $L_i(\varepsilon)$ be the corresponding Lyapunov constants. Using the method explained in Section 2.4 we compute the Taylor series of these Lyapunov constants up to first-order with respect to ε, $L_i^{[1]}(\varepsilon)$, and we write $L_i(\varepsilon) = L_i^{[1]}(\varepsilon) + \mathcal{O}_2(\varepsilon)$. We get $L_2^{[1]}(0) = \cdots = L_8^{[1]}(0) = 0$ and $L_9^{[1]}(0) \neq 0$. Hence, as the matrix formed with the coefficients of $(L_2^{[1]}, \ldots, L_8^{[1]})$ with respect to ε has rank 7, we obtain eight hyperbolic crossing limit cycles of small amplitude bifurcating from the origin adding the trace parameter and using the Implicit Function Theorem and then the derivation-division algorithm (see again [29]). Finally, adding the sliding parameter we obtain the ninth hyperbolic crossing limit cycle of small amplitude. ◻

Proposition 19. *Consider the system*

$$\begin{aligned}\dot{x} &= -\frac{4}{5}y - \frac{13}{8}z - \frac{5}{2}xz + \frac{4}{5}y^2 - \frac{59}{8}yz - z^2,\\ \dot{y} &= \frac{4}{5}x + 2z - \frac{4}{5}xy + 8xz - 2yz,\\ \dot{z} &= \frac{13}{8}x - 2y + \frac{5}{2}x^2 - \frac{5}{8}xy + xz + 2y^2.\end{aligned} \tag{24}$$

(a) There exists a continuous quadratic perturbation of (24) *in* $\mathcal{X}$ *such that at least* 5 *hyperbolic crossing limit cycles of small amplitude bifurcate from the equilibrium point* $(0,1,0)$ *on* $\mathbb{S}^2_1$.

(b) There exists a refractive quadratic perturbation of (24) *in* $\mathcal{X}$ *such that at least* 6 *hyperbolic crossing limit cycles of small amplitude bifurcate from the equilibrium point* $(0,1,0)$ *on* $\mathbb{S}^2_1$.

(c) There exists a piecewise quadratic perturbation of (24) *in* $\mathcal{X}$ *such that at least* 10 *hyperbolic crossing limit cycles of small amplitude bifurcate from the equilibrium point* $(0,1,0)$ *on* $\mathbb{S}^2_1$.

Proof. It was proved in [6] that system (19) has a center at the origin when its coefficients satisfy the conditions $a_4 = 0$, $w \neq 1$, $a_1 = \dfrac{w^2-1}{w^2+1}a_8, a_5 = \dfrac{w^2+1}{w^2-1}a_{11}$, and $a_7 = \dfrac{1}{w^2+1} - \dfrac{1}{(w^2+1)}a_8^2 - \dfrac{w^2+1}{(w^2-1)^2}a_{11}^2$, exhibiting an inverse integral factor for the system. Moreover, there exists an analytical perturbation inside family (19) such that at least 3 small amplitude limit cycles bifurcate from the equilibrium point $(0,1,0)$ on $\mathbb{S}^2_1$. Thus, as system (24) is obtained doing $a_1 = 4/5, a_4 = 0, a_5 = 5/2, a_7 = -5/8, a_8 = 1, a_{11} = 2$, and $w = 3$ in (19) it has a center at $(0,1,0)$. So, we take the parameter values $(a_1, a_5, a_7, a_8, a_{11}, w)$ satisfying it and we consider the piecewise smooth perturbation $(a_1, a_5, a_7, a_8, a_{11}, w) = (4/5 + \varepsilon_1^\pm, 5/2 + \varepsilon_2^\pm, -5/8 + \varepsilon_3^\pm, 1 + \varepsilon_4^\pm, 2 + \varepsilon_5^\pm, 3 + \varepsilon_6^\pm)$ in the projected system (20). As we saw before, we consider the separation set $\{(u,v) \in \mathbb{R}^2 : v = 0\}$ of the projected system and then we consider the perturbative parameter $\varepsilon^+ = (\varepsilon_1^+, \ldots, \varepsilon_6^+)$ for $v > 0$, $\varepsilon^- = (\varepsilon_1^-, \ldots, \varepsilon_6^-)$ for $v < 0$ and $\varepsilon = (\varepsilon_1^+, \ldots, \varepsilon_6^+, \varepsilon_1^-, \ldots, \varepsilon_6^-)$. We denote by $L_i(\varepsilon)$, the corresponding Lyapunov constants. When $\varepsilon = 0$ the origin is a center and then $L_i(0) = 0$ for all i. Using the method explained in Section 2.4 we compute the Taylor series of these Lyapunov constants up to first-order with respect to ε, $L_i^{[1]}(\varepsilon)$, and we write $L_i(\varepsilon) = L_i^{[1]}(\varepsilon) + \mathcal{O}_2(\varepsilon)$.

In the case (a) (resp. (b)) we consider a continuous (resp. refractive) perturbation of this center family. As we saw above, it implies that $\varepsilon_i^+ = \varepsilon_i^-$, for $i = 1,2,3,4$ (resp. $\varepsilon_i^+ = \varepsilon_i^-$, for $i = 2,3,4$). With this assumption, the matrix formed by the coefficients of $(L_2^{[1]}, \ldots, L_7^{[1]})$ with respect to ε has rank 5 (resp. 6). Adding the trace parameter and using the Melnikov theory, as we have explained in the Section 2.4, we obtain 5 (resp. 6) hyperbolic crossing limit cycles of small amplitude bifurcating from the origin.

Finally, in the case (c), the proof follows because the matrix formed by the coefficients of $(L_2^{[1]}, \ldots, L_{12}^{[1]})$ with respect to ε has rank 9, so adding the trace parameter and using the Melnikov theory, as we explained in the Section 2.4, we get 9 hyperbolic crossing limit cycles of small amplitude bifurcating from the origin. Adding the sliding or escaping segments we obtain one more crossing limit cycle and the proof follows. $\square$

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