FLOWS WITH CONSTANT SLOPE ON THE KLEIN BOTTLE

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Abstract. We consider smooth flows with constant slope on the 2-dimensional torus and on the Klein bottle. It is well known that for a flow with constant slope on the torus either all orbits are periodic or all orbits are dense. We obtain a similar classification for the orbits of a flow with constant slope on the Klein bottle: again either all orbits are periodic or all orbits are dense. We shall see that this requires considering flows on the unit normal bundle of the Klein bottle.

1. Introduction

Our main aim is to describe a complete classification of the orbits of a smooth flow with constant slope on the Klein bottle, which is a nonorientable surface. So that the result and its proof are appreciated fully, we shall first describe a corresponding classical result for the 2-dimensional torus. This will also be useful to introduce some basic constructions that then will be used for the Klein bottle, such as the process of continuing the flow when we reach the boundary of a fundamental domain, which we will always take to be a square with the appropriate identifications, both for the torus and for the Klein bottle. It turns out that for both surfaces a smooth flow with constant slope either all orbits are periodic or all orbits are dense, depending on the slope.

We note that while the notion of a smooth flow with constant slope on the 2-dimensional torus is quite natural, to the best of our knowledge no similar construction exists in the literature for the Klein bottle. In fact, even the notion of a flow with "constant slope" first needs to be clarified for a nonorientable surface. For the torus this amounts to consider straight lines that after reaching the boundary of the fundamental domain at a point p continue through a point q that is identified with p with the same slope. On the other hand, in the case of the Klein bottle we first need to understand what should happen to the flow after the gluing process at the boundary of the fundamental domain. In fact, this process reverses the orientation on the vertical sides of the square $\{(x, y) \in \mathbb{R}^2 : 0 \le x, y \le 1\}$, namely for $x = 1$. This causes that if we continue to travel *on the fundamental domain* (that is, on the square) with the same slope as before after such an identification, then in fact we are changing the flow abruptly (see Figure 3 together with a corresponding description in Section 3). In other words, this procedure would certainly give a continuous flow, but coming from a nonsmooth vector field on the Klein bottle (in Appendix A we classify all orbits of such a continuous flow).

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