MORE LIMIT CYCLES FOR COMPLEX DIFFERENTIAL EQUATIONS WITH THREE MONOMIALS

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Abstract. In this paper we improve, by almost doubling, the existing lower bound for the number of limit cycles of the family of complex differential equations with three monomials, $\dot{z} = Az^k\bar{z}^l + Bz^m\bar{z}^n + Cz^p\bar{z}^q$, being k, l, m, n, p, q non-negative integers and $A, B, C \in \mathbb{C}$. More concretely, if $N = \max(k+l, m+n, p+q)$ and $H_3(N) \in \mathbb{N}\cup\{\infty\}$ denotes the maximum number of limit cycles of the above equations, we show that for $N \geq 4$, $H_3(N) \geq N-3$ and that for some values of N this new lower bound is $N+1$. We also present examples with many limit cycles and different configurations. Finally, we show that $H_3(2) \geq 2$ and study in detail the quadratic case with three monomials proving in some of them non-existence, uniqueness or existence of exactly two limit cycles.

1. Introduction and Main Results

In this paper we study lower bounds for the number of limit cycles of complex differential equations with three monomials,

$$\dot{z} = Az^k\bar{z}^l + Bz^m\bar{z}^n + Cz^p\bar{z}^q,$$

with k, l, m, n, p, q non-negative integers and $A, B, C \in \mathbb{C}$. Let $N = \max(k+l, m+n, p+q)$ and denote by $H_3(N) \in \mathbb{N} \cup \{\infty\}$ the maximum number of limit cycles of the above equations.

It is known that when $ABC = 0$ then the maximum number of limit cycles is 1, see [1]. It is also known that for $N \geq 3$ odd, $H_3(N) \geq (N+3)/2$, see [17]. Moreover, in the given differential equations reaching these bounds, each one of these limit cycles surrounds a different critical point. In fact, in [10] one more limit cycle is proved to exist and it surrounds all the other limit cycles, showing that $H_3(N) \geq (N+3)/2 + 1$.

In this work, we prove that the existing lower bound can in fact be almost doubled, see next Theorems A and B. Moreover, while the essential techniques used in [17] are the rotational symmetries and the Abelian integrals, in this paper we also use the computation of the Lyapunov quantities and the properties of the transformation $w = z^n$.

As a matter of fact, we need to compute in many situations the first Lyapunov quantity, L_1, for a weak focus that is not in the usual normal form, namely $\dot{z} = \alpha i z + O_2(z, \bar{z})$, with $0 \neq \alpha \in \mathbb{R}$. For this reason we have decided to include an appendix where L_1 is given in full generality. The expression that we obtain coincides with the one of the classical book [2, p. 253] attributed to Bautin. Moreover, for the sake of completeness,

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we present both its real and complex expressions, see Theorems D and E in that appendix, together with a version of the classical Andronov–Hopf bifurcation theorem. The Lyapunov quantities L_2 and L_3 are also computed in some examples.

Next three statements collect our results on $H_3(N)$ for arbitrary N.

Theorem A. *For $N \geq 4$ there exist differential equations of degree N with three monomials having at least $N-3$ limit cycles, each one of them surrounding one critical point, that is $H_3(N) \geq N-3$.*

Next result slightly improves the previous lower bound but only for a sequence of values of N tending to infinity.

Theorem B. *For $j \geq 1$ there exist differential equations of degree $N = 4j-1$ with three monomials having at least $N+1$ limit cycles, each one of them surrounding one critical point. Hence, for these values of N, $H_3(N) \geq N+1$.*

In the two previous results, each of the existing limit cycles surrounds a single different critical point. To show that there are other different configurations with many limit cycles we prove next result:

Proposition 1.1. *(i) For $j \geq 1$ there exist differential equations of degree $N = 3j-1$ with three monomials having at least $2(N+1)/3$ limit cycles. In this case, for j different critical points there exist two nested limit cycles surrounding each one of them.*

(ii) For $j \geq 1$ there exist differential equations of degree $N = 4j-1$ with three monomials having at least $3(N+1)/4$ limit cycles. In this case, for j different critical points there exist three nested limit cycles surrounding each one of them.

Its proof is based on similar ideas to those of the previous theorems but needs as a starting point the computations of the first three Lyapunov quantities, L_1, L_2 and L_3 of a quadratic or a cubic 3-monomial equation (the case $j = 1$ of both items of our proposition) having a critical point that is not the origin. As we will see, these equations act as seeds for the results of higher degrees.

While we have not been able to prove that the maximum number of small amplitude limit cycles bifurcating from a weak focus for the differential equations (3.1) is three, we believe this assertion to be true. The proof provided in item (ii) of the aforementioned proposition establishes that at least three small amplitude limit cycles do appear for $N = 4j-1, j \geq 1$.

As a second part of this work we focus our attention on the case $N = 2$. It is known that the study of the limit cycles of any planar quadratic system (QS) can be reduced to the 4-monomial differential equation

$$\dot{z} = Rz + Az^2 + Bz\bar{z} + C\bar{z}^2, \tag{1.1}$$

with $R, A, B, C \in \mathbb{C}$, see for instance [7]. Moreover, it is known that QS can have at least four limit cycles, see [4, 22]. Because of the extreme difficulty that this problem entails, we will focus on QS with 3 monomials.

There are $\binom{6}{3} = 20$ families of QS having 3 monomials. Among them, it is well-known that the linear equations, $\dot{z} = A + Bz + C\bar{z}$, and the homogeneous QS, $\dot{z} = Az^2 + Bz\bar{z} + C\bar{z}^2$ do not have limit cycles. Hence it remains to study 18 families of QS, 9 of them with exactly one non-linear term and 9 with exactly two non-linear terms. Our results about their number of limit cycles are summarized in next theorem.

Theorem C. *Consider the differential equations*

$$\dot{z} = AM_1 + BM_2 + CM_3,$$

with $A, B, C \in \mathbb{C}$ and M_1, M_2 and M_3, are 3 different fixed monomials, $M_j \in \{1, z, \bar{z}, z^2, z\bar{z}, \bar{z}^2\}$. Then their number of limit cycles are given in Tables 1 and 2. In both tables, when an integer number ℓ appears it means that the full family with the corresponding 3 monomials has at most ℓ limit cycles, taking into account their multiplicities, and moreover there are equations in this family with exactly ℓ nested hyperbolic limit cycles. If it appears the expression $\geq \ell$, it means that ℓ nested hyperbolic cycles do appear but it is not proved that ℓ is the upper bound. Finally, if it is written $1+1$, then it means that we are in the first situation with $\ell = 2$ but that the two limit cycles are not nested.

Monomials	$1, z$	$1, \bar{z}$	$z, \bar{z}$
z^2	0	≥ 1	≥ 1
$z\bar{z}$	1	1	1
$\bar{z}^2$	0	0	0

TABLE 1. Number of limit cycles for QS with three monomials, one of them being quadratic.

Monomials	$z^2, z\bar{z}$	$z^2, \bar{z}^2$	$z\bar{z}, \bar{z}^2$
1	$1+1$	$1+1$	$1+1$
z	≥ 1	≥ 2	≥ 1
$\bar{z}$	≥ 1	≥ 1	≥ 1

TABLE 2. Number of limit cycles for QS with three monomials, two of them being quadratic.

Next corollary is a straightforward consequence of Theorems B and C.

Corollary 1.2. *It holds that $H_3(2) \geq 2$ and $H_3(3) \geq 4$.*

Remark 1.3. In Lemma 3.8 we have also proved that the maximum number of limit cycles of both families

$$\dot{z} = A + B\bar{z} + Cz^2 \quad \text{and} \quad \dot{z} = Az + B\bar{z} + Cz^2$$

coincide. As a consequence, the full case of 3-monomial QS with only one non-linear monomial showed in Table 1 would be totally solved if we were able to complete the study of the differential equation $\dot z = Az + B\bar z + Cz^2$. We think that the maximum number of limit cycles for this family is one but for the moment we only know that this fact holds when $|B| \le |\operatorname{Re}(A)|/2$, see [17, Thm. B] and in some cases corresponding to the ones treated in item (v) of Proposition 3.11. In Section 3.4 we present that proposition and some more insights on this equation.

We suspect that $H_3(2) = 2$ but all our attempts to prove this or to find more than 2 limit cycles have been unsuccessful. In Section 3.5, we present several approaches to obtaining limit cycles. Specifically, we emphasize that the study of Melnikov functions up to order 6 for the perturbation of $\dot z = \mathrm{i}z$ yields at most 1 limit cycle around the origin for each of the 3-parameter families, as detailed in Proposition 3.12. Furthermore, we have investigated the first-order perturbations with three different monomials of the equation $\dot z = \mathrm{i}z + z^2$, which possess two simultaneous centers. From this study, we prove that only 1 limit cycle appears from one of the two centers, as discussed in Subsection 3.5.2.

2. Proof of Theorems A and B

Proof of Theorem A. For each integer $j \ge 1$, let us consider next 3-monomials differential equation of degree $N = j + 3 \ge 4$,

$$\dot z = (A+B)z - Az^{j+1} - Bz^{j+2}\bar z = Az(1 - z^j) + Bz(1 - \bar z z^{j+1}), \qquad (2.1)$$

being $A = j + 1 + a + i$ and $B = -j + i$. The critical points of this equation are $z = 0$ and the points $z = w_s$ such that $w_s^j = 1$ for $s = 1, \ldots, j$.

Let w_s be a critical point of Equation (2.1), $s = 1, \ldots, j$. Observe that this equation is invariant by the change of the dependent variable $u = w_s^{j-1} z$ for all $s = 1, \ldots, j$. By this change, the critical point w_s of the original equation is transformed into the critical point $u = 1$. Hence, varying s we get that all the critical points w_s of the original equation have the same character and stability as $z = 1$. Let us study this critical point.

Following the results in [17] we have that

$$\begin{aligned}
\operatorname{div}(X)_{z=1} &= 2\operatorname{Re}\left(\frac{\partial}{\partial z}F(z,\bar z)\right)_{z=1} = -2ja,\\
\det(dX)_{z=1} &= \left|\frac{\partial}{\partial z}F(z,\bar z)\right|^2_{z=1} - \left|\frac{\partial}{\partial \bar z}F(z,\bar z)\right|^2_{z=1}\\
&= j|A|^2 + j|B|^2 + j(j+1)|A||B| > 0.
\end{aligned}$$

Hence, when $a \ne 0$ the critical point $z = 1$ is a strong focus, while if $a = 0$ it is a weak focus. Let us prove that $L_1 \ne 0$. In order to apply Theorem E of the Appendix to compute its first Lyapunov quantity L_1, we perform the translation $w = z - 1$ and change the sign of the vector field $(t \to -t)$, arriving to the differential equation

$$\dot w = -(A+B)(w+1) + A(w+1)^{j+1} + B(w+1)^{j+2}(\bar w + 1).$$

We can now apply Theorem E. After many computations we get

$$L_1 = \frac{(5 + 2j - j^2)j^3}{9j^2 + 8j + 3}.$$

Notice that $L_1 > 0$ for $j = 1, 2, 3$ and $L_1 < 0$ for $j \geq 4$. Hence, because of the change of time, we know that the point $z = 1$ of Equation (2.1) is an attractor when $j \leq 3$, and a repellor otherwise.

Using Corollary 3.13 in the Appendix, we also know the existence of an Andronov–Hopf bifurcation by moving slightly the parameter a and taking it with the suitable sign. One gets a hyperbolic limit cycle born from the critical point $(1, 0)$ of the original Equation (2.1). As this equation has a $j-$symmetry, from each one of the j non-zero critical points of Equation (2.1), a limit cycle is born at the same time. Thus, the equation has at least $j = N - 3$ hyperbolic limit cycles for each $j \in \mathbb{N}$. The limit cycles exist for $|a|$ small enough: when $a < 0$ and $j = 1, 2, 3$ they are stable, while when $a > 0$ and $j \geq 4$ they are unstable. □

Proof of Theorem B. We start proving the result for $N = 3$. As we will see this result will be a seed for proving all the other cases.

Let us consider the following cubic equation that posses a symmetry, with respect the origin, of order 4:

$$\dot{z} = Az + Bz^2\bar{z} + C\bar{z}^3.$$

As it was proved in [25], the previous equation can have four limit cycles, that is $N + 1$, each one surrounding one critical point. In order to be self-contained, we are going to explain briefly how to produce these four limit cycles. We consider $A = -B - C$, $B = b_1 + 5\mathrm{i}$ and fix $C = 1 - \mathrm{i}/2$. Then we get that the non-zero critical points of the equation are located at $1, \mathrm{i}, -1, -\mathrm{i}$. As the equation has a symmetry of order four, it is enough with studying one of the critical points. The divergence at $z = 1$ is $2(b_1 - 1)$. Hence to have a weak focus at this point we impose that $b_1 = 1$. Moreover it is easy to see that it has index $+1$. In order to generate a limit cycle from it we apply Theorem E to obtain its first Lyapunov quantity. First we move the critical point at $z = 1$ to the origin. After some computations we obtain that $L_1 = -576/71$. Hence, we are under the hypotheses of the classical Andronov–Hopf bifurcation (see Corollary 3.13) and for $|b_1 - 1|$ small enough and $b_1 - 1 > 0$ the differential equation has a limit cycle surrounding $z = 1$. Hence, by its symmetry, the original equation has four limit cycles, each one surrounding one of its critical points.

Observe that these limit cycles, as they are born by an Andronov–Hopf bifurcation, can be as small as we want. As a consequence, we can fix the b_1 parameter in the original equation in such a way that there exist the four limit cycles, and they are as close as we wish to its corresponding critical point.

To consider the general case $N = 4j - 1, j \geq 1$, we perform the non-invertible transformation $z = u^j$ and the change of time $\mathrm{d}t/\mathrm{d}s = (u\bar{u})^{j-1}$. With these changes, Equation (2.1) becomes

$$u' = \frac{du}{ds} = \frac{A}{j}u^j\bar{u}^{j-1} + \frac{B}{j}u^{2j}\bar{u}^{2j-1} + \frac{C}{j}\bar{u}^{4j-1}.$$

Observe that because of the form of the transformation, each one of the original critical points is transformed into j critical points. Observe also that these critical points cannot coincide because they are the $j-$roots of the angles $\alpha\pi/2$, with $\alpha \in \{0, 1, 2, 3\}$. The same happens with the j limit cycles because each one of them surrounds and is very close to the critical point from which it has born. Hence, the new equation has $4j$ critical points, each one of them surrounded by a single limit cycle. Thus, it has $4j = N+1$ limit cycles and the result is proved. □

In order to prove item (i) (resp. item (ii)) of Proposition 1.1, we are going to use the same idea as before but we will choose as a seminal equation a quadratic (resp. cubic) one having two (resp. three) limit cycles surrounding one critical point.

Proof of Proposition 1.1. (i) We start with Equation (3.1), studied in the forthcoming Proposition 3.1:

$$\dot{z} = Az + Bz^2 - (A+B)\bar{z}^2,$$

with some complex values A and B such that it has two limit cycles surrounding the critical point $z = 1$. If we apply the non-invertible transformation $z = u^j$ and the change of time $dt/ds = (u\bar{u})^{j-1}$, we get the following equation:

$$u' = \frac{du}{ds} = \frac{A}{j}u^j\bar{u}^{j-1} + \frac{B}{j}u^{2j}\bar{u}^{j-1} - \frac{(A+B)}{j}\bar{u}^{3j-1}.$$

It has degree $N = 3j - 1$ and j critical points located at each of the j-th roots of the unity. From the symmetry of the transformation and using the ensuing equation, there exist values of the parameters such that each one of these critical points has at least two limit cycles surrounding it and hence, the equations has at least $2j = 2(N+1)/3$ limit cycles.

(ii) Consider the complex differential equation

$$\dot{z} = Az + B\bar{z} - (A+B)z^3. \tag{2.2}$$

Notice that $z = 1$ is one of its critical points. Consider $A = a_1 + a_2\mathrm{i}$ and $B = b_1 + b_2\mathrm{i}$. The divergence of the vector field at $z = 1$ is $-4a_1 - 6b_1$. To have a weak focus at the origin we impose that $a_1 = -3b_1/2$. We fix $a_2 = 2 + a$, $b_1 = \sqrt{7/2}$ and $b_2 = 5 + b$, where a and b are small real parameters. It can be seen that, for these values of the parameters, the point $z = 1$ is a weak focus. From Theorem E we obtain L_1 and, following the same procedure and after many computations, we also obtain L_2 and L_3. It turns out that:

$$\begin{aligned}
L_1 =& L_1(a,b) = -\frac{\sqrt{14}}{1501}(1527a - 510b) + O_2(a,b),\\
L_2 =& L_2(a,b) = -\frac{3\sqrt{14}}{13035571751440}(293140117939a + 746b) + O_2(a,b),\\
L_3 =& L_3(a,b) = -\frac{2268\sqrt{14}}{194275} + O_1(a,b),
\end{aligned}$$

where notice that the linear parts of L_1 and L_2 are linearly independent. We also observe that, in particular, for $a = b = 0$ the point $z = 1$ is a weak focus of order three because $L_1 = L_2 = 0$ and $L_3 < 0$. Hence we can choose a and b small enough, such that $L_1 < 0$, $L_2 > 0$, $L_3 < 0$, and

$$|L_1| \ll |L_2| \ll |L_3|.$$

In this way two limit cycles bifurcate from the critical point. Finally we can choose $a_1 = -3b_1/2 + c$, obtaining $L_0 = \operatorname{div}(X)_{z=1} = 4a_1 - 6b_1 = -4c$.

With this final bifurcation, with $c < 0$ and $|c|$ small enough, a third limit cycle is obtained, proving that the cubic Equation (2.2) has three limit cycles surrounding $z = 1$. In a few words, we have proved that this equation exhibits a codimension 3 Andronov–Hopf bifurcation at $z = 1$.

This proves item (ii) when $j = 1$, that is $N = 3$. By the same procedure used in the proof of item (i), namely by using the transformation $z = u^j$ and the change of time $\mathrm{d}t/\mathrm{d}s = (u\bar{u})^{j-1}$, we prove the result for all $N = 4j - 1$. □

3. Proof of Theorem C

Before giving the proof of the theorem we present two preliminary sections, one about lower bounds of the number limit cycles for QS with three monomials, and another one containing some upper bounds.

3.1. **Lower bounds for the number of limit cycles.** In this section we prove three propositions collecting all the lower bounds that we have obtained for QS by using Andronov–Hopf type bifurcations.

Proposition 3.1. *Consider the complex differential equation*

$$\dot{z} = Az + Bz^2 - (A + B)\bar{z}^2. \tag{3.1}$$

There exist coefficients $A, B \in \mathbb{C}$ such that the previous equation has two limit cycles surrounding the critical point $z = 1$.

Proof. Observe that $z = 1$ is a critical point of the equation. Consider $A = a_1 - 2\mathrm{i}$ and $B = b_1 + 4\mathrm{i}$. The divergence of the vector field at this point is $2(a_1 + 2b_1)$. To have a weak focus at the origin we impose $a_1 = -2b_1$ and to have a point of index $+1$, $|b_1| < \sqrt{5}$.

We want to obtain two limit cycles bifurcating from this point. Hence we need to compute two Lyapunov quantities L_1 and L_2 and prove that there are values of b_1 such that $L_1 = 0$ and $L_2 \neq 0$. From Theorem E it holds that

$$L_1 = \frac{4b_1\left(2 - b_1^2\right)}{b_1^2 + 22}.$$

To compute the second one we skip all the details but we follow the same procedure developed in the Appendix to obtain L_1. See also the Appendix to know how these two quantities can be used to obtain two limit cycles. We arrive to

$$L_2 = -\frac{\left(96b_1^{10} - 3648b_1^8 - 86408b_1^6 + 74640b_1^4 + 913200b_1^2 - 2156000\right)b_1}{\left(675b_1^2 + 14850\right)\left(b_1^6 - 75b_1^2 + 250\right)}.$$

Notice that if we write $b_1 = \sqrt{2} + b$, where b is a small parameter, it holds that

$$L_1\big|_{b_2=\sqrt{2}+b} = -\frac{2b}{3} + O_2(b) \quad \text{and} \quad L_2\big|_{b_2=\sqrt{2}+b} = \frac{4}{9}\sqrt{2} + O_1(b).$$

Hence, by taking $b_1 = \sqrt{2} + b$, with $b > 0$, and a value of a_1 such that $a_1 + 2b_1 < 0$ and moreover $|a_1 + 2b_1| \ll |b| \ll 1$, we get an equation with $L_0 = \operatorname{div}(X)_{z=1} = 2(a_1 + 2b_1) > 0$, $L_1 < 0$ and $L_2 > 0$ and satisfying $|L_0| \ll |L_1| \ll |L_2|$. Therefore the equation has two limit cycles bifurcating from $z = 1$, by a codimension two Andronov–Hopf bifurcation. For more details see [6] and its references, and also the Appendix. $\square$

Proposition 3.2. *Consider the next families of QS with three monomials*

$$\text{(i) } \dot{z} = A + Bz^2 + Cz\bar{z}, \text{ (ii) } \dot{z} = A + Bz^2 + C\bar{z}^2, \text{ (iii) } \dot{z} = A + Bz\bar{z} + C\bar{z}^2.$$

There exist values of $A, B, C \in \mathbb{C}$ such that each of the previous QS has two limit cycles. Moreover, each one of the limit cycles surrounds a different critical point.

Proof. In all the three cases we take $C = -(A + B)$. Then, $z = \pm 1$ are critical points. These families are symmetric with respect to the origin (observe that they are invariant by the change $(z, t) \to (-z, -t)$). Hence, modulus a change of orientation, what happens around $z = 1$ is the same that happens around $z = -1$.

Let us see that for each one of the former cases there is an Andronov–Hopf bifurcation around $z = 1$. We will apply Theorem E and Corollary 3.13.

For case (i) we take $A = a_1 - \mathrm{i}$ and $B = b_1 + \mathrm{i}$. Then $\operatorname{div}(X)_{z=1} = 2(b_1 - a_1)$ and when $a_1 = b_1$ and $|b_1| < 1$ it holds that $z = 1$ is a weak focus and

$$L_1 = \frac{2b_1\left(b_1^2 + 1\right)}{b_1^2 + 2}.$$

Hence, perturbing slightly the coefficients, one limit cycle is born from each of the two foci. Both limit cycles are hyperbolic and they have different stabilities. Hence, the equation has at least two limit cycles, each one of them surrounding a different critical point, as we wanted to prove.

In case (ii) we take the same values of A and B. Then, $\operatorname{div}(X)_{z=1} = 4b_1$ and when $b_1 = 0$ and $|a_1| < 1$ it holds that $z = 1$ is a weak focus and

$$L_1 = \frac{2a_1\left(a_1^2 - 1\right)}{a_1^2 + 2}.$$

Then the result follows again by Corollary 3.13.

Case (iii) is proved by using the same approach. In this situation we take $A = a_1 - \mathrm{i}$ and $B = b_1 + 2\mathrm{i}$. Then $\operatorname{div}(X)_{z=1} = 2b_1$ and when $b_1 = 0$ and again $|a_1| < 1$ it holds that $z = 1$ is a weak focus and

$$L_1 = \frac{2\left(3 - a_1^2\right)a_1}{a_1^2 + 2}.$$

Hence, by using again Corollary 3.13 the result follows in this case. $\square$

Next result collects all our achievements obtained by using Andronov–Hopf type bifurcations.

Proposition 3.3. *The following results for 3-monomials QS hold:*

(i) The maximum cyclicity of a weak focus is two. Furthermore, this maximum cyclicity occurs only in Equation (3.1)*, studied in Proposition 3.1.*

(ii) The maximum joint cyclicity of two weak foci is two and when it occurs one limit cycle bifurcates simultaneously from each of the two weak foci. Furthermore, this maximum cyclicity occurs only in the families studied in Proposition 3.2.

(iii) *In 14 of the 18 cases considered in Theorem C, at least one limit cycle appears by an Andronov–Hopf bifurcation. These are the cases of the differential equations appearing in Tables 1 and 2 not having associated a zero.*

Proof. Consider any of the 18 differential equations

$$\dot{z} = AM_1 + BM_2 + CM_3, \tag{3.2}$$

with $A, B, C \in \mathbb{C}$ and M_1, M_2 and M_3 different monomials and satisfying $M_j \in \{1, z, \bar{z}, z^2, z\bar{z}, \bar{z}^2\}$. When the monomial 1 does not appear, $z = 0$ is a critical point and we can easily impose that it is a focus and study its first Lyapunov quantity by using Theorem E.

When we are interested in studying a non zero critical point $z = z_0 \neq 0$, the key point is that it is always possible to make a change of variables of the form $u = Dz$, with $D \in \mathbb{C}$, in such a way that now the critical point is $u = 1$. Equivalently we can assume without loss of generality that $A + B + C = 0$ in Equation (3.2). Then, with the translation $u = z - 1$ the new equation has the critical point at the origin and we can proceed as in the first case.

In all cases, but the one of item (i), when $L_1 = 0$ we already get a center; so, at most one limit cycle is generated from the point. Moreover, if we impose that the QS has two weak focus simultaneously, only the cases of item (ii) give rise to two simultaneous bifurcations of one limit cycle.

To conclude the proof of case (i), we need to calculate L_2 for the entire family (3.1) and demonstrate that when the divergence at the critical point vanishes and $L_1 = L_2 = 0$, then $L_3 = 0$ as well. This conclusion will establish that the critical point is a center. This is so because for a weak focus of a QS, if its first three Lyapunov quantities are zero, then it is indeed a center, see [3]. We have verified these assertions using the tools outlined in the Appendix, but for brevity, we have omitted the detailed calculations. □

3.2. **Upper bounds for the number of limit cycles.** In this section we include a result on non-existence of limit cycles (Lemma 3.4) and two theorems (Theorems 3.6 and 3.7) giving an upper bound for the number of limit cycles for QS with four monomials. These theorems, as we will see, are adaptations of known results on QS.

This first lemma collects the 3-monomials differential equations without limit cycles and has a straight proof.

Lemma 3.4. *The following families of QS with three monomials do not have limit cycles:*

(i) $\dot{z} = A + Bz + C\bar{z}$, (ii) $\dot{z} = A + Bz + Cz^2$,

(iii) $\dot{z} = A + B\bar{z} + C\bar{z}^2$, (iv) $\dot{z} = A + Bz + C\bar{z}^2$,

(v) $\dot{z} = Az + B\bar{z} + C\bar{z}^2$, (vi) $\dot{z} = Az^2 + Bz\bar{z} + C\bar{z}^2$,

Proof. Case (i) is trivial because it is an affine differential equation.

The differential equation of case (ii) is holomorphic and it is well-known that general holomorphic differential equations $\dot{z} = f(z)$ do not have limit cycles, see for instance [12] and their references. A simple proof is to realize that $1/(f(z)\overline{f(z)})$ is an integrating factor of the equation.

The proof for cases (iii), (iv) and (v) is a straightforward consequence of Dulac criterion because the respective divergences of the differential equations are $0, 2\,\mathrm{Re}(A)$ and $2\,\mathrm{Re}(B)$ and they either vanish identically or do not change sign. Hence no limit cycle can appear.

Case (vi) corresponds to a homogeneous QS and it is well-known that general homogeneous planar vector fields do not have limit cycles because periodic orbits never appear isolated. Moreover, when the differential equations are homogeneous of even degree, they do not have periodic orbits. □

The following result is a well-known result on QS. It is due to Coppel and its proof can be found in [8].

Theorem 3.5 ([8])**.** *Suppose a QS satisfies one of the following conditions:*

(1) it has an invariant straight line,

(2) the highest degree terms are proportional,

Then, the QS has at most one limit cycle and when it exists it is hyperbolic.

From this theorem we will obtain two key results to prove the upper bounds stated in Theorem C.

Next we prove a theorem that presents a family of QS having none or two limit cycles. This result is due to Suo Guangjian, see [19]. For the sake of completeness and because it is a work not easily accessible we include here a proof inspired by the one of the original paper. As we will see, this result is a consequence of some clever use of several changes of variables and of the previous well-known Theorem 3.5.

Theorem 3.6 ([19])**.** *The equation $\dot z = A + Bz^2 + Cz\bar z + D\bar z^2$ either does not have limit cycles or it has exactly two limit cycles, γ and $-\gamma$. Moreover, in this latter case they are hyperbolic, with different stabilities and each one of them surrounds a different critical point.*

Although it is not stated in the theorem, in Proposition 3.2 we have already seen that when one of the parameters B, C, D is zero there are differential equations of the above form having at least two limit cycles γ and $-\gamma$. As usual, given a subset $\mathcal{S}$ of $\mathbb{R}^2$, we denote as $-\mathcal{S} = \{-x \,:\, x \in \mathcal{S}\}$.

Proof of Theorem 3.6. It is convenient to write the equation in (x, y) variables, where $z = x + yi$. It reads as

$$\begin{cases} \dot x = a + a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2, \\ \dot y = b + b_{2,0}x^2 + b_{1,1}xy + b_{0,2}y^2, \end{cases} \tag{3.3}$$

where all the involved parameters are real. Obviously we can assume that it has some critical point, say (x_0, y_0), because otherwise the theorem is proved and the system does not have limit cycles. Moreover we can also suppose that $(x_0, y_0) \neq (0, 0)$, because otherwise $(a, b) = (0, 0)$ and the system is homogeneous and of even degree. As we have mentioned above, it is well-known that these systems can not have periodic orbits.

In short, the system has a critical point (x_0, y_0) with for instance $x_0 \neq 0$. By introducing the new coordinates $X = x/x_0$, $Y = y - y_0x/x_0$, the system keeps the same form but the critical point is moved to be $(X, Y) = (1, 0)$.

By renaming X and Y again as x and y, and keeping the same name for the parameters, we get that System (3.3) writes as

$$\begin{cases} \dot{x} = a - ax^2 + a_{1,1}xy + a_{0,2}y^2, \\ \dot{y} = b - bx^2 + b_{1,1}xy + b_{0,2}y^2. \end{cases} \tag{3.4}$$

Before continuing, we remark an interesting property of System (3.4) that we will use in the sequel. This property is that it is invariant by the change of variables and time $(x, y, t) \longrightarrow (-x, -y, -t)$, or in other words, that if $(x(t), y(t))$ is a solution, $(-x(-t), -y(-t))$ it is also. Hence, its limit cycles appear in couples and with stabilities interchanged. In fact, we have already used this property in the proof of Proposition 3.2. In particular, the following property holds: *System* (3.3) *cannot have only one limit cycle.*

When $b = 0$, System (3.4) is a QS with an invariant straight line, $y = 0$. Then by Theorem 3.5 it has at most one limit cycle and hence, as we have argued above, this fact implies that the systems does not have limit cycles.

Assume now that $b \neq 0$. To continue our proof, let us perform a new change of variables to System (3.4), that keeps the critical point $(1, 0)$ fixed: $X = x - ry$, $Y = y$, with $r \in \mathbb{R}$ to be determined. Then it writes again as in (3.4) but with new coefficients. After some computations we obtain that the new coefficient of Y^2 in $\dot{X}$ is

$$A_{0,2}(r) = br^3 - (a + b_{1,1})\, r^2 + (a_{1,1} - b_{0,2})\, r + a_{0,2}.$$

The polynomial $A_{0,2}(r)$ is cubic and for sure it has a real root. If we take r as one of its real roots we get that System (3.4) is transformed into

$$\begin{cases} \dot{x} = a - ax^2 + a_{1,1}xy, \\ \dot{y} = b - bx^2 + b_{1,1}xy + b_{0,2}y^2, \end{cases} \tag{3.5}$$

where for simplicity we keep the old names for the variables and coefficients.

When $a = 0$ in System (3.5), we can argue as in the case $b = 0$ above and prove that the system does not has limit cycles. So it only remains to study the case $a \neq 0$ in System (3.5). By rescaling the time by a suitable constant we have reduced the proof to study the number of limit cycles of system

$$\begin{cases} \dot{x} = 1 - x^2 + a_{1,1}xy, \\ \dot{y} = b - bx^2 + b_{1,1}xy + b_{0,2}y^2, \end{cases} \tag{3.6}$$

where, once more, we keep the same names for the parameters.

When $a_{1,1} = 0$ in System (3.6) again it does not have periodic orbits. For instance it suffices to note that the first differential equation writes as $\dot{x} = 1 - x^2$, which clearly does not have periodic solutions. When $a_{1,1} \neq 0$, by introducing the new variables $X = x$, $Y = a_{1,1}y$ we arrive to the final reduced system:

$$\begin{cases} \dot{x} = 1 - x^2 + xy, \\ \dot{y} = b - bx^2 + b_{1,1}xy + b_{0,2}y^2, \end{cases} \tag{3.7}$$

Notice that $\dot{x}\big|_{x=0} = 1 > 0$ and hence the line $x = 0$ is without contact. In particular the possible limit cycles of the system are either contained in $x > 0$ or in $x < 0$. Moreover, by the symmetry property explained above we know that in each of the half planes the number of limit cycles is the same. As a consequence the theorem will be demonstrated if we prove that in $x>0$, System (3.7) has at most one limit cycle and that when it exists it is hyperbolic.

To prove this fact we perform for System (3.7), and in the region $x>0$, a final non-linear change of variables

$$X=x^2,\quad Y=1-x^2+xy,\quad \text{with inverse}\quad x=\sqrt{X},\quad y=\frac{X+Y-1}{\sqrt{X}},$$

and a change of time $\mathrm{d}t/\mathrm{d}s=x=\sqrt{X}$ where s is the new time. In these new variables the system writes as

$$\begin{cases} X' = & 2XY,\\ Y' = & b_{0,2}+(b-2\,b_{0,2}-b_{1,1})\,X-(2\,b_{0,2}+1)\,Y+(-b+b_{0,2}+b_{1,1})\,X^2\\ & +(2\,b_{0,2}+b_{1,1}-1)\,XY+(b_{0,2}+1)\,Y^2. \end{cases} \tag{3.8}$$

Once more we have arrived to a QS with an invariant straight line $X=0$. Hence, by Theorem 3.5, it has at most one limit cycle (which is hyperbolic) and the desired result holds. □

Theorem 3.7. *The equation $\dot z = A+Bz+C\bar z+Dz\bar z$, has at most one limit cycle. Moreover, when it exists it is hyperbolic.*

Although it is not stated in the theorem, in Proposition 3.3 we have already seen that when one of the parameters A,B,C is zero there are differential equations of the above form with $D\neq 0$, having at least one limit cycle.

Proof of Theorem 3.7. By doing a rotation one can always consider that the parameter D is real. Hence, passing the complex equation to cartesian coordinates and denoting $A=a_1+a_2\mathrm{i}, B=b_1+b_2\mathrm{i}, C=c_1+c_2\mathrm{i}, D=d$, one gets:

$$\begin{cases} \dot x = a_1+(b_1+c_1)x+(-b_2+c_2)y+d(x^2+y^2),\\ \dot y = a_2+(b_2+c_2)x+(b_1-c_1)y. \end{cases}$$

The terms of highest degree of the system are $(P_2(x,y),Q_2(x,y))=(d(x^2+y^2),0)$. Applying item (2) of Coppel's Theorem 3.5 one concludes the result. □

3.3. **Proof of Theorem C.** We start proving the results in Table 1. The zeroes appearing in that table are consequence of the results of Lemma 3.4 concerning the four cases (ii)-(v). The 1's are a corollary of Theorem 3.7 and Proposition 3.3. The remaining two cases that have the symbol ≥ 1 are also a consequence of Proposition 3.3.

Let us prove the results in Table 2. The cases of the first row, that have the symbol $1+1$, follow from Theorem 3.6. All the other 6 cases are again consequence of Proposition 3.3.

3.4. **On the open cases in Table 1.** To completely end the QS case with three monomials it remains to complete the study of eight of the cases in Tables 1 and 2. Let us see that the two remaining cases of Table 1 reduce to a single one.

Lemma 3.8. *The maximum number of limit cycles ℓ_1 of the family $\dot{z} = A + B\bar{z} + Cz^2$ coincides with the maximum number of limit cycles ℓ_2 of the family $\dot{z} = Az + B\bar{z} + Cz^2$.*

Proof. Let us prove first that $\ell_1 \leq \ell_2$. If $\dot{z} = A + B\bar{z} + Cz^2$ does not have critical points then, of course, it does not have periodic orbits. Otherwise it has at least a critical point $z = z_0$. Then $A + B\bar{z}_0 + Cz_0^2 = 0$. By doing the change of variables $u = z - z_0$ it holds that

$$\dot{u} = \dot{z} = a + B\bar{z} + Cz^2 = A + B(\bar{u} + \bar{z}_0) + C(u + z_0)^2 = 2z_0Cu + B\bar{u} + Cu^2.$$

Since this new differential equation is of the form $\dot{z} = Az + B\bar{z} + Cz^2$, although with other values of A, B and C, the result follows.

The proof that $\ell_2 \leq \ell_1$ is similar and we skip the details. □

We continue studying the differential equation

$$\dot{z} = Az + B\bar{z} + Cz^2. \tag{3.9}$$

Proposition 3.9. *Let $\ell \geq 1$ be the maximum number of limit cycles surrounding the origin of any equation of type* (3.9) *with $A, B, C \in \mathbb{C}$. Then the maximum number of limit cycles of equation* (3.9) *is 2ℓ.*

Proof. If $z = 0$ is the unique critical point, there is nothing to prove. Otherwise the differential equation has a second critical point $z = z_0$. The change of variables given by the involution $z = z_0 - w$ switches the critical points $z = 0$ and $z = z_0$ and transforms equation (3.9) into

$$\dot{w} = (A + 2Cz_0)w + B\bar{w} - Cw^2.$$

This equation keeps the same monomials as the original one.

Let $z = \zeta$ be one of the critical points of Equation (3.9). As a consequence of former result, using the hypothesis of this proposition we deduce that the number of limit cycles surrounding only $z = \zeta$ is at most ℓ.

It is well-known that for QS each limit cycle surrounds exactly a single critical point and that there are at most two nests of limit cycles, see [7]. Hence the maximum number of limit cycles of each Equation (3.9) is 2ℓ. □

Indeed, in [21] the author proves that if a QS has limit cycles surrounding two different foci, then around one of them the maximum number of limit cycles is one. By using his result the global upper bound given in Proposition 3.9 could be decreased to be $\ell + 1$, but in order to be self-contained we have preferred to state and prove the above version. In fact, as we have already said, we believe that the maximum number of limit cycles of this equation will be ℓ.

Aiming to obtain ℓ we write Equation (3.9) in real variables. After several changes of variables, and keeping in mind Proposition 3.9, we prove that ℓ *is also the maximum number of limit cycles surrounding the origin of the QS*

$$\begin{cases} \dot{x} = & x + ay + cx^2 + 2xy = P(x, y), \\ \dot{y} = & x + by + cx^2 + y^2 = Q(x, y), \end{cases} \tag{3.10}$$

where $a, b, c \in \mathbb{R}$, $c \leq 0$ and $(b - 1)^2 + 4a < 0$. Notice that the second inequality ensures that the origin is a critical point of index $+1$ with complex eigenvalues. Recall, that for QS these critical points are the only ones that can be surrounded by limit cycles. The condition $c \le 0$, appears from the computations.

Although we have not obtained ℓ, we have some partial results on System (3.10) proving that in some regions of the parameter space the system has either no limit cycles or it has exactly one, that surrounds the origin. We will use next well-known proposition that allows controlling the sign of quadratic polynomials in $\mathbb{R}[x,y]$, see [9, pp. 306–308].

Proposition 3.10. *Set* $H(x,y) = px^2 + 2qxy + ry^2 + 2sx + 2ty + u$ *and define*

$$\Delta_1 = \begin{vmatrix} p & q \\ q & r \end{vmatrix}, \; \Delta_2 = \begin{vmatrix} p & s \\ s & u \end{vmatrix}, \; \Delta_3 = \begin{vmatrix} r & t \\ t & u \end{vmatrix}, \; \Delta = \begin{vmatrix} p & q & s \\ q & r & t \\ s & t & u \end{vmatrix}.$$

Then:

(i) $H(x,y) > 0$ *if and only if* $p > 0, \Delta_1 > 0$ *and* $\Delta > 0$.
(ii) $H(x,y) < 0$ *if and only if* $p < 0, \Delta_1 > 0$ *and* $\Delta < 0$.
(iii) $H(x,y) \ge 0$ *if and only if* $p \ge 0, r \ge 0, u \ge 0$, $\Delta_1 \ge 0$, $\Delta_2 \ge 0$, $\Delta_3 \ge 0$, *and* $\Delta \ge 0$.
(iv) $H(x,y) \le 0$ *if and only if* $p \le 0, r \le 0, u \le 0$, $\Delta_1 \ge 0$, $\Delta_2 \ge 0$, $\Delta_3 \ge 0$, *and* $\Delta \le 0$.

Proposition 3.11. *Consider System* (3.10) *with* $a, b, c \in \mathbb{R}$ *and* $(b-1)^2 + 4a < 0$. *The following holds:*

(i) For $c > 0$ *there are cases with at least two limit cycles surrounding the origin.*
(ii) For $c \le 0$ *there are cases with at least one limit cycle surrounding the origin.*
(iii) When $c(1+c) \ne 0$ *and the polynomial* $ry^2 + 2ty + u$ *does not change sign, where*

$$r = \frac{3(1-b-ac)}{(1+c)\,c}, \qquad u = \frac{\left(b^2-1\right)(ac-cb-1)}{2\,(1+c)\,c^2},$$
$$t = \frac{a^2c^2 - c(bc-2b+2)a + (1-b)(2bc+c+2)}{2(1+c)c^2},$$

the system does not have limit cycles.
(iv) When $(b-1)c \ne 0$ *and the polynomial* $ry^2 + 2ty + u$ *does not change sign, where*

$$r = \frac{2(2+c)(4+c)}{(b-1)c}, \; u = \frac{1+b}{c}, \; t = \frac{(b-a)c^2 + (3b+1)c + 4b}{(b-1)c},$$

the system does not have limit cycles.
(v) In a certain semialgebraic set (described in the proof) of the parameter space $\mathcal{P} \subset \{(a,b,c) \in \mathbb{R}^3 : (b-1)^2 + 4a < 0\}$, *which has open interior, the system has at most one limit cycle, and when it exists, it surrounds the origin and it is hyperbolic. Moreover, in an open subset of* $\mathcal{P} \cap \{c < 0\}$ *this limit cycle do exist.*

Proof. (i)-(ii) To prove the existence of limit cycles, as in other situations in this work, we will compute the Lyapunov quantities of the origin and apply the tools developed in the Appendix.

The divergence at the origin is $b+1$ and when $b=-1$ the first Lyapunov quantities are

$$L_1 = \frac{2\,(2-ac)\,\big(c(1+a)+1\big)}{3a^2-2a+7},$$
$$L_2 = \frac{-2\,(2-ac)\,(4\,a+3)}{(a-1)\,(5\,a^2+2\,a+17)\,(1+a)^2}, \qquad L_3 = 0.$$

When $2-ca=0$ it can be seen that the origin is a center. Moreover, when $b=-1$ the hypotheses of the theorem imply that $1+a<0$. Hence, when $2-ca\neq 0$ and $c\leq 0$ we obtain that $L_1\neq 0$ and the origin is a weak focus of order one. Then, it is easy to see that only a single limit cycle can bifurcate from the origin. Otherwise, when $2-ca\neq 0, 4a+3\neq 0$ and $c(a+1)+1=0$ we get that $L_1=0$ and $L_2\neq 0$. In this situation, two limit cycles can bifurcate from the critical point.

To prove items (iii)-(v) we follow the approaches developed in [13, 14] to look for a Dulac function that is suitable to apply Bendixson–Dulac Theorem to our system. For any $0\neq w\in\mathbb{R}$, and any $\mathcal{C}^1$ function $V(x,y)$, we have that

$$\operatorname{div}\Big(\frac{P(x,y)}{|V(x,y)|^{1/w}}, \frac{Q(x,y)}{|V(x,y)|^{1/w}}\Big) = \frac{-\operatorname{sign}(V(x,y))M_w(x,y)}{w|V(x,y)|^{1+1/w}}, \qquad (3.11)$$

where

$$M_w(x,y) = V_x(x,y)P(x,y)+V_y(x,y)Q(x,y)-w(P_x(x,y)+Q_y(x,y))V(x,y).$$

To apply Bendixson–Dulac Theorem we need the function M_w not to change sign.

To search for a suitable V, with its corresponding w, that allows to prove items (iii) and (iv), we try with functions of the form

$$V(x,y) = d+ex+fy+gx^2.$$

After some computations we get that

$$M_w(x,y) = N_w(x,y) - 2gc(w-1)x^3 - 4g(w-1)x^2y,$$

where N_w is a quadratic polynomial on x and y. To control the sign of M_w it is a necessary to cancel its cubic terms. This condition gives us two sets of plausible requirements:

$$(\mathrm{I}): w=1; \qquad (\mathrm{II}): g=0.$$

Notice that when M_w does not change sign on $\mathbb{R}^2$, the divergence neither does on $\mathcal{U} := \mathbb{R}^2 \setminus \{V(x,y)=0\}$. Moreover, since

$$M_w(x,y)|_{V(x,y)=0} = V_x(x,y)P(x,y)+V_y(x,y)Q(x,y)$$

does not change sign, the periodic orbits of System (3.10) cannot cut the set $\{V(x,y)=0\}$. Therefore, all the periodic orbits are contained in one of the connected components of $\mathcal{U}$. Because of the shape of $\{V(x,y)=0\}$ all these connected components are simply connect. Hence System (3.10) will not have periodic orbits at all. This follows by applying the usual Bendixson–Dulac Theorem, with Dulac function $|V|^{-1/w}$, to each one of them, because the theorem implies that none of these components can contain periodic orbits.

To prove our results, let us continue studying each of the cases by imposing more conditions on w and the parameters of V forcing that the corresponding M_w does not change sign.

Case (I): Recall that $w = 1$. Moreover, it is not restrictive to suppose that $g \neq 0$ because, otherwise, we are in case (II). Furthermore, there is no loss of generality in considering $g = 1$. With these assumptions we get that

$$M_1(x, y) = (1 - b - ec + fc)\, x^2 + 2\,(a - e - fc)\, xy - 3fy^2 \\ + (f - eb - 2cd)\, x + (ea - 4d - f)\, y - d(b + 1).$$

To control the sign of M_1 we will apply items (iii) and (iv) of Proposition 3.10 to $H = M_1$. We impose M_1 not to change sign because it is less restrictive than imposing it is sign-definite. From condition $p = 1-b-ec+fc = 0$, when $c \neq 0$, we fix $f = e + (b - 1)/c$. Then

$$\Delta_1 = -\left(1 + a - b - e(1 + c)\right)^2.$$

Since Δ_1 must be non-negative we force that, when $1 + c \neq 0$, $\Delta_1 = 0$ by taking $e = (1 + a - b)/(1 + c)$. Then, some computations give that

$$\Delta = \frac{3\,(ac + b - 1)\left(2\,(1 + c)\, c^2 d + (b - 1)\,(ac - cb - 1)\right)^2}{4\,(1 + c)^3\, c^3}.$$

Imposing $\Delta = 0$ we fix

$$d = \frac{(b - 1)\,(1 + cb - ac)}{2\,(1 + c)\, c^2}.$$

Fixing the previous values on d, e and f we have that, in the notation of Proposition 3.10, $p = \Delta_1 = \Delta = 0$. We compute all the other quantities appearing in that proposition and we obtain:

$$\Delta_2 = 0, \quad r = \frac{3(1 - b - ac)}{(1 + c)\, c}, \quad u = \frac{\left(b^2 - 1\right)(ac - cb - 1)}{2\,(1 + c)\, c^2},$$

and

$$\Delta_3 = ru - t^2, \text{ where } t = \frac{a^2c^2 - c(bc - 2b + 2)a + (1 - b)(2bc + c + 2)}{2(1 + c)c^2}.$$

Hence the conditions

$$r \geq 0, \quad u \geq 0 \quad \text{and} \quad \Delta_3 \geq 0$$

imply that $M_1 \geq 0$, and the conditions

$$r \leq 0, \quad u \leq 0 \quad \text{and} \quad \Delta_3 \geq 0$$

imply that $M_1 \leq 0$, as we wanted to prove. Indeed, it holds that

$$M_1(x, y) = ry^2 + 2ty + u$$

and its discriminant $4(t^2 - ru)$ is non-positive.

Case (II): Recall that $g = 0$. In this situation we take $d = 1$ and the quadratic polynomial of which we have to control the sign is

$$\begin{aligned} M_w(x, y) = {} & c\,(e + f - 2ew)\,x^2 + (2e - 2cfw - 4ew)\,xy + (1 - 4w)\,fy^2 \\ & + (e + f - ewb - 2cw - ew)\,x \\ & + (ea + bf - bfw - fw - 4w)\,y - (b + 1)w. \end{aligned}$$

By doing similar considerations to those of the previous case, we arrive at the values of w, f and e that should be taken are

$$w = -\frac{1}{c}, \quad f = \frac{2(2 + c)}{b - 1}, \quad e = \frac{2c}{1 - b}.$$

Then, we get that

$$M_w(x, y) = ry^2 + 2ty + u,$$

where

$$r = \frac{2(2 + c)(4 + c)}{(b - 1)c}, \quad u = \frac{1 + b}{c} \quad \text{and} \quad t = \frac{(b - a)c^2 + (3b + 1)c + 4b}{(b - 1)c}$$

and item (iv) is proved.

In order to prove item (v) we follow a similar approach as in the previous items. We construct a suitable Dulac function, $|V(x, y)|^{-1/w}$, but this time with V vanishing at the origin. We will prove that in a semi-algebraic region $\mathcal{P}$ of the parameters, that has open interior, the associated function $M_w(x, y)$ given in (3.11) does not change sign. As a consequence, we will prove that in $\mathcal{P}$, the system will have at most one limit cycle. Furthermore, we will show that at some points of the interior of $\mathcal{P}$ an Andronov–Hopf bifurcation takes place and hence, a hyperbolic limit cycle do exist. Let us give some details.

We consider $V(x, y)$ as:

$$V(x, y) = -x^2 + ay^2 + (1 - b)xy + \frac{1}{q}(v_3x^3 + v_2x^2y + v_1xy^2 + v_0y^3),$$

where

$$\begin{aligned} q &= (2b^2 + 9a - 5b + 2)(a - b), \\ v_3 &= -2\left(2b^2 + 3a - 3b + 1\right)(a - b)\,c - 4b^2 - 12a + 6b - 2, \\ v_2 &= \left(-2b^3 + 3ab + 7b^2 + 3a - 7b + 2\right)(a - b)\,c \\ &\quad - 2\,(b - 2)\left(2b^2 + 6a - 3b + 1\right), \\ v_1 &= 2a\,(2b - 1)\,(b - 2)\,(a - b)\,c + \left(2b^2 + 6a - 3b + 1\right)(3a + b - 2)\,, \\ v_0 &= -2a^2\,(b - 2)\,(a - b)\,c - \left(ab + 3b^2 + 13a - 7b + 2\right)a. \end{aligned}$$

With this function and $w = 1$ we compute its associated function $M_1(x, y)$, given in (3.11), and we get:

$$M_1(x, y) = \frac{1}{q}(m_4x^4 + m_3x^3y + m_2x^2y^2 + m_1xy^3 + m_0y^4), \tag{3.12}$$

being m_j polynomials in the parameters a, b, c, for $j = 1, \ldots, 4$, that we do not specify for the sake of shortness. We remark that $V(x, y)$ is constructed precisely to force that $M_1(x, y)$ is a homogeneous polynomial. This approach has been suggested to us by our colleague and friend Hector Giacomini.

As $M_1(x,y)$ is a homogeneous polynomial of degree 4, its zero set can be algebraically computed from the zeroes of $M_1(1,y)$, that is a single variable polynomial of degree 4. Indeed, the conditions that characterize when a polynomial of degree 4 does not change sign are well-known and semialgebraic in its coefficients. Hence the conditions for the above polynomial, $M_1(1,y)$, to force that it does not change sign form a semi-algebraic set $\mathcal{P}$, that is the one introduced in the statement of (v). We omit the explicit expressions of these inequalities because of their size.

Hence we can apply Bendixson–Dulac theorem for the values of $(a,b,c) \in \mathcal{P}$ to get an upper bound for the number of limit cycles of the system. To know which is this upper bound we need to study the shape of each of the connected components of $\mathcal{U} = \mathbb{R}^2 \setminus \{V(x,y) = 0\}$, see [13, 14]. To do this, we study first the shape of $\mathcal{V} := \{V(x,y) = 0\}$. Notice that in polar coordinates $V(x,y) = V_2(\theta)r^2 + V_3(\theta)r^3 = 0$, where for $j = 1,2$, $V_j(\theta)$ are homogeneous trigonometric polynomials of degree j and, moreover $V_2(\theta) < 0$ for all θ, because $(b-1)^2+4a < 0$. Hence $\mathcal{V}$ reduces to be an isolated point, the origin $(r = 0)$, and the piece (or pieces) of the unbounded curve $r = -V_2(\theta)/V_3(\theta)$ for the values of θ where $-V_2(\theta)/V_3(\theta) > 0$. In any case, this second curve does not have ovals because the denominator is of degree 3 and changes sign. As a consequence, all connected components of $\mathcal{U}$ are simple connected, but one, precisely the one having the origin as a hole. Hence, the system, when $(a,b,c) \in \mathcal{P}$, has at most one limit cycle and if it exists, it is in this holey connected component, surrounds the hole (the origin) and it is hyperbolic.

To end the proof of item (v) we only need to find an open subset of values $(a,b,c) \in \mathbb{R}^3$, that is in $\mathcal{P} \cap \{c < 0\}$ where a limit cycle exists.

For instance, if we take $a = -2, b = -1, c = -2$, it turns out that

$$M_1(1,y) = -\frac{8}{3}y^4 + \frac{32}{3}y^3 - 20y^2 + \frac{32}{3}y - \frac{8}{3}.$$

It is a straightforward computation proving that the previous polynomial is strictly negative for all values of $y \in \mathbb{R}$. The coefficients of $M_1(1,y)$ in (3.12) are rational functions in the parameters a,b,c and moreover, at the point $(a,b,c) = (-2,-1,-2)$, the function q does not vanish. In particular, these coefficients are continuous functions at this point. Hence, moving slightly (a,b,c) in a neighborhood of the point $(-2,-1,-2)$, the corresponding function $M_1(x,y)$ will not change sign. In short, (a,b,c) is in the interior of $\mathcal{P}$.

In order to prove that in the interior of $\mathcal{P}$ there exist values of the parameters exhibiting an Andronov–Hopf bifurcation, observe that for $b = -1$, the divergence of the system is zero and the first Lyapunov quantity for $a = -2, c = -2$ is $L_1 = -12/23$. If the parameter b is slightly moved in such a way that the divergence is positive (that is, $b \gtrsim -1$) then a limit cycle appears by an Andronov–Hopf bifurcation. Moreover, as we have already proved that in this region of the parameters the function $M_1(x,y)$ does not change sign, this limit cycle is unique and hyperbolic. In short, for each $\varepsilon > 0$ small enough, $(-2,-1+\varepsilon,-2)$ belongs to the interior of $\mathcal{P}$ and for a whole neighborhood of this point in $\mathcal{P}$ our system has exactly one limit cycle, which surrounds the origin and is hyperbolic.

Of course, open sets in $\mathcal{P}$ with the same property can be constructed starting at points $(a,-1,c)$ where the system has a weak focus of order one ($L_1 \neq 0$), $q \neq 0$, and the corresponding $M_1(1,y)$ does not vanish.

We end this proof by illustrating how to determine the boundaries of $\mathcal{P}$ in a particular case. As an example, we study the set $\mathcal{P} \cap \{(a,b,c) \in \mathbb{R}^2 : a = c = -2\}$. For these values of the parameters, the condition $(b+1)^2+4c<0$ is equivalent to saying that $b \in (b^-,b^+) = (1-2\sqrt{2}, 1+2\sqrt{2}) \simeq (-1.828, 3.828)$. Next we have to characterize whether the polynomial $M_1(1,y)$ given in (3.12) does not change sign. We define $N_b(y)$ as the numerator of $M_1(y)$ and, once more, for simplicity we do not give it explicitly. Following the approach developed in [11, App. II] we compute the discriminant of the polynomial N_b with respect to b. We get that it is a polynomial of degree 22 in b, being one of its factors

$$4b^{10} + 88b^9 + 417b^8 + 139b^7 - 3629b^6 - 8928b^5 \\ - 1393b^4 + 19868b^3 + 19194b^2 - 2635b - 1525.$$

By computing its Sturm sequence we know that it has has six real zeroes, being all of them simple. Their approximate values are

$$\begin{array}{lll} b_1 \simeq -15.699, & b_2 \simeq -2.452, & b_3 \simeq -0.247, \\ b_4 \simeq 0.306, & b_5 \simeq 1.537, & b_6 \simeq 2.792. \end{array}$$

From these values, and by using [11, Prop 6] we obtain that

$$\mathcal{P} \cap \{(a,b,c) \in \mathbb{R}^2 : a = c = -2\} = (b^-, b_3] \cup [b_5, b_6].$$

Hence, for $a = c = -2$ and these values of b, which of course include the value $b = -1$, the system has at most one limit cycle, which when exists it is hyperbolic. $\square$

3.5. **Other approaches to generate limit cycles for QS.** In this section we try other approaches to get limit cycles for our families. None of them has provided more limit cycles than the ones given in Theorem C.

3.5.1. *Higher order Melnikov functions.* Consider a perturbation of a hamiltonian system of the form

$$\begin{cases} \dot{x} = -H_y + \sum_{j=1}^M \varepsilon^j P_j(x,y), \\ \dot{y} = H_x + \sum_{j=1}^M \varepsilon^j Q_j(x,y), \end{cases}$$

with a center at the origin when $\varepsilon = 0$. We parameterize a transversal section near the origin by the energy $H(x,y) = h$, $h \in (0,h_0)$, where $h = 0$ corresponds to the origin. For ε small enough the origin keeps being a center and its return map R writes as

$$R(h,\varepsilon) = h + \epsilon^k M_k(h) + O_{k+1}(\varepsilon),$$

where M_k is not identically zero and it is called the k-th Melnikov function. It is known that each simple zero $h = h^* \in (0,h_0)$ of M_k gives rise to a hyperbolic limit cycle for $|\varepsilon|$ small enough, that tends to the oval $H(x,y) = h^*$ when ε tends to 0, see [5, 18] for more details.

In general, it is konwn that by imposing $M_1 = .. = M_{n-1} = 0$ and studying the number of simple zeroes of M_n, this number increases (or at least does not decrease) when n increases and, so does the number of limit cycles of the family. We will apply this approach to

$$\dot{z} = \mathrm{i}z + \sum_{j=1}^{6} \varepsilon^j F_j(z, \bar{z}) \tag{3.13}$$

where $F_j(z, \bar{z}), j = 1, 2, \ldots, 6$ are of one of the following forms:

(i) $A_j z + B_j z^2 + C_j z\bar{z}$, (ii) $A_j z + B_j z^2 + C_j \bar{z}^2$, (iii) $A_j z + B_j z\bar{z} + C_j \bar{z}^2$.

After many computations, following the method proposed in [18] we get the following result:

Proposition 3.12. *The maximum numbers of positive simple zeroes of the function* $M_j(h), j = 1, 2, \ldots 6$, *associated to* (3.13) *are, respectively:*

(i) *For case (i):* $0, 1, 1, 1, 1, 1$ *zeroes.*
(ii) *For case (ii):* $0, 0, 0, 0, 0, 0$ *zeroes.*
(iii) *For case (iii):* $0, 0, 0, 1, 1, 1$ *zeroes.*

Hence at most one limit cycle surrounding the origin for Equation (3.13) *is obtained by this approach.*

It is worth commenting that doing similar computations but for the complete QS (1.1) the number of limit cycles obtained are: $0, 1, 1, 2, 2, 3$, giving the maximum number of limit cycles known surrounding a critical point of a QS, see [20].

Notice that although case (ii) does not produce limit cycles around the origin, the differential equations with these three monomials do have at least two limit cycles surrounding other critical points, see Table 2.

3.5.2. *Perturbation of non-linear centers.* A similar approach that in the previous subsection could be applied by studying perturbations of QS with two monomials and a center at the origin like for instance:

$$\dot{z} = \mathrm{i}z + Bz^2, \quad \dot{z} = \mathrm{i}z + Bz\bar{z}, \quad \dot{z} = \mathrm{i}z + \mathrm{i}b_2\bar{z}^2,$$

where $B \in \mathbb{C}$ and $b_2 \in \mathbb{R}$. Although these centers are not hamiltonian they admit explicit integrating factors, see for instance [15].

We focus our attention on the first case with $B = 1$, that has simultaneously two centers at $z = 0$ and $z = -\mathrm{i}$, and we study the simultaneous bifurcation from both sets of periodic orbits. More concretely, we will compute the first order Melnikov functions associated to the period annuli of $z = 0$ and $z = -\mathrm{i}$, that we will call $\mathcal{M}$ and $\mathcal{N}$, respectively. We will follow the method developed in [12]

Firstly, we consider the differential equation

$$\dot{z} = \mathrm{i}z + z^2 + \varepsilon\big(A + Bz + C\bar{z} + Dz^2 + Ez\bar{z} + F\bar{z}^2\big) \tag{3.14}$$

and, as usual, we write

$$\begin{aligned} A =& a_1 + \mathrm{i}a_2, \quad B = b_1 + \mathrm{i}b_2, \quad C = c_1 + \mathrm{i}c_2, \\ D =& d_1 + \mathrm{i}d_2, \quad E = e_1 + \mathrm{i}e_2, \quad F = f_1 + \mathrm{i}f_2, \end{aligned}$$

with the twelve parameters being real constants.

The key point of the computations in [12] is that the birrational transformation $w = z/(1 + z)$ transforms the holomorphic differential equation

$\dot{z} = \mathrm{i}z + z^2$ into the linear isochronous center $\dot{w} = \mathrm{i}w$, and Equation (3.14) into a rational perturbation of the linear center. In this way, after several computations, we get the first Melnikov function $\mathcal{M}$ associated to the basin of attraction of $z = 0$, obtaining that $\mathcal{M}(\rho) = -2\pi\rho^2 M(\rho)$, where

$$M(\rho) = (c_1 + e_2)\,\rho^2 + 2a_2 - b_1, \tag{3.15}$$

with $0 < \rho < 1$.

Since the change of variables $Z = -z - i$ transforms the differential equation $\dot{z} = \mathrm{i}z + z^2$ into $\dot{Z} = \mathrm{i}Z + Z^2$ and interchanges the critical points $z = 0$ and $z = -\mathrm{i}$, we can use it to obtain $\mathcal{N}$ from $\mathcal{M}$. After some computations we arrive to $\mathcal{N}(\rho) = -2\pi\rho^2 N(\rho)$, where

$$N(\rho) = (-c_1 + 2f_2)\,\rho^2 + 2a_2 - b_1 + 2c_1 + e_2 - 2f_2, \tag{3.16}$$

and again $\rho \in (0, 1)$. In this way it is easy to chose the parameters in (3.14) in such a way that M and N have in $(0, 1)$ either 1 and 0, or 0 and 1, or 1 and 1 simple zeroes, respectively. These results give rise to the configurations 1, or $1 + 1$ for the number of limit cycles of Equation (3.14), recovering known results about this equation.

We want to particularize the above results when in (3.14) there are only three monomials. A case by case study shows that the configuration $1 + 1$ never appears in these situations, because never both function M and N have simultaneously 1 zero in $(0, 1)$.

We only detail one of the cases. Consider $A = E = F = 0$. Then Equation (3.14) has only the three monomials $\{z, \bar{z}, z^2\}$ and writes as

$$\dot{z} = \mathrm{i}z + z^2 + \varepsilon\big(Bz + C\bar{z} + Dz^2\big).$$

Hence

$$M(\rho) = c_1\rho^2 - b_1, \quad N(\rho) = -c_1\rho^2 - b_1 + 2c_1.$$

Finally, notice that if $\rho = \sqrt{b_1/c_1}$ is a zero of M in $(0, 1)$ then the zero of N, that is $\sqrt{2 - b_1/c_1}$, can not be in $(0, 1)$, because $0 < b_1/c_1 < 1$ implies that $\sqrt{2 - b_1/c_1} > 1$. In other words, the configuration $1 + 1$ never appears by using the first order Melnikov function for both centers.

Appendix: General expression of the first Lyapunov quantity L_1 and Adronov–Hopf type bifurcations

In Theorems D and E we give, in real or complex coordinates respectively, the general expression of the first Lyapunov quantity L_1 (sometimes also called V_3) of the origin when it is a weak focus but it is not written in any special normal form. We recover the formula for L_1 given in [2, p. 253] calculated by a different method. In that book L_1 was obtained by computing the Taylor's series of the return map near the origin while our approach uses a small modification of Lyapunov method.

We thank our colleague and friend Joan Torregrosa who gave us the key idea to compute L_1, and also all subsequent $L_j, j \geq 2$, by using a clever modification of Lyapunov procedure to find a local suitable Lyapunov function.

In our proof we will only give some details of how we have obtained L_1, but the same ideas work for any $L_j, j \geq 2$. In fact we have prepared a Maple's code that obtains several L_j in a fast an efficient way.

Recall that a weak focus is a critical point of index $+1$ such that the divergence of the vector field on it is zero.

Theorem D. *Consider a $\mathcal{C}^4$ planar differential equation defined in a neighborhood of the origin:*

$$\begin{cases} \dot{x} = ux + vy + \sum_{j+k=2}^{3} a_{j,k} x^j y^k + O_4(x,y) = P(x,y), \\ \dot{y} = wx - uy + \sum_{j+k=2}^{3} b_{j,k} x^j y^k + O_4(x,y) = Q(x,y), \end{cases}$$

where all the involved parameters are real, $w > 0$, $u^2 + vw < 0$, and $O_4(x,y)$ denotes terms of order at least 4. Then the origin is a weak focus and its first Lyapunov quantity is

$$L_1 = \frac{L}{4u^2 + 3v^2 - 2vw + 3w^2}, \tag{3.17}$$

where

$$\begin{aligned} L = \; & \big(a_{1,1}a_{2,0} - b_{0,2}b_{1,1}\big)(2u^2 - vw) \\ & + \big(a_{1,1}b_{2,0} - 2{a_{2,0}}^2 + a_{2,0}b_{1,1} + 2b_{0,2}b_{2,0} + {b_{1,1}}^2\big)\, uv \\ & + \big(2a_{0,2}a_{2,0} + a_{0,2}b_{1,1} + {a_{1,1}}^2 + a_{1,1}b_{0,2} - 2{b_{0,2}}^2\big)\, uw \\ & - b_{2,0}\,(2a_{2,0} + b_{1,1})\, v^2 + a_{0,2}\,(a_{1,1} + 2b_{0,2})\, w^2 \\ & - \big(2\,(a_{2,1} + b_{1,2})\, u - (3a_{3,0} + b_{2,1})\, v + (a_{1,2} + 3b_{0,3})\, w\big)(u^2 + vw). \end{aligned}$$

Proof. Before particularizing to our planar system we explain the general method for obtaining several Lyapunov quantities L_j. Write the differential system as

$$\begin{cases} \dot{x} = \sum_{m=1}^{N} P_m(x,y) + O_{N+1}(x,y) = P(x,y), \\ \dot{y} = \sum_{m=1}^{N} Q_m(x,y) + +O_{N+1}(x,y) = Q(x,y), \end{cases}$$

where $P_1(x,y) = ux + vy$, $Q_1(x,y) = wx - uy$, $w > 0$, $u^2 + vw < 0$, P_m and Q_m are homogeneous polynomials of degree m and N is big enough. Consider $H(x,y) = \sum_{k\geq 2} H_k(x,y)$, where

$$H_2(x,y) = -\frac{v}{2}y^2 - uxy + \frac{w}{2}x^2,$$

and H_k are homogeneous polynomials of degree k. Notice that H_2 is a first integral of the linear part of the above system which corresponds to a center and it is positive definite.

Then the Lyapunov's method consists in proving that there exist $H_k, k \geq 3$ (not unique), such that

$$H_x(x,y)P(x,y) + Q(x,y)H_y(x,y) = \sum_{m=1}^{M} L_m (x^2+y^2)^{m+1} + O_{2M+3}(x,y).,$$

for a suitable M, where recall that $H_x P + H_y Q = \dot{H}$. Independently of the choice of the polynomials H_k, the values $L_m, m \geq 1$, are called Lyapunov quantities. Moreover, for general vector fields with a focus at the origin, L_0

is by definition the real part of the eigenvalues associated to the equilibrium point, and it is a positive multiple of the divergence of the vector field at the origin. If $L_0 \neq 0$ its sign gives the stability of the origin.

It is clear by the classical theory of Lyapunov functions that the sign of the first non-zero $L_j, j \geq 1$ also controls the stability of the origin. In fact, to simplify the computations, the functions L_j are usually reduced, giving expressions of them when all the previous ones L_k, $k < j$, vanish, because only in this situation they have a dynamical meaning.

Moreover, if the system is analytic and all the Lyapunov quantities vanish, according to the Poincaré's linearization theorem, it is locally integrable and the origin is a center, see for instance [23].

Let us start computing L_1. Since H_2 is already known, the function H_3 is determined by imposing that

$$(H_2 + H_3)_x P + (H_2 + H_3)_y Q = O_4(x, y).$$

Because $(H_2)_x P_1 + (H_2)_y Q_1 = 0$ we get that H_3 has to satisfy

$$(H_3)_x P_1 + (H_3)_y Q_1 = -(H_2)_x P_2 - (H_2)_y Q_2,$$

where notice that the right hand side is known and the equality provides a linear system for the coefficients of H_3. It is compatible and determined and has a unique solution. In fact, $H_3(x,y) = \sum_{j+k=3} h_{j,k} x^j y^k$ where

$$\begin{aligned}
h_{3,0} &= \frac{u^2 b_{2,0} - 2uwa_{2,0} - uwb_{1,1} + vwb_{2,0} - w^2 a_{1,1} - 2\,w^2 b_{0,2}}{3(u^2+vw)},\\
h_{2,1} &= \frac{u^2 a_{2,0} + u^2 b_{1,1} + uwa_{1,1} + 2uwb_{0,2} - vwa_{2,0}}{u^2+vw},\\
h_{1,2} &= -\frac{u^2 a_{1,1} + u^2 b_{0,2} - 2uva_{2,0} - uvb_{1,1} - vwb_{0,2}}{u^2+vw},\\
h_{0,3} &= -\frac{u^2 a_{0,2} + uva_{1,1} + 2uvb_{0,2} - 2v^2 a_{2,0} - v^2 b_{1,1} + vwa_{0,2}}{3(u^2+vw)}.
\end{aligned}$$

Now fixed H_2 and H_3, and following Lyapunov idea, we look for H_4 such that

$$(H_2 + H_3 + H_4)_x P + (H_2 + H_3 + H_4)_y Q = L_1(x^2+y^2)^2 + O_5(x, y).$$

In this case, we need to add a new unknown L_1 because without it the linear system associated to the above equality, with unknowns the coefficients of H_4, would be incompatible. With this trick we obtain several possibilities for H_4 but a given quantity corresponding to L_1. This is so, because it can be seen that the solution of the associated linear system is not unique. By taking any of the solutions we obtain a good H_4 and the expression L_1 of the statement of the theorem. We have preferred to not include the details due to the length of the involved expressions.

This procedure can be continued if N is big enough and turns out that the H_{2k+1} are always uniquely determined while we have some freedom to choose the H_{2k}. In any case, any choice is good for obtaining useful L_j. □

In next corollary we state the classical Andronov–Hopf bifurcation theorem adapted to our setting.

Corollary 3.13 (Andronov–Hopf bifurcation)**.** *Consider the 1-parametric family of differential equations*

$$\begin{cases} \dot{x} = P(x,y) + rx = (u+r)x + vy + O_2(x,y), \\ \dot{y} = Q(x,y) = wx - uy + O_2(x,y), \end{cases}$$

where P and Q are the ones given in Theorem D and $r \in \mathbb{R}$. Then, if L is the expression appearing in that theorem, then:

(i) When $r = 0$ and $L \neq 0$, the weak focus $(0,0)$ is an attractor (resp. a repellor) if $L > 0$ (resp. $L < 0$).

(ii) If $r \neq 0$ the origin is strong focus with non-zero divergence r. Hence it is an attractor when $r < 0$ and a repellor when $r > 0$. Moreover, if $rL < 0$ and $|r|$ is small enough, in a sufficiently small neighborhood of the origin, the system has a unique limit cycle surrounding it, which is hyperbolic and with opposite stability to the one of the origin.

Remark 3.14. (i) In the statement of Corollary 3.13 the expression L can be replaced by L_1.

(ii) Notice that when the origin is a weak focus written in normal form, that is $u = 0$, $v = -1$, $w = 1$, then $L_1 = L/8$, where

$$\begin{aligned} L =& a_{1,1}a_{2,0} - b_{0,2}b_{1,1} + a_{0,2}\,(a_{1,1} + 2b_{0,2}) - b_{2,0}\,(2a_{2,0} + b_{1,1}) \\ &+ 3a_{3,0} + a_{1,2} + b_{2,1} + 3b_{0,3}. \end{aligned}$$

The above expression of L, or the one corresponding to $u = 0$ and $v = -w$, are the ones appearing in most text books, see for instance [2].

For completeness we also include some words about more degenerate Andronov–Hopf type bifurcations because they can give rise to more limit cycles. Consider a smooth enough parametric family of vector fields, with parameters $\lambda \in \Lambda \subset \mathbb{R}^j$ and having a weak focus at the origin, $\dot{x} = P(x,y,\lambda)$, $\dot{y} = Q(x,y,\lambda)$. Compute several Lyapunov quantities, $L_j, j \geq 1$, that are functions of the parameters λ of the family. Moreover, if these functions satisfy:

(c1) for some $m = M \geq 1$ and some $\lambda = \lambda^*$, $L_M(\lambda^*) \neq 0$ and $L_j(\lambda^*) = 0$ for all $j < M$,

(c2) the map defined on a neighborhood of $\lambda = \lambda^*$,

$$\lambda \to (L_1(\lambda), \ldots, L_{M-1}(\lambda))$$

fills a complete neighborhood of the origin,

(c3) and we add a new independent real parameter λ_0 that controls the sign of L_0, for instance like $\dot{x} = P(x,y,\lambda) + \lambda_0 x$, $\dot{y} = Q(x,y,\lambda)$,

then this extended family presents a degenerate Andronov–Hopf bifurcation for $(\lambda, \lambda_0) = (\lambda^*, 0)$ at the origin. In particular, there are differential systems in the family having at least M hyperbolic limit cycles in a small neighborhood of the origin and surrounding it, see for instance [6] and the references therein.

Next result presents the expression of L_1 given in Theorem D when the initial differential equation is written in complex coordinates.

Theorem E. *Consider a $\mathcal{C}^4$ real planar vector differential equation defined in a neighborhood of the origin:*

$$\dot{z} = Rz + S\bar{z} + Az^2 + Bz\bar{z} + C\bar{z}^2 + Dz^3 + Ez^2\bar{z} + Fz\bar{z}^2 + G\bar{z}^3 + O_4(z, \bar{z})$$

where all the involved parameters are complex, $R = r_1 + \mathrm{i}r_2$, $S = s_1 + \mathrm{i}s_2$. When $r_1 = 0$, $S\bar{S} - R\bar{R} < 0$ and $\operatorname{Im}(R + S) > 0$ the origin is a weak focus and its first Lyapunov quantity is

$$L_1 = \frac{\operatorname{Im}(M)}{2R\bar{R} + S\bar{S}}, \tag{3.18}$$

where

$$\begin{aligned} M = &\big(2RE - S(3D + \bar{F})\big)(R\bar{R} - S\bar{S}) + \big(AB + 2\bar{A}C + BC\big)\bar{S}^2 \\ &+ \big(2AC - 2A^2 + \bar{A}B + B^2 + \bar{B}C\big)R\bar{S} - A^2(S - \bar{S})(R - \bar{R}) \\ &- AB\big(2R\bar{R} + S\bar{S} + \bar{S}^2\big). \end{aligned}$$

Remark 3.15. (i) Notice that when the origin is a weak focus written in normal form, that is $R = \mathrm{i}$ and $S = 0$, then $L_1 = \operatorname{Re}(E) - \operatorname{Im}(AB)$, a well-known and nice expression, see for instance [16].

(ii) In the notation and hypotheses of Theorem D the bifurcation of Andronov–Hopf happens when instead of $r_1 = \operatorname{Re}(R) = 0$ we take $|r_1| \neq 0$ small enough and $\operatorname{Re}(R)\operatorname{Im}(M) < 0$.

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