MORE LIMIT CYCLES FOR COMPLEX DIFFERENTIAL EQUATIONS WITH THREE MONOMIALS

M.J. ÁLVAREZ, B. COLL, A. GASULL, AND R. PROHENS

Abstract. In this paper we improve, by almost doubling, the existing lower bound for the number of limit cycles of the family of complex differential equations with three monomials, $\dot{z} = Az^k\bar{z}^l + Bz^m\bar{z}^n + Cz^p\bar{z}^q$, being k, l, m, n, p, q non-negative integers and $A, B, C \in \mathbb{C}$. More concretely, if $N = \max(k+l, m+n, p+q)$ and $H_3(N) \in \mathbb{N}\cup\{\infty\}$ denotes the maximum number of limit cycles of the above equations, we show that for $N \geq 4$, $H_3(N) \geq N-3$ and that for some values of N this new lower bound is $N+1$. We also present examples with many limit cycles and different configurations. Finally, we show that $H_3(2) \geq 2$ and study in detail the quadratic case with three monomials proving in some of them non-existence, uniqueness or existence of exactly two limit cycles.

1. Introduction and Main Results

In this paper we study lower bounds for the number of limit cycles of complex differential equations with three monomials,

$$\dot{z} = Az^k\bar{z}^l + Bz^m\bar{z}^n + Cz^p\bar{z}^q,$$

with k, l, m, n, p, q non-negative integers and $A, B, C \in \mathbb{C}$. Let $N = \max(k+l, m+n, p+q)$ and denote by $H_3(N) \in \mathbb{N} \cup \{\infty\}$ the maximum number of limit cycles of the above equations.

It is known that when $ABC = 0$ then the maximum number of limit cycles is 1, see [1]. It is also known that for $N \geq 3$ odd, $H_3(N) \geq (N+3)/2$, see [17]. Moreover, in the given differential equations reaching these bounds, each one of these limit cycles surrounds a different critical point. In fact, in [10] one more limit cycle is proved to exist and it surrounds all the other limit cycles, showing that $H_3(N) \geq (N+3)/2 + 1$.

In this work, we prove that the existing lower bound can in fact be almost doubled, see next Theorems A and B. Moreover, while the essential techniques used in [17] are the rotational symmetries and the Abelian integrals, in this paper we also use the computation of the Lyapunov quantities and the properties of the transformation $w = z^n$.

As a matter of fact, we need to compute in many situations the first Lyapunov quantity, L_1, for a weak focus that is not in the usual normal form, namely $\dot{z} = \alpha iz + O_2(z, \bar{z})$, with $0 \neq \alpha \in \mathbb{R}$. For this reason we have decided to include an appendix where L_1 is given in full generality. The expression that we obtain coincides with the one of the classical book [2, p. 253] attributed to Bautin. Moreover, for the sake of completeness,

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