The 16th Hilbert problem
A simple version of algebraic limit cycles

Xiang Zhang

Shanghai Jiao Tong University

xzhang@sjtu.edu.cn

New Trends in Dynamical Systems

Salou, Spain, October 1-5, 2012

Talk in October 2 at 15:30

Outline of the talk

- Background of the problem.
- Statement of the main results.
- Sketch proof of the main results.
- Open questions.

Background of the problem

- Hilbert's 16th problem consists of two similar problems in different branches of mathematics:
  - The first part of the problem:
    is to study the relative position of the branches of real algebraic curves of degree n (and similarly for algebraic surfaces).
  - The second part of the problem:
    is to determine the upper bound for the number of limit cycles in polynomial vector fields of degree n and an investigation of their relative positions.

Root of the first part

- In 1876 Harnack investigated algebraic curves and found that curves of degree n could have no more than

$$\frac{n^2-3n+4}{2}$$

separate components in the real projective plane.
- Furthermore he showed how to construct curves that attained that upper bound, and thus that it was the best possible bound.
- Curves with that number of components are called M-curves.

- Hilbert had investigated the M-curves of degree 6, and found that the 11 components always were grouped in a certain way.
- His challenge to the mathematical community now was to completely investigate the possible configurations of the components of the M-curves.
- This is the first part of the Hilbert's 16th problem.

ALGEBRAIC GEOMETRY

A simple version of Hilbert's 16th problem

- It is well known that the Hilbert's 16th problem remains open for both of the two parts

Connection between the two parts: algebraic limit cycles

- Algebraic limit cycle is a limit cycle which is contained in an oval of an invariant algebraic curve.
- Invariant algebraic curve is an algebraic curve which is invariant under the flow of a give vector field
- Algebraic curve is the set of zeros of a polynomial.

On algebraic limit cycles of planar polynomial differential systems

- There appears a simple version of the Hilbert' 16th problem (see Llibre *et al*, JDE 248 (2010), 1401–1409.):

  Is there an upper bound on the number of algebraic limit cycles of all real planar polynomial vector fields of a given degree?

History on this simple version

Quadratic differential systems: the results are rich, the problem remains open too

- Algebraic limit cycles of degree 2:

  Qin Yuanxun [1958] Acta Math. Sinica **8** 23 – 35: proved the uniqueness of limit cycles for quadratic differential systems (QDS) having an algebraic limit cycle of degree 2.

- Algebraic limit cycles of degree 3:

  Evdokimenco[1970,1974, 1979] Differential Equations **6** 1349–1358; **9** 1095–1103, **15** 215–221: proved the nonexistence of algebraic limit cycles of degree 3 for quadratic differential systems,

  For simple proofs, see for instance

  - Chavarriga, Llibre and Moulin Ollagnier [2001] LMS J. Comput. Math. **4** 197 – 210, or
  - Z [2003] Sci. China Ser. A **46** 271–279.

- Algebraic limit cycles of degree 4:
  - Yablonskii[1966] Differential equations **2** 335–344: found the first family of QDS having an algebraic limit cycle of degree 4
  - Piliptsov[1973] Differential Equations **9** 983–986: found a new family of QDS having an algebraic limit cycle of degree 4
  - Chavarriga[1999] A preprint: found a third family of QDS having an algebraic LC of degree 4
  - Chavarriga, Llibre and Sorolla [2004] JDE **200**, 206–244: Completed the classification of QDS having an algebraic LC of degree 4
  - Chavarriga, Giacomini and Llibre [2001] JMAA **261** 85–99: proved the uniqueness of limit cycles for QDS having an algebraic LC of degree 4.

Remark:

- For QDS the classification and uniqueness of ALC of degree 2,3,4 were done.
- In all the other cases for concrete systems, there are only partial results.

Recent general results on simple version

- Llibre, Ramirez and Sadovskaia [2010] JDE **248** 1401–1409: proved that

  For a real planar polynomial vector field of degree m having all its irreducible invariant algebraic curves generic, the maximal number of algebraic limit cycles is at most

  - $1+(m-1)(m-2)/2$ if m is even, and
  - $(m-1)(m-2)/2$ if m is odd, and
  - the upper bounds can be reached.

For generic invariant algebraic curves,

Generic is defined by satisfying the 5 conditions:

- All the curve $f_j = 0$ are nonsingular, (i.e. there are no points of $f_j = 0$ at which f_j and its first derivative all vanish);
- The highest order homogeneous terms of f_j have no repeated factors;
- If two curves intersect at a point in the affine plane, they are transversal at this point;
- There are no more than two curves $f_j = 0$ meeting at any point in the affine plane;
- There are no two curves having a common factor in the highest order homogeneous terms.

For only nonsingular invariant algebraic curves,

- Llibre, Ramirez and Sadovskaia [2011] JDE **250** 983–999: proved that

  For a real planar polynomial vector field of degree m having all its irreducible invariant algebraic curves nonsingular, the maximal number of algebraic limit cycles is at most $m^4/4+3m^2/4+1$.

Remak:

- Both of the results by Llibre *et al* require a sufficient condition that all the invariant algebraic curves of a prescribed vector field are nonsingular, and so they cannot be self – intersected.
- They have a conjecture: Is $1+(m-1)(m-2)/2$ the maximal number of algebraic limit cycles that a polynomial vector field of degree m can have?

Statement of our results

Consider real planar polynomial vector fields of degree m

$$\mathscr{X} = p(x,y)\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y},$$

with

- $p(x,y), q(x,y) \in \mathbb{R}[x,y]$ the ring of real polynomials in x, y and
- $m = \max\{\deg p, \deg q\}$

Our results will

- verify the conjecture by Llibre *et al* in JDE [2010] in a very general conditions.

Theorem 1

If a real planar polynomial vector field $\mathscr{X}$ of degree m has only nodal invariant algebraic curves taking into account the line at infinity, then the following hold.

(a) The maximal number of algebraic limit cycles of the vector fields is at most

- $1+(m-1)(m-2)/2$ when m is even, and
- $(m-1)(m-2)/2$ when m is odd.

(b) There exist vector fields $\mathscr{X}$ which have the maximal number of algebraic limit cycles.

Recall that

- an algebraic curve S (not necessary irreducible) is nodal if all its singularities are of normal crossing type, that is at any singularity of S there are exactly two branches of S which intersect transversally.

Remark that

- Our assumptions are on the singularities of S.
- The assumptions only satisfy the third and fourth conditions of the generic conditions in Llibre *et al* [JDE 2010].

Next we consider more general real planar polynomial vector fields of degree m

$$\mathscr{Y} = (p(x,y) + xr(x,y))\frac{\partial}{\partial x} + (q(x,y) + yr(x,y))\frac{\partial}{\partial y},$$

with

- $p(x,y), q(x,y), r(x,y) \in \mathbb{R}[x,y]$ the ring of real polynomials in x, y and
- $m =$ max$\{\deg p, \deg q, \deg r\}$

Theorem 2

If a real planar polynomial vector field $\mathscr{Y}$ of degree m has all its invariant algebraic curves non-dicritical, then the following hold.

(a) If $r(x,y) \equiv 0$, the maximal number of algebraic limit cycles of the vector fields is at most

- $1+m(m-1)/2$ when m is even, and
- $m(m-1)/2$ when m is odd.

(b) If $r(x,y) \not\equiv 0$, the maximal number of algebraic limit cycles of the vector fields is at most

- $1+(m+1)m/2$ when m is even, and
- $(m+1)m/2$ when m is odd.

Note: these results can be found in [JDE, 2011] by Z.

Recall that

- A singularity of a vector field is non–dicritical if there are only finitely many invariant integral curves passing through it.
- Otherwise, the singularity is called dicritical.
- An invariant algebraic curve is non–dicritical if there is no dicritical singularities on it.
- Clearly, a non–dicritical algebraic curve can be singular.

Tools of the proof

Invariant algebraic curves and their degree

Consider a holomorphic singular foliation $\mathscr{F}$ of degree m.

- In the projective coordinates, $\mathscr{F}$ can be written as the closed one–form

$$\widetilde{\omega} = P(X,Y,Z)dX + Q(X,Y,Z)dY + R(X,Y,Z)dZ,$$

with $P,Q,R \in \mathbb{C}[X,Y,Z]$ homogeneous polynomials of degree $m+1$ satisfying the projective condition

$$XP + YQ + ZR = 0.$$

- In the affine coordinates, $\mathscr{F}$ can be written as the one-form

$$\omega = -(q(x,y)+yr(x,y))dx+(p(x,y)+xr(x,y))dy,$$

or as the vector field

$$\mathscr{Y} = (p(x,y)+xr(x,y))\frac{\partial}{\partial x}+(q(x,y)+yr(x,y))\frac{\partial}{\partial y},$$

with $p,q,r \in \mathbb{C}[x,y]$ and $\max\{\deg p, \deg q, \deg r\} = m$ and $r(x,y)$ homogeneous polynomial of degree m or is naught.

Definition:

- An algebraic curve S defined by a reduced homogeneous polynomial $F(X,Y,Z) \in \mathbb{C}[X,Y,Z]$ is called invariant by $\mathscr{F}$ if $\widetilde{\omega} \wedge dF = F\theta$, where θ is a two - form.
- a reduced polynomial is the one which has no repeat factors.

Note:

- It is easy to prove that F is an invariant algebraic curve of $\mathscr{F}$ if and only if $Xf = kf$ for some $k \in \mathbb{C}[x,y]$, where $f = F|_{Z=1}$.

The known results will be used in the proof of our results:

Lemma 1.1: Cerveau and Lins Neto Theorem [Ann. Inst. Fourier 1991]

Let $\mathscr{F}$ be a foliation in $\mathbb{C}P(2)$ of degree m, having S as a nodal invariant algebraic curve with the reduced homogeneous equation $F = 0$ of degree n. Then

- $n \leq m+2$
- If $n = m+2$ then F is reducible and the foliation $\mathscr{F}$ is of logarithmic type, that is given by a rational closed form $\sum_i \lambda_i \frac{dF_i}{F_i}$, where $\lambda_i \in \mathbb{C}$ and F_i are the irreducible homogeneous components of F and $\sum_i \lambda_i \mathrm{deg} F_i = 0$.

Lemma 1.2: Harnack's Theorem

- The number of ovals of a real irreducible algebraic curve of degree n is at most

$$1+(n-1)(n-2)/2-\sum_{p} v_p(S)(v_p(S)-1), \text{ if } n \text{ is even,}$$

or

$$(n-1)(n-2)/2-\sum_{p} v_p(S)(v_p(S)-1), \text{ if } n \text{ is odd,}$$

where p runs over all the singularities of $\mathscr{F}$ on S, and $v_p(S)$ is the order of S at the singular point p.
- Moreover these upper bounds can be reached for convenient algebraic curves of degree n.

Lemma 1.3: Giacomini, Llibre and Viano [Nonlinearity 1996]

Let $\mathscr{X}$ be a C^1 vector field defined in the open subset U of $\mathbb{R}^2$, and let $V : U \to \mathbb{R}$ be an inverse integrating factor of $\mathscr{X}$.

- If γ is a limit cycle of $\mathscr{X}$, then γ is contained in $\{(x,y) \in U : V(x,y) = 0\}$.

Ideal of the proof of Theorem 1

Recall Theorem 1

Theorem 1

If a real planar polynomial vector field $\mathscr{X}$ of degree m has only nodal invariant algebraic curves taking into account the line at infinity, then the following hold.

(a) The maximal number of algebraic limit cycles of the vector fields is at most

- $1+(m-1)(m-2)/2$ when m is even, and
- $(m-1)(m-2)/2$ when m is odd.

(b) There exist vector fields $\mathscr{X}$ which have the maximal number of algebraic limit cycles.

- Write system $\mathscr{X}$ in the one–form

$$q(x,y)dx - p(x,y)dy.$$

Its projective one–form is

$$\omega_0 = ZQdX - ZPdY + (YP - XQ)dZ,$$

where X, Y, Z are the homogeneous coordinates and

$$P = Z^m p(X/Z, Y/Z), \quad Q = Z^m q(X/Z, Y/Z).$$

- Consider the holomorphic singular foliation $\mathscr{F}_0$ induced by the one–form ω_0.
- Clearly $\mathscr{F}_0$ has the infinity as an invariant line.

Proof of Theorem 1 (a):

Under the assumption of Theorem 1, by Cerveau and Lins Neto theorem the total degree n of all invariant algebraic curves of the foliation $\mathscr{F}_0$ is no more than $m+2$.

Case 1: $n = m+2$. Again using Cerveau and Lins Neto theorem, it follows that

- F is reducible, saying $F = F_1 \cdot \ldots \cdot F_k$ the irreducible decomposition with $k \geq 2$.
- The one–form ω_0 has the expression

$$\omega_0 = F \sum_{i=1}^{k} \lambda_i \frac{dF_i}{F_i},$$

where $\lambda_i \in \mathbb{C}$.

- The one–form ω_0 has the inverse integrating factor F, and consequently Darboux integrable with the Darboux first integral

$$H(X,Y,Z) = F_1^{\lambda_1} \cdot \ldots \cdot F_k^{\lambda_k}.$$

- ω_0 is projective $\Longrightarrow \lambda_1 \deg F_1 + \ldots + \lambda_k \deg F_k = 0$.

Subcase 1. $k = 2$. The foliation $\mathscr{F}_0$ has a rational first integral

$$H(X,Y,Z) = F_1^k F_2^{-l}, \qquad \text{with } k,l \in \mathbb{N},\ (k,l) = 1.$$

$$\Downarrow$$

$\exists$ infinitely many invariant algebraic curves. A contradiction.

Subcase 2. $k \geq 3$. By Harnack's theorem that each invariant algebraic curve has at most

$$(\deg F_i - 1)(\deg F_i - 2)/2 + a_i \text{ ovals},$$

where $a_i = 1$ if $\deg F_i$ is even, and $a_i = 0$ if $\deg F_i$ is odd.

$$\Downarrow$$

- The total number of ovals contained in F_i for $i = 1, \ldots, k$ is no more than

$$\sum_{i=1}^{k} \left(\frac{(\deg F_i - 1)(\deg F_i - 2)}{2} + a_i \right) \leq \frac{(m+2-k)(m+1-k)}{2} + \sum_{i=1}^{k} a_j.$$

- The equality holds if and only if one of the F_i's has the degree $m+3-k$ and the others all have degree 1.

Set

$$M(k)=\frac{(m+2-k)(m+1-k)}{2}+\sum_{j=1}^{k}a_j.$$

$$\Downarrow$$

The maximum of the $M(k)$ for $k\in\{3,\ldots,m+2\}$ takes place at $k=3$.

For $k=3$, the three invariant algebraic curves have respectively the degrees 1,1 and m, the maximum is

$$\frac{(m-1)(m-2)}{2}+a,$$

where $a=1$ if m is even and $a=0$ if m is odd. The proof is finished for $n=m+2$.

Case 2: $n<m+2$. The proof is similar to Case 1, but is more easier.

Proof of Theorem 1 (b): The proof is constructive.

Case 1: $m+1$ is the total degree of the invariant algebraic curves in the affine plane.

$\Downarrow$ by the Harnack's theorem

$\exists$ a nonsingular algebraic curve of degree m having the maximal number, i.e. $(m-1)(m-2)/2+a$, of ovals, where $a=1$ if m is even, or $a=0$ if m is odd.

- Denote by f_1 this curve.
- Choose a straight line, called f_2, as the line at infinity such that which is outside the ovals of f_1 and intersects f_1 transversally.
- Choose another straight line, called f_3, which is outside the ovals of f_1 and intersects f_1 and f_2 transversally and does not meet the intersection points of f_1 and f_2.

- Let F_1, F_2 and F_3 be the projectivization of f_1, f_2 and f_3.
- Taking $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ non–zero such that

$$\lambda_1 m + \lambda_2 + \lambda_3 = 0 \text{ and } \lambda_i/\lambda_j \notin \{r \in \mathbb{Q}; r < 0\}.$$

Then the foliation $\mathscr{F}_m$ induced by the projective one–form

$$\lambda_1 F_2 F_3 dF_1 + \lambda_2 F_1 F_3 dF_2 + \lambda_3 F_1 F_2 dF_3$$

- has only the three invariant algebraic curves F_1, F_2, F_3.
- and is of degree m.

$$\Downarrow$$

- $\mathscr{F}_m$ has exactly $(m-1)(m-2)/2 + a$ algebraic limit cycles.
- Since $\mathscr{F}_m$ has the inverse integrating factor $F_1 F_2 F_3$, and so it has no other limit cycles.

Case 2. m is the total degree of the invariant algebraic curves in the affine plane.

The proof was given by Christoper and Llibre *et al*. Indeed,

- As the above proof, $\exists$ a nonsingular algebraic curve of degree m having the maximal number of ovals.
  Denote it by $g(x,y)$.
- Choose a linear function $h(x,y)$ such that $h = 0$ does not intersect the ovals of $g = 0$, and choose $a, b \in \mathbb{R}$ satisfying $ah_x + bh_y \neq 0$.

Then the real planar differential system

$$\dot{x} = ag - hg_y, \qquad \dot{y} = bg + hg_x,$$

- is of degree m and
- has all the ovals of $g = 0$ as hyperbolic limit cycles.

Moreover the system has no other limit cycles. $\square$

Ideal of the proof of Theorem 2

Recall Theorem 2

Theorem 2

If a real planar polynomial vector field $\mathscr{Y}$ of degree m has all its invariant algebraic curves non-dicritical, then the following hold.

(a) If $r(x,y) \equiv 0$, the maximal number of algebraic limit cycles of the vector fields is at most

- $1+m(m-1)/2$ when m is even, and
- $m(m-1)/2$ when m is odd.

(b) If $r(x,y) \not\equiv 0$, the maximal number of algebraic limit cycles of the vector fields is at most

- $1+(m+1)m/2$ when m is even, and
- $(m+1)m/2$ when m is odd.

- Write system $\mathscr{Y}$ in the one–form

$$(q(x,y)+yr(x,y))dx-(p(x,y)+xr(x,y))dy.$$

Its projective one–form is

$$\omega_1=(ZQ+YR)dX-(ZP+XR)dY+(YP-XQ)dZ,$$

where X,Y,Z are the homogeneous coordinates and

$$P=Z^m p(X/Z,Y/Z),\quad Q=Z^m q(X/Z,Y/Z),\quad R=Z^m r(X/Z,Y/Z).$$

- Consider the holomorphic singular foliation $\mathscr{F}_1$ induced by the one–form ω_1.
- Distinguish $r(x,y)\equiv 0$ or not, $\mathscr{F}_1$ has the infinity as an invariant line or not.

Lemma 1.4: Carnicer [Ann. Math. 1994]

Let $\mathscr{F}$ be a holomorphic singular foliation of degree m in $\mathbb{C}P(2)$. Assume that

- S is an algebraic curve which is invariant by $\mathscr{F}$, and
- S is given by a reduced polynomial F of degree n.

If there are no dicritical singularities of $\mathscr{F}$ on S, then $n \leq m+2$.

Then we can prove Theorem 2 by

- distinguishing $r(x, y) \equiv 0$ or not, and
- using the methods given in the proof of Theorem 1, and
- combining the Carnicer Theorem.

Open problems

- Is it unique that the number of limit cycles of quadratic differential systems having an invariant algebraic curve?
- In the non–dicritical case, we obtain an upper bound, but we do not get the exact one,
  What is the exact upper bound of the maximum number of algebraic limit cycle?
- How to solve the simple version of the Hilbert's 16th problem in the dicritical case?

Happy Birthday, Jaume!

Thank You Very Much!