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Introduction

For complex numbers \( \alpha, \beta \) and \( \gamma \), we consider the following family of birational maps \( f : \mathbb{C}^2 \to \mathbb{C}^2 \) for \( \alpha \neq 0 \):

\[
f(x, y) = \left( \frac{\alpha x + \beta y}{1 - \gamma x} \right).
\]

(1)

- This family has Dynamical Degree \( D = 1 \), where

\[
D = \lim_{n \to \infty} \left( \log |f^n(f)|^2 \right).
\]

(2)

The Dynamical Degree in (2) is a canonical quantity that gives us the idea about the dynamics of the birational map. It is an algebraic number which is when equal to one, the map is expected to have a simpler and more predictable dynamics.

The birational map \( f \) in (1) is a member of the subfamilies of \( g \) in (3) with \( D = 1 \). A. Cima and S. Zafar in [1] studied the below given family \( g : \mathbb{C}^2 \to \mathbb{C}^2 \) and located all of its subfamilies with \( 1 \leq D \leq 2 \). The 9 parameter family of birational maps studied in [1] is the following:

\[
g(x, y) = \left( \alpha_i + \alpha x + \beta_i y, \frac{\beta_i x + \beta y}{\alpha_i + \gamma_i x + \gamma y} \right)
\]

where \( \alpha_i, \beta_i \) and \( \gamma_i \) for \( i = 0, \ldots, 2 \) are complex numbers.

Theorem

Theorem 1. Let \( f : \mathbb{C}^2 \to \mathbb{C}^2 \) be the birational map in (1) then if for some \( p \in \mathbb{N}^+ \), \( f^n(-\alpha, \beta, 0) = (-\beta, \gamma) \), then \( f \) is a 2p+2 periodic map.

Sketch of Proof

Consider the imbedding \( (x, y) \mapsto [1 : x : y] \in \mathbb{P}^2 \) into projective space and consider the induced map \( F : \mathbb{P}^2 \to \mathbb{P}^2 \) of \( f \) given by

\[
F[x_0 : x_1 : x_2] = [\gamma x_1 + x_2 x_0 : \alpha x_1 x_0 + x_2 : -x_0 (x_0 + x_2)].
\]

(4)

- The indeterminacy locus of \( F \) is \( \mathcal{I} = \{ \alpha, \beta, 0 \} \) and \( F^{-1} \mathcal{I} = \{ \alpha_i, \beta_i, 1 \} \).
- The exceptional curves of \( F \) are \( \mathcal{E} = \{ S_i, S_j, S_k \} \) and \( F^{-1} \mathcal{E} = \{ T_i, T_j, T_k \} \).
- All these exceptional curves are distinct.

There is a relation between indeterminacy points and exceptional curves of \( F \) and \( F^{-1} \) in this family. That is \( S_i \to A_i \) and \( T_i \to O_i \) for \( i = 0, 1, 2 \) as described in the following picture.

We call this map \( F \) an Algebraically stable map if no \( S_i \) or any of it’s iterate under \( F \) reach an indeterminacy point of \( F \).

The result generalized by Fornaes andibu [7] is as follows:

- If for every exceptional curve \( C \) and for all \( n \geq 0 \), \( F^n(C) \in \mathbb{P}^2 \), then \( F \) is an Algebraically stable, where \( F \) is the extension of \( f \) we get after blowing up operation.

In order to regularize \( F \) we blow up all the points \( P \) reaching any indeterminacy point of \( F \). Trivially \( A_0 = O_2 = [1 : 0 : 1] \), \( A_1 = O_1 = [0 : 1 : 0] \) and \( A_2 = O_0 = 1 : -\beta : -\gamma \). We impose the following condition on the orbit of \( A_2 \) such that \( P^n(A_2) = O_i \), \( p \in \mathbb{N} \) for \( A_2 = [\gamma : -\alpha : \beta] \) and \( O_i = [-1 : -\beta : -\gamma] \).

We have \( p + 2 \) points to blow up which are \( A_0 = O_2 \), \( A_1 = O_1 \), \( A_2 = O_0 \), \( e_2, \ldots, e_2 \) and \( O_i \). After blowing up we get \( E_{2,2} \) fibres on \( p + 2 \) points.

This results in the expansion of \( \mathbb{P}^2 \) and we get the new expanded space \( X \) with the induced map \( F_X \).

Furthermore \( F_{2,2}^2(S_0) = S_1 \) and \( F_{2,2}^2(S_2) = S_0 \). We thus conclude here that \( F_X \) is bijective.

By the consequence of the imposed condition \( F_X(A_i) = O_i \) we find that \( F_X(\mathcal{E}) = \mathcal{I} \). This gives us that \( F_X \mathcal{E} = \mathcal{I} \) for \( \mathbb{P}^2 \) which implies that \( F_X^2(S_0) = E_2 \). This behavior of orbit of \( S_2 \) can be observed in the front figure.

References


The pictures included in this poster other than those related to the proof of the theorem are the depiction of Mobius transformation, Periodic Orbits and Fibonacci series. This is to show their existence in nature and some of their uses to mankind. These families and many more exist inside the family (3) studied by A. Cima and S. Zafar in [1] and also by E. Bedford and K. Kim in [2].