A Particular Family of Globally Periodic Birational Maps

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Introduction

For complex numbers α, β and γ, we consider the following family of birational maps $f : \mathbf{C}^2 \to \mathbf{C}^2$ for $\alpha \neq 0$:

$$f(x, y) = \left(\alpha x, \frac{\beta + x}{\gamma + y}\right). \tag{1}$$

- This family has *Dynamical Degree* $D = 1$, where

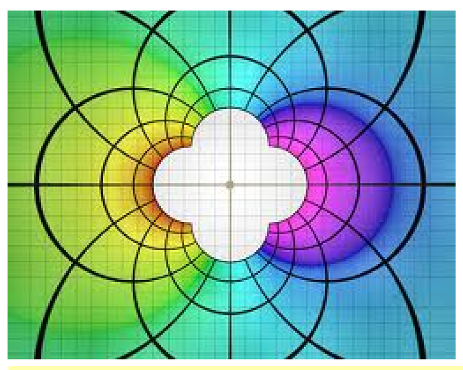
$$D = \lim_{n\to\infty} (\deg(f^n))^{\frac{1}{n}}. \tag{2}$$

The **Dynamical Degree** in (2) is a canonical quantity that gives us the idea about the dynamics of the birational map. It is an algebraic number which is when equal to one, the map is expected to have a simpler and more predictable dynamics.

The birational map f in (1) is a member of the subfamilies of g in (3) with $D = 1$. A. Cima and S. Zafar in [1] studied the below given family $g : \mathbf{C}^2 \to \mathbf{C}^2$ and located all of its subfamilies with $1 \leq D \leq 2$. The 9 parameter family of birational maps studied in [1] is the following:

$$g(x, y) = \left(\alpha_0 + \alpha_1 x + \alpha_2 y, \frac{\beta_0 + \beta_1 x + \beta_2 y}{\gamma_0 + \gamma_1 x + \gamma_2 y}\right), \tag{3}$$

where α_i, β_i and γ_i for $i = 0, ..., 2$ are complex numbers.



Theorem

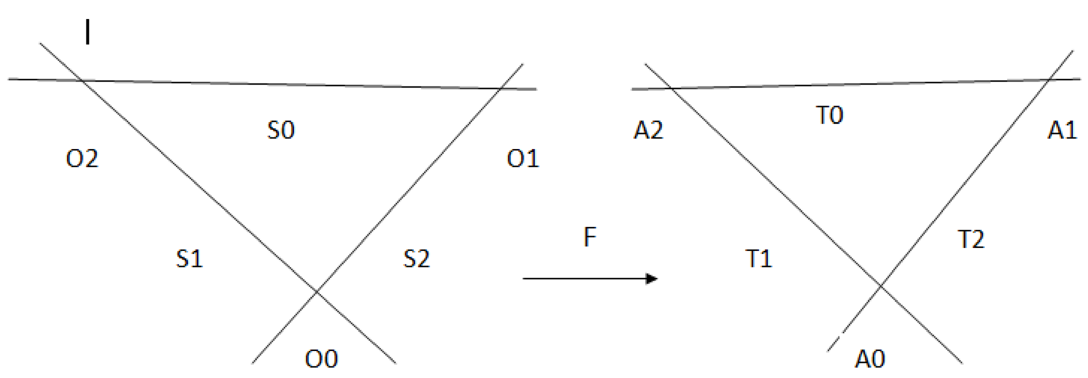
Theorem 1 *Let $f : \mathbf{C}^2 \to \mathbf{C}^2$ be the birational map in (1) then if for some $p \in \mathbf{N}$, $f^p(-\alpha\,\beta, 0) = (-\beta, \gamma)$, then f is a $2p + 2$ periodic map.*

Sketch of Proof

Consider the imbedding $(x, y) \mapsto [1 : x_1 : x_2] \in \mathbf{P}^2$ into projective space and consider the induced map $F : \mathbf{P}^2 \to \mathbf{P}^2$ of f given by

$$F[x_0, x_1, x_2] = [(\gamma x_0 + x_2)x_0 : \alpha x_1(\gamma x_0 + x_2) : x_0(\beta x_0 + x_1)]. \tag{4}$$

- The **indeterminacy locus** of F is $\mathcal{I} = \{O_0, O_1, O_2\}$ and F^{-1} is $\mathcal{I}' = \{A_0, A_1, A_2\}$.
- The **exceptional curves** of F are $\mathcal{E} = \{S_0, S_1, S_2\}$ and F^{-1} are $\mathcal{E}' = \{T_0, T_1, T_2\}$.
- All these exceptional curves are **distinct**.
- There is a relation between indeterminacy points and exceptional curves of F and F^{-1} in this family. That is $S_i \to A_i$ and $T_i \to O_i$ for $i = 0, 1, 2$ as described in the following picture.

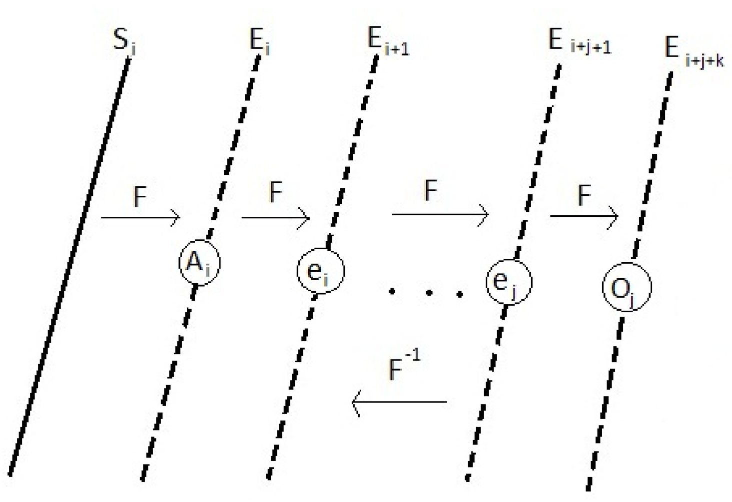


$S_i \to A_i$ and $T_i \to O_i$ for $i = 0, 1, 2$.

We call this map F an **Algebraically stable map** if no S_i or any of it's iterate under F reach an indeterminacy point of F.

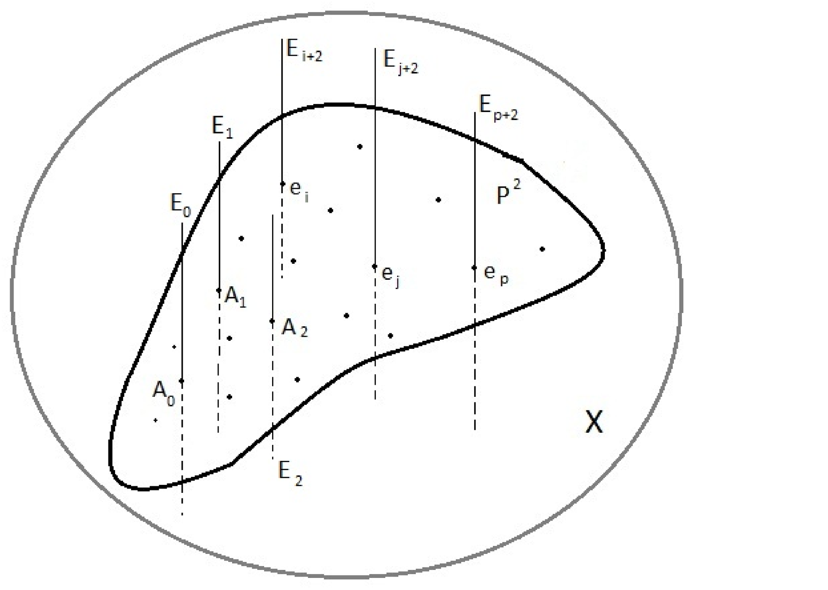
The result generalized by **Fornaes** and **Sibony** in [5] is as follows:

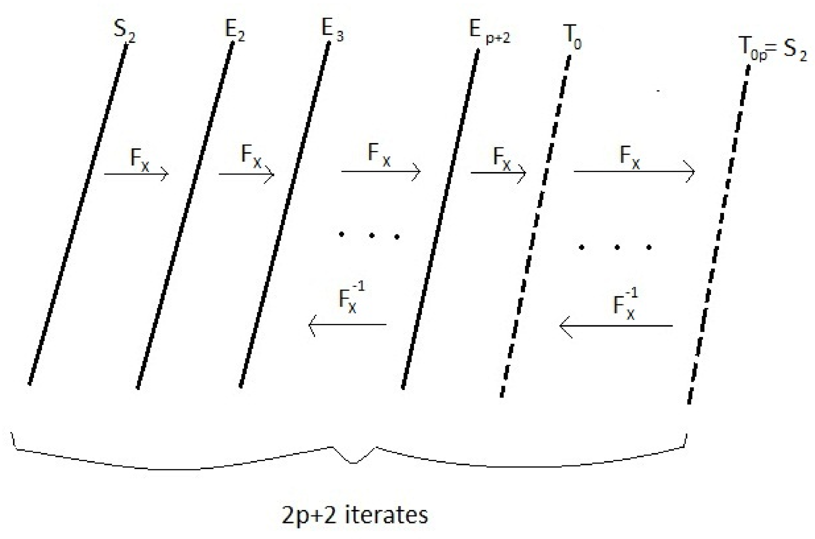
- If for every exceptional curve C and for all $n \geq 0, \tilde{F}^n(C) \notin \mathcal{I}(\tilde{F})$ then $\tilde{F}$ is *Algebraically stable*, where $\tilde{F}$ is the extension of F we get after blowing up operation.



In order to regularize F we blow up all the points p reaching any indeterminacy point of F. Trivially $A_0 = O_2 = [0 : 1 : 0]$, $A_1 = O_1 = [0 : 0 : 1]$ but A_2 is not equal to any O_i. We impose the following condition on the orbit of A_2 such that $F^p(A_2) = O_0$, $p \in \mathbf{N}$ for $A_2 = [1 : -\alpha\,\beta : 0]$ and $O_0 = [1 : -\beta : -\gamma]$.

We have $p + 2$ points to blow up which are $A_0 = O_2$, $A_1 = O_1$, A_2, e_1, e_2, ..., $e_p - 1 = O_0$. After blowing up we get E_{p+2} fibres on $p + 2$ points. This results in the expansion of $\mathbf{P}^2$ and we get the new expanded space X with the induced map F_X.





By the consequence of the imposed condition $F^p(A_2) = O_0$ we find that $F^p(T_0) = S_2$. This gives us that ${F_X}^{2p+2}(S_2) = S_2$ for $p > 2$ which implies that ${F_X}^{2p+2}(E_2) = E_2$. This behavior of orbit of S_2 can be observed in the front figure.

Furthermore ${F_X}^{2p+2}(S_0) = S_0$ and ${F_X}^{2p+2}(S_1) = S_1$. We thus conclude here that F_X is **bijective**.

Diller and Favre in [2] showed that when F is an automorphism then the sequence of degrees of F does not grow linearly. On the other hand by using the results found by E. Bedford and K. Kim in [2] we find that the characteristic polynomial associated to F_X is $(x - 1)^2(x + 1)(x^{p+1} + 1)$. This implies that

$$d_n = a_0(-1)^n + a_1 + a_2 n + a_3(\lambda_1)^n ... + a_{p+3}(\lambda_{p+1})^n,$$

where $\lambda_1, \lambda_2, ..., \lambda_{p+1}$ are the roots of $(x^{p+1} + 1)$.

By using the above result of Diller and Favre we know that $a_2 = 0$. This implies that $d_{2p+2+n} = d_n, \forall n$. For $n = 0$ we know that $d_{2p+2} = d_0 = 1$. Thus F^{2p+2} is **linear**. This implies that $f^{2p+2}(x, y) = (x, \delta_0 + \delta_1 x + \delta_2 y)$.

When F_X is algebraically stable we consider its induced map ${F_X}^* : \mathcal{P}ic(X) \to \mathcal{P}ic(X)$, where $\mathcal{P}ic(X)$ is the Picard group generated by the generic line $L \in \mathbf{P}^2$.

It can be seen that ${F_X}^*(E_i) = E_i$ for $i = 2, 3, 4$. This implies that F^{2p+2} fixes the base points of E_i for $i = 2, 3, 4$.

Since the base points of E_2, E_3, E_4 are A_2, $F(A_2)$, $F^2(A_2)$ respectively. We find that there are three points in $\mathbf{C}^2$ which are fixed by f^{2p+2} and these three points are non collinear. Hence f^{2p+2} must be the **identity map**. This implies that f is a $2p + 2$ periodic map provided that $F^p(A_2) = O_0$ for some $p > 2 \in \mathbf{N}$.

We treat the cases for $p = 0, 1, 2$ separately. We give here the periodic families for $p = 0$ and $p = 1$.

For $p = 0$ we have the following $2-$periodic mapping:

$$f(x, y) = \left(x, \frac{\beta + x}{y}\right).$$

For $p = 1$ we have the following $4-$periodic mapping:

$$f(x, y) = \left(-x, \frac{-(\gamma - 2x)}{2\gamma + y}\right).$$

References

[1] A. Cima, S. Zafar. *Dynamical classification of a family of birational maps via Dynamical Degree*, pre print.

[2] E. Bedford, K. Kim. *On the degree growth of birational mappings in higher dimension*.J. Geom. Anal. **14** (2004), 567-596.

[3] J. Diller, C. Favre. *Dynamics of bimeromorphic maps of surfaces*.Amer. J. of Math., **123** (2001), 1135-1169.

[4] J. Diller. *Dynamics of Birational Maps of* $\mathbf{P}^2$. Indiana Univ. Math. J. 45, no. 3, 721-772 (1996).

[5] J-E. Fornaes, N. Sibony. *Complex dynamics in higher dimension. II*. Modern methods in complex analysis. Ann. of Math. Stud. 137, Princeton Univ. Press (1995), pp. 135-182.


The pictures included in this poster other than those related to the proof of the theorem are the depiction of Mobius transformation, Periodic Orbits and Fibonacci series. This is to show their existence in nature and some of their uses to mankind. These families and many more exist inside the family (3) studied by A. Cima and S. Zafar in [1] and also by E. Bedford and K. Kim in [2].