The existence of Traveling Wave of FitzHugh-Nagumo System

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Introduction

We consider the well-known FitzHugh-Nagumo system

$$\begin{aligned} u_t &= u_{xx} - f(u) - v, \\ v_t &= d(u - bv) \end{aligned} \tag{1}$$

where $f(u) = u(u-1)(u-a)$, $0 < a < 1/2, b > 0, d > 0$, a, b, d are parameters, which is a simplification of the classical Hodgkin-Huxley model of the spike dynamics in a biological neuron that describes the propagation of action potentials in the nerve axon of the squid. System (1) was suggested in 1961 by FitzHugh [1], and constructed in 1962 by Nagumo[2]. The dynamical behaviors of (1) have been widely studied in the past, especially the traveling wave solutions, see [3-7] and the references therein. For example, Jones [7] studied stability of the traveling wave solution of the FitzHugh-Nagumo system; Gao and Wang [3] proved the existence of wavefronts and impulse to FitzHugh-Nagumo Equations; Llibre and C. Valls [5] gave the analytic first integrals of the FitzHugh-Nagumo systems.

In this paper, we study the existence of traveling wave with small amplitude of FitzHugh-Nagumo equations. We will want to claim our main theorem

Theorem 1 *System (1) has at least three traveling periodic solutions with small amplitude.*

In fact, we consider the traveling wave solution of (1), $(u,v)(x,t) = (u,v)(x+ct)$, where c represents the wave speed. Setting $x = u, y = u - bv, z = \dot{u}$, we obtain a three-dimensional nonlinear ordinary differential system from (1)

$$\begin{aligned} \dot{x} &= z \\ \dot{y} &= z - \tfrac{bd}{c} y \\ \dot{z} &= f(x) + \tfrac{1}{b}(x - y) + cz. \end{aligned} \tag{2}$$

where $f(x) = x(x-1)(x-a)$. We may also point out that there are a number of studies on the Hopf bifurcation with relation to (1) [4][6]. However in this article we consider the degenerate Hopf bifurcation to occur in (1). Specially by the center manifold reduction theorem and Lyapunov coefficient method, the first three focal values is given around some singular points of (2).

1. Local analysis of singular point

First we note that (2) possesses singular points $(x^*, 0, 0)$, where x^* satisfies the equation

$$x[x^2 - (a+1)x + (a + \frac{1}{b})] = 0.$$

Observing the discriminant $\Delta = (a-1)^2 - \frac{4}{b}$, we classify the singular points into the following three cases,

(i) If $\Delta < 0$, i.e. $(a-1)^2 - \frac{4}{b} < 0$, (2) has a unique singular point $O(0,0,0)$,

(ii) If $\Delta = 0$ i.e. $(a-1)^2 - \frac{4}{b} = 0$, (2) has two singular points $O(0,0,0), P(\frac{1}{2}(a+1), 0, 0)$,

(iii) If $\Delta > 0$, $(a-1)^2 - \frac{4}{b} > 0$, (2) has three singular points $O(0,0,0), P_1(\frac{1}{2}(a+1+\sqrt{\Delta}), 0, 0), P_2(\frac{1}{2}(a+1-\sqrt{\Delta}), 0, 0)$.

For the case (i), Hopf bifurcation does not occur, because the linearized system of (2) at the unique singular point $O(0,0,0)$ has not any pair of pure imaginary eigenvalue.
Similar to the case (i), for the case (ii), Hopf bifurcation also does not occur near the singular point $O(0,0,0)$ of (2). However, there exists complicated dynamical behavior near the singular point $P(\frac{1}{2}(a+1), 0, 0)$ because the linearized system of (2) at P has a pair of pure imaginary eigenvalue $\pm\sqrt{\frac{1}{b} - bd}i$ and a zero eigenvalue for $\frac{1}{b} - bd > 0$ and $(\frac{bd}{c} - c) = 0$. We won't consider it here.

We are most interested in the case (iii). Similar to the case (i), for the case (iii), Hopf bifurcation also does not occur near the singular point $O(0,0,0)$ of (2). At the singular point P_1, the Jacobian of (2) is

$$\mathcal{F}(P_1) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -\frac{bd}{c} & 1 \\ \frac{1}{2}\sqrt{\Delta}(a+1+\sqrt{\Delta}) & -\frac{1}{b} & c \end{bmatrix}$$

whose characteristic equation is

$$\lambda^3 + (\frac{bd}{c} - c)\lambda^2 + [\frac{1}{b} - bd - \frac{1}{2}\sqrt{\Delta}(a+1+\sqrt{\Delta})]\lambda - \frac{bd}{2c}\sqrt{\Delta}(a+1+\sqrt{\Delta}) = 0 \tag{3}$$

.

It is easy to get the necessary condition as follows which guarantees (3) has pure imaginary eigenvalue,

$$\begin{aligned} &\frac{1}{b} - bd - \frac{1}{2}\sqrt{\Delta}(a+1+\sqrt{\Delta}) > 0 \\ &(\frac{bd}{c} - c)[\frac{1}{b} - bd - \frac{1}{2}\sqrt{\Delta}(a+1+\sqrt{\Delta})] = -\frac{bd}{2c}\sqrt{\Delta}(a+1+\sqrt{\Delta}) \end{aligned} \tag{4}$$

and the eigenvalues are

$$\lambda_1 = -(\frac{bd}{c} - c)$$

$$\lambda_{2,3} = \pm\sqrt{\frac{1}{b} - bd - \frac{1}{2}\sqrt{\Delta}(a+1+\sqrt{\Delta})}\,i$$

Eq. (4) implies

$$\sqrt{\Delta}(a+1+\sqrt{\Delta}) = 2(\frac{1}{b} - bd)(1 - \frac{bd}{c^2}) \tag{5}$$

Hence, the eigenvalue are simplified as

$$\lambda_1 = c - \frac{bd}{c}, \quad \lambda_{2,3} = \pm\sqrt{\frac{bd}{c^2}(\frac{1}{b} - bd)}\,i.$$

Let

$$(x - \frac{1}{2}(a+1+\sqrt{\Delta}), y, z)^T = M(u,v,w)^T,$$

where

$$M = \begin{bmatrix} 0 & \sqrt{\frac{c^2}{d(1-b^2d)}} & \frac{c}{c^2-bd} \\ bc & \frac{c^2}{d}\sqrt{\frac{d(1-b^2d)}{c^2}} & \frac{1}{c} \\ 1 & 0 & 1 \end{bmatrix}$$

then the system (2) is changed into the following normal form

$$\begin{aligned} \dot{u} &= -sv + Ak(u,v,w) \\ \dot{v} &= su + Bk(u,v,w) \\ \dot{w} &= pw + Ek(u,v,w) \end{aligned} \tag{6}$$

where
$s = \sqrt{\frac{d(1-b^2d)}{c^2}}, p = c - \frac{bd}{c}, A = \frac{d(-bc^2+1)}{c^4-2bdc^2+d}$,
$B = -\frac{c^3}{c^4-2bdc^2+d}\sqrt{\frac{d(1-b^2d)}{c^2}}, E = \frac{c^2(c^2-bd)}{c^4-2bdc^2+d}$,
$F = \frac{a+1+3\sqrt{\Delta}}{2}, k(u,v,w) = (\frac{v}{s} + \frac{w}{p})(\frac{v}{s} + \frac{w}{p} + F)$.

We need to study Hopf bifurcation of the system (6) at the singular point $(0,0,0)$. Noting that for $c - \frac{bd}{c} < 0$, the linearized system of Sys.(6) at $(0,0,0)$ has a negative eigenvalue and a pair of pure imaginary eigenvalues. Hence there exists a center manifold in the neighborhood of $(0,0,0)$. See [8]

Lemma 1 *If the parameters a, b, c, d are located in*

$$\Sigma = \{c - \frac{bd}{c} < 0,\ \frac{1}{b} - bd - \frac{1}{2}\sqrt{\Delta}(a+1+\sqrt{\Delta}) = \frac{bd}{c^2}(\frac{1}{b} - bd)\}$$

there exists a 1-dimensional stable manifold and a 2-dimensional center manifold in the neighborhood of $(0,0,0)$ of (2).

2. Hopf bifurcation of the traveling wave of FitzHugh-Nagumo

We assume that the two dimensional center manifold can be denoted by $W^c_{loc}(O)$ satisfying

$$\{(u,v,w) \in W^c_{loc}(O) | \quad w = h(u,v), |u| + |v| \ll 1, h(0,0) = \partial_u h(0,0) = \partial_v h(0,0) = 0\}$$

In the standard Lyapunov constant method to compute the center manifold and the focal value of higher order, we obtain by a hard computation

$$\begin{aligned} h(u,v) = &\ h_{20}u^2 + h_{11}uv + h_{02}v^2 + h_{30}u^3 + h_{21}u^2v + h_{12}uv^2 + h_{03}v^3 \\ &+ h_{40}u^4 + h_{31}u^3v + h_{22}u^2v^2 + h_{13}uv^3 + h_{04}v^4 + \ldots \end{aligned} \tag{7}$$

where

$$h_{20} = -\frac{2EF}{p(4s^2+p^2)},\ h_{02} = -\frac{EF(2s^2+p^2)}{s^2p(4s^2+p^2)},\ h_{11} = -\frac{2EF}{s(4s^2+p^2)},$$

$$h_{30} = -\frac{2\varphi_1(s,p)}{p\psi(s,p)}, h_{21} = -\frac{2\varphi_2(s,p)}{s\psi(s,p)}, h_{12} = \frac{\varphi_3(s,p)}{s^2p\psi(s,p)}, h_{03} = -\frac{\varphi_4(s,p)}{s^3p\psi(s,p)} \cdots$$

and

$$\begin{aligned} \psi(s,p) &= 9s^4 + 10p^2s^2 + p^4, \\ \varphi_1(s,p) &= \varphi_2(s,p) \\ &= 3EF(h_{20} + 2h_{02})s^2 - F(6Bh_{02} + 3h_{11}A - 2Eh_{11})ps \\ &\quad -F(Bh_{11} - Eh_{20} + 2h_{20}A)p^2 + 3Ep, \end{aligned}$$

$$\begin{aligned} \varphi_3(s,p) = &-18EFh_{02}s^4 - 3F(2Eh_{11} - 6Bh_{02} - 3h_{11}A)ps^3 - 3p^3E \\ &+[F(4Eh_{20} + 3Bh_{11} + 6h_{20}A - 6Eh_{02})p^2 + 9Ep)s^2 \\ &-F(2Eh_{11} - 6Bh_{02} - 3h_{11}A)p^3s + F(Bh_{11} + 2h_{20}A)p^4, \end{aligned}$$

$$\begin{aligned} \varphi_4(s,p) = &-6Eh_{11}Fs^5 + F(3Bh_{11} + 14Eh_{02} + 4Eh_{20} + 6h_{20}A)ps^4 \\ &-F(7h_{11}A + 2Eh_{11} + 14Bh_{02})p^2s^3 + (F(Bh_{11} + 2h_{02} \\ &+2h_{20}A)p^3 + 7Ep^2)s^2 - F(2Bh_{02} + h11A)p^4s + p^4E \end{aligned}$$

Furthermore, the first three focal values

$$V_1 = \frac{l_1(m_{12}F^2 + m_{10})}{n_1} \tag{8}$$

where $l_1 = c^7\sqrt{\frac{d(1-b^2d)}{c^2}}$,
$m_{12} = 2c^2[bc^4 - 2(2b^2d-1)c^2 - bd(5b^2d-6)]$,
$m_{10} = 3(b^2d-1)(c^2-bd)(c^4 - 2bdc^2 - 3b^2d^2 + 4d)$,
$n_1 = 4d(b^2d-1)^3(bd-c^2)(c^4 - 2bdc^2 - 3b^2d^2 + 4d)(c^4 - 2bdc^2 + d)$.

$$V_2 = \frac{l_2(m_{24}F^4 + m_{22}F^2 + m_{20})}{n_2} \tag{9}$$

where
$l_2 = c^{11}\sqrt{\frac{d(1-b^2d)}{c^2}}$

$$\begin{aligned} m_{20} = &-9d(db^2-1)^2(5c^8 + 12d^3b^2 - 20d^2c^2b + 24c^2d^3b^3 + 38dc^4 \\ &-32d^2c^4b^2 - 12dc^6b - 15d^2)(bd-c^2)^3(b^2d^2 - c^4 + 2dc^2b - 4d)^3 \end{aligned}$$

$n_2 = 24d^4(2dc^2b - c^4 - d)^3(db^2-1)^5(3b^2d^2 - c^4$
$+ 2dc^2b - 4d)^3(2dc^2b - c^4 + 8b^2d^2 - 9d)(bd - c^2)^3$,

$m_{24} = 2c^4[35bc^{24} - (430db^2 - 70)c^{22} + 5db(299db^2 + 36)c^{20} + \cdots (10\ lines)$
$m_{22} = c^2(bd-c^2)(3b^2d^2 - c^4 + 2bdc^2 - 4d)((115db^2 - 105)c^{20} + \cdots (9\ lines)$

$$V_3 = \frac{l_3(m_{36}F^8 + m_{34}F^6 + m_{32}F^2 + m_{30})}{n_3} \tag{10}$$

which is so long up to 170 lines.

Remark Generally, we can get our main theorem, i.e. three limit cycles in the center manifold by perturbing the parameters $(a,b,c,d) \in \Sigma$ around $V_1 = V_2 = V_3 = 0$. However, it is another hard job to choose a suitable parameters.
In addition, we can study Hopf bifurcation around the P_2 in the same way as P_1, and get the similar formulas of focal values.

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