Integrals of rational 1-forms over algebraic cycles

The “forgotten” case of the Infinitesimal Hilbert 16th Problem

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- Or does it?

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What about the sets defined by vanishing of Abelian integrals?

γ_t a pencil of cycles, ω fixed. How many zeros may have $I(t) = I(\gamma_t, \omega)$?

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First return map (…, Poincaré, Andronov, Pontryagin, Melnikov, …)

Let T be a cross-section of U, transversal to $\mathscr{F}_0$, with the chart $t = H|_T$. Then the Poincaré first return map of $\mathscr{F}_\varepsilon$ is linearizable,

$$\Delta(t,\varepsilon) = t + \varepsilon I_1(t) + O(\varepsilon^2), \qquad I_1(t) = \oint_{\gamma_t} \omega.$$

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Corollary. $I_1 \not\equiv 0 \implies \#\{I_1 = 0\} \geqslant \#\{\text{compact leaves of } \mathscr{F}_\varepsilon \text{ in } U\}$.

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Give the answer in terms of $\deg H$ and $\deg \omega$.

Une autre rencontre: the period function

Added after the talk by M. Sabatini

Assume a rational vector field

$$\dot{x} = R(x,y), \qquad \dot{y} = S(x,y), \qquad R,S \in \mathbb{R}(x,y)$$

has a rational first integral H and a period annulus $U \subseteq \mathbb{R}^2$.
Then the period of circulation along the cycles $\sigma_c \subset U$ is given by the Abelian integral,

$$T(c) = T(\sigma_c) = \oint_\sigma \frac{\mathrm{d}x}{R(x,y)} = \oint_\sigma \frac{\mathrm{d}y}{S(x,y)}, \qquad \sigma_t \subseteq \{H = c\}.$$

The same is true for all higher derivatives of T with respect to c.
The rational 1-forms ω_k yielding the kth derivative of T, are the iterated Gelfand–Leray derivatives of $\omega_0 = \mathrm{d}x/R$:

$$\omega_1 = \frac{\mathrm{d}\omega_0}{\mathrm{d}H}, \quad \omega_2 = \frac{\mathrm{d}\omega_1}{\mathrm{d}H}, \quad \dots$$

and all degrees $\deg \omega_k$ can be instantly computed.

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- The perturbed foliation is the portrait of the polynomial vector field

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First formulated in the late 1960-ies by Ilyashenko and Arnold, existence of finite bound proved by Varchenko and Khovanskii (1984), lots of particular low degree cases studied since then, the general explicit solution given by the authors (GB, DN, SY) in 2010.

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The ramification is linear and monodromic: $\Delta X(t) = X(t)M$ for any closed path avoinding Σ. Hence $\mathrm{d}X \cdot X^{-1}$ is a rational matrix 1-form on $\mathbb{C}P^\ell$.

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$$\mathrm{d}X(t) = \Omega(t) \cdot X(t) \iff \mathrm{d}x_{\alpha\beta}(t) = \sum\nolimits_{\gamma} \omega_{\alpha\gamma}(t) \cdot x_{\gamma\beta}(t), \qquad t \in \mathbb{C}P^{\ell}.$$

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4. The system is defined over $\mathbb{Q}$.

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Main theorem (Binyamini–Novikov–SY, 2010)

1. The system is integrable, $\mathrm{d}\Omega = \Omega \wedge \Omega$,
2. The singularity on Σ is "polar-like" (moderate, regular),
3. The monodromy operators for any "small loop" around Σ has eigenvalues on the unit circle,
4. The system is defined over $\mathbb{Q}$.

Under these assumptions the "number of roots of the solution $X(t)$" is bounded by an explicit double exponential expression in the relevant parameters $\dim t$, $\dim X$, $\deg \Omega$, $\mathrm{bitl}\, \Omega$.

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The first $\frac{3}{4}$ of assumptions of the Main Theorem can be easily verified for the period matrix of Abelian integrals.

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Example. $\dim t = 1$, $\dim X = 2$, $\deg \Omega = 1$, $\operatorname{bitl}(\Omega) = N \to +\infty$

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \frac{1}{t}\begin{pmatrix} 0 & \\ & N \end{pmatrix} \cdot X.$$

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$$\frac{\mathrm{d}X}{\mathrm{d}t} = \frac{1}{t}\begin{pmatrix} 0 & \\ & N \end{pmatrix} \cdot X. \qquad X(t) = \begin{pmatrix} a & \\ & bt^N \end{pmatrix}, \qquad \#\{X = 0\} = N \to +\infty.$$

Example: Picard–Fuchs system for Elliptic Integrals

Collection of all elliptic curves tangent to the infinite line:

$$H(x,y) = y^2 - (x^3 + t_1 x + t_2)$$

Monomial forms (basis of the cohomology):

$$\omega_1 = \frac{\mathrm{d}x}{y}, \qquad \omega_2 = \frac{x\,\mathrm{d}x}{y}$$

Picard–Fuchs system in the affine chart $\mathbb{C}^2 = \{(t_1, t_2)\}$ on $\mathbb{C}P^2$:

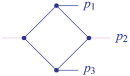
$$\Omega = \frac{1}{4t_1^3 + 27t_2^3} \cdot \left[\begin{pmatrix} -t_1^2 & \frac{9}{2}t_2 \\ \frac{3}{2}t_1 t_2 & t_1^2 \end{pmatrix} \mathrm{d}t_1 + \begin{pmatrix} -\frac{9}{2}t_2 - 3t_1 \\ -t_1^2 + \frac{9}{2}t_2 \end{pmatrix} \mathrm{d}t_2 \right]$$

[F. Pham, Singularités des Systèmes Différentiels de Gauss–Manin, p. 8.]

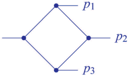
Differential equations for Feynman integrals

Example:

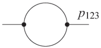
$$\frac{\partial}{\partial s_{123}}$$



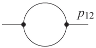
$$= \frac{D-4}{2(s_{12}+s_{23}-s_{123})}$$



$$+\frac{2(D-3)}{(s_{123}-s_{12})(s_{123}-s_{12}-s_{23})}\left[\frac{1}{s_{123}}\right.$$



$$-\frac{1}{s_{12}}$$



$$\left.\vphantom{\frac{1}{s}}\right]$$

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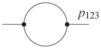

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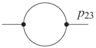

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Kotikov, Remiddi, Gehrmann, Laporta

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Theorem [D. Novikov–S.Y., 2006]

If $H(x, y) = \sum t_{ij} x^i y^j$ is the "universal polynomial" of degree $d + 1$, then there exist monomial forms of degree $\leqslant 2d$ such that the logarithmic derivative $\Omega(t) = \mathrm{d}X \cdot X^{-1}$ of the respective period matrix $X(t)$ is defined over $\mathbb{Q}$.

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Corollary.

The number of isolated zeros of the period of a polynomial form of degree $\leqslant d$ over algebraic ovals of degree $\leqslant d + 1$ does not exceed $2^{2^{\mathrm{Poly}(d)}}$, where $\mathrm{Poly}(d)$ is an explicit polynomial expression of degree $\leqslant 61$.

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Of course, the main difficulty occurs on/near $\{g = 0\}$, yet "purely fast" cycles are tracked by the integral $\oint g^{-1}(p\,dx + q\,dy)$ with a rational 1-form over algebraic ovals $\{H = t\}$ disjoint with $\{g = 0\}$.

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1. The first three conditions seem to be automatically verified as well.
2. The strongest setback: there seems to be no explicit algorithm producing the Picard–Fuchs system.
3. But once you start looking at the details attentively, there are lots of problems here and there.

Nasty nature of integrals of rational 1-forms

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Consider the class of multivalued functions

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- Integrals display some new analytic features, like ramification points of finite order, which don't occur in the polynomial case.

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- Increases the dimension of the ambient space by 1, making most of the standard “computational algorithms” impractical or outright impossible;
- Blurs any distinction between the 1-cycle and the 1-form.

Sampler of difficulties 1: effective Grothendieck theorem

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Cohomology of planar algebraic curves

Let $\{H(x,y)=0\}$ be a *generic* affine algebraic curve of degree $n+1$. Then its 1-cohomology over $\mathbb{Q}$ is generated by n^2 monomial forms whose differentials are $x^i y^j \, \mathrm{d}x \wedge \mathrm{d}y$ with $0 \leqslant i,j \leqslant n-1$.

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Proof: a version of the "long division" algorithm for univariate polynomials.

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Some partial results were achieved very recently by Peter Scheiblechner, `arXiv:1203.5706, 1112.2489`, but the general answer seems unclear.

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In other words, the order is restored, and Abelian integrals of rational 1-forms behave not too badly compared with integrals of polynomial 1-forms.

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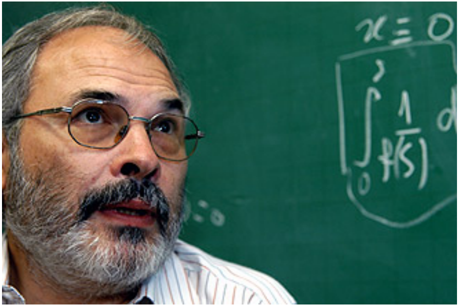
3. **Going beyond algebra:** can similar bounds be proved for integrable vector fields and their perturbations in the class of trigonometric polynomials, living on $\mathbb{C}P^1 \times \mathbb{T}^1$ or on $\mathbb{T}^1 \times \mathbb{T}^1$?

Final message



Till 120, Jaume!