Interval Translation Maps

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New Trends in Dynamical Systems

Salou 2012

Interval Translation Maps

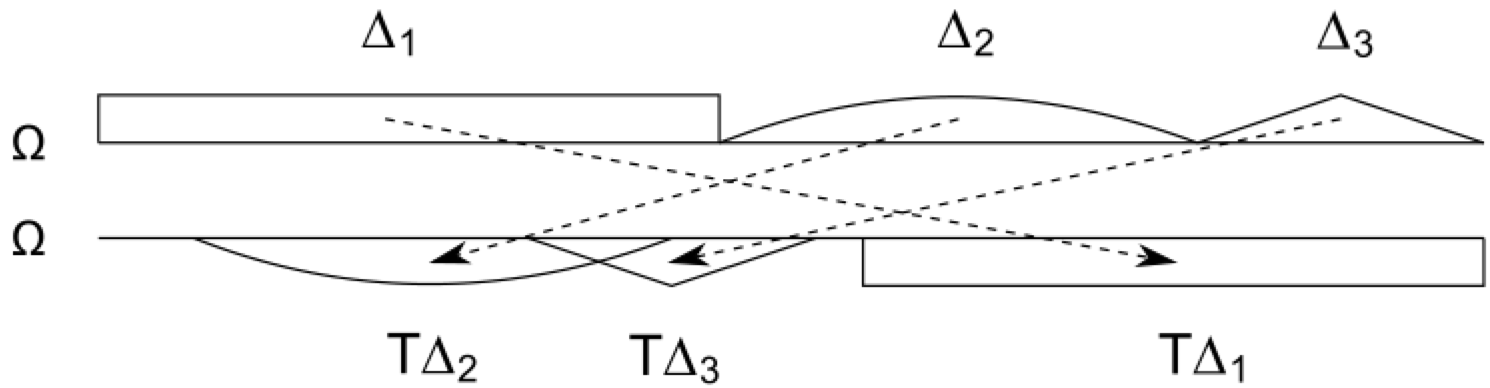


FIGURE: An Interval Translation Map of $d = 3$ intervals.

$$0 = \beta_0 < \beta_1 < \cdots < \beta_d = 1, \quad \Delta_j := [\beta_{j-1}, \beta_j),$$

$$\Omega := [0, 1), \quad \Omega = \sqcup_{j=1}^{d} \Delta_j.$$

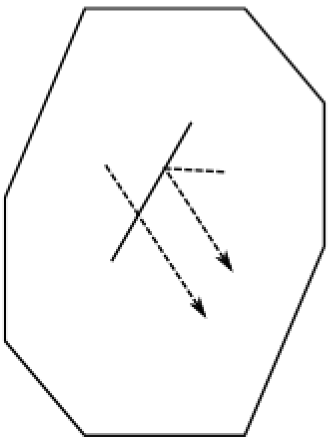
An *interval translation* $T : \Omega \to \Omega$ is a map given by a translation on each of Δ_j:

$$T|_{\Delta_j} : x \mapsto x + \gamma_j,$$

for some $(\gamma_1, \ldots, \gamma_d) \in \mathbb{R}^d$.

Motivation

Introduced in 1995 by Boshernitzan, Kornfeld.



- Non-invertible generalizations of Interval Exchange Transformations
- Polygonal billiards with semi-permeable walls
- Applications: digital filters, conductivity of crystals in a magnetic field

Lebesgue measure is no longer invariant. New effects due to this.

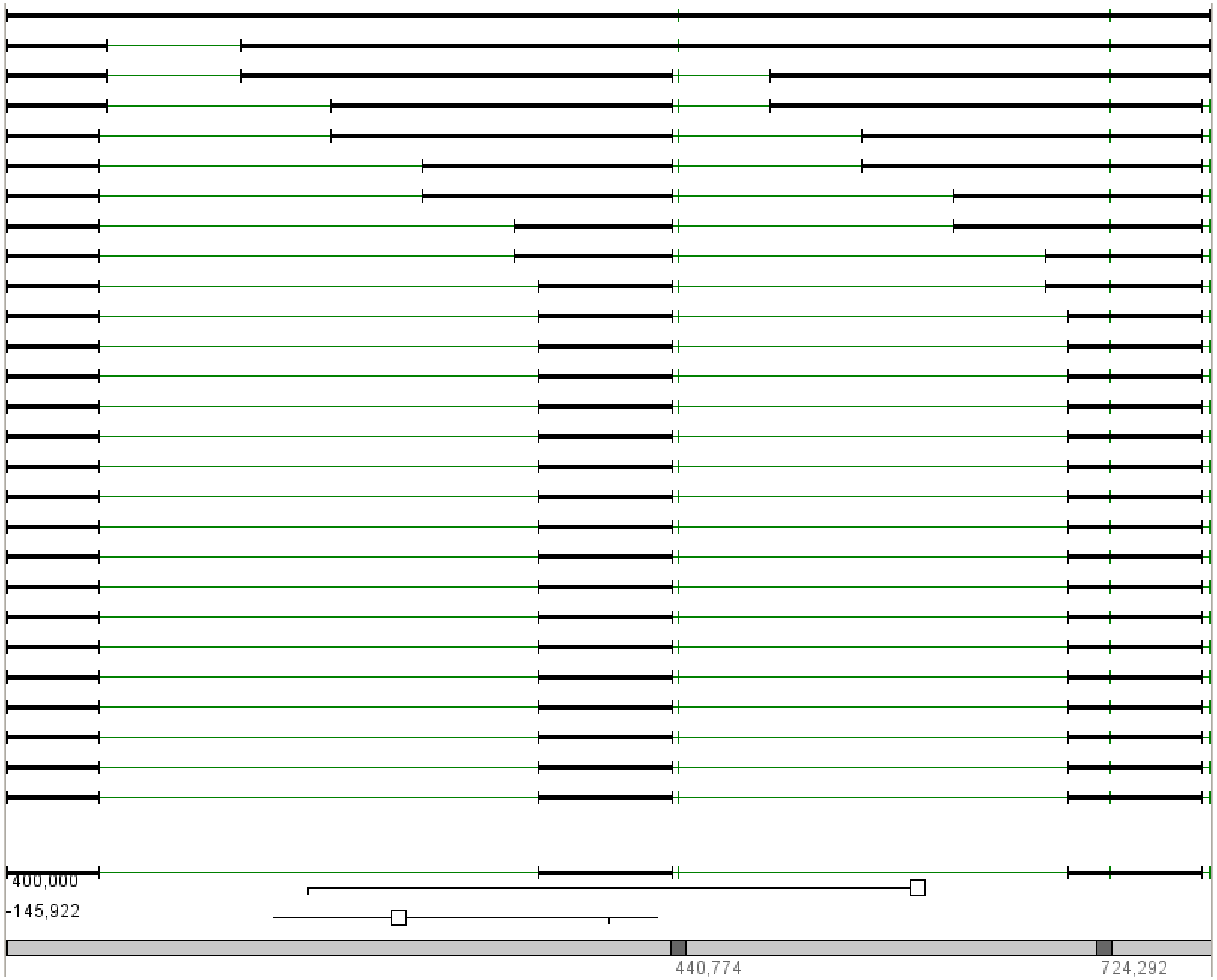
Limit set

Let $\Omega_0 = \Omega$, $\Omega_n = T\Omega_{n-1}$ for $n \geq 1$.
The *limit set* X is the closure of $\cap_{n=1}^{\infty} \Omega_n$.

An interval translation map is called of *finite type* if $\Omega_{n+1} = \Omega_n$ for some n, otherwise it is called of *infinite type*.

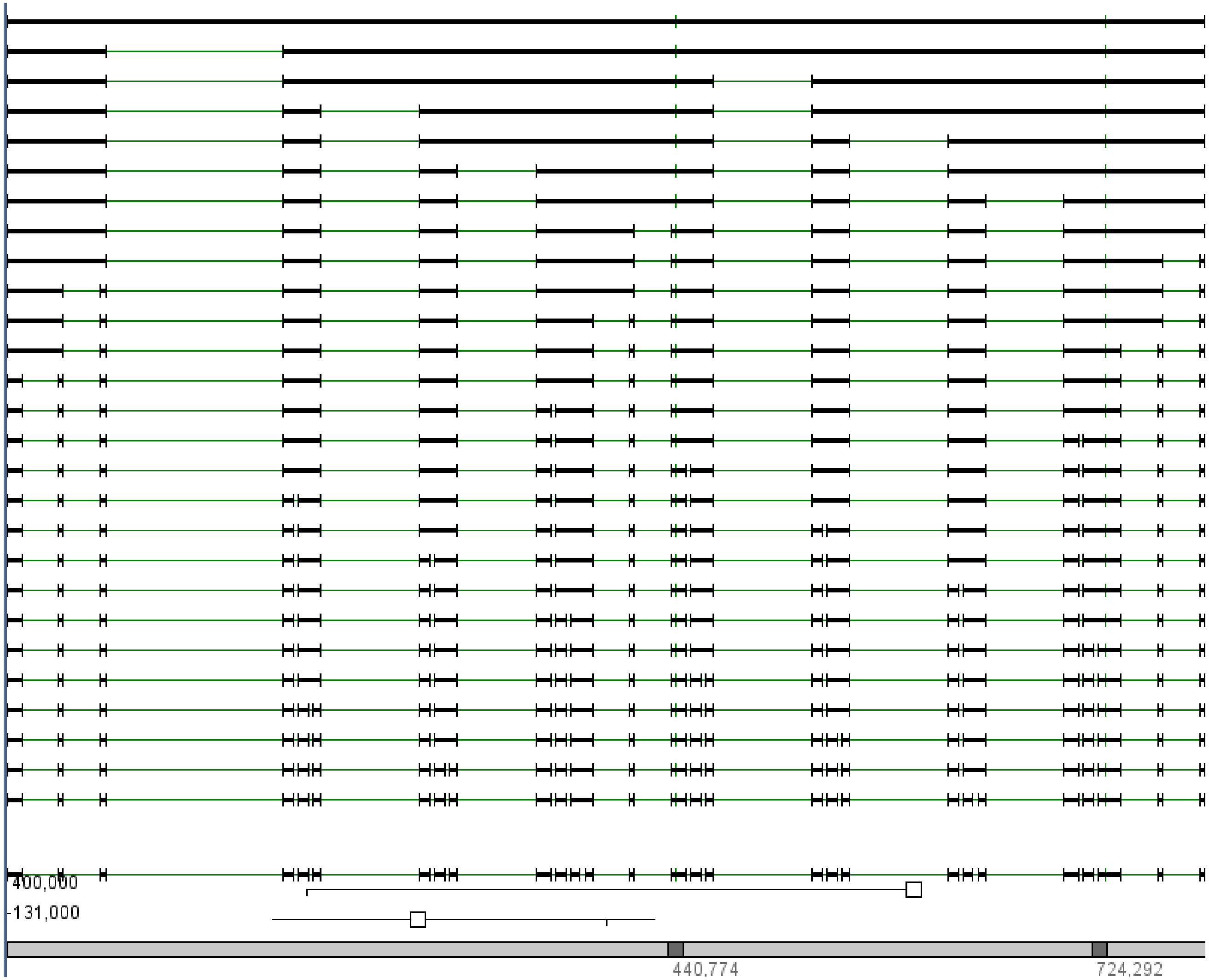
Denote the set of infinite type ITMs by $\mathcal{S}$.

Iterations: example 1



X is 3 intervals.

Iterations: example 2



X is 28 intervals.

Finiteness dichotomy

THEOREM (BOSHERNITZAN, KORNFELD, 1995)

- $\mathrm{rk}(\beta_i, \gamma_i)_{\mathbb{Q}} \leq 2 \Rightarrow T$ *is of finite type.*
- *There exists a translation map of three intervals of infinite type.*

THEOREM (SCHMELING, TROUBETZKOY, 1998)

- *Finite type* $\Leftrightarrow X$ *is a finite union of intervals,* $T|_X$ *is IET.*
- *Infinite type,* $T|_X$ *is transitive* $\Rightarrow X$ *is a Cantor set.*

Parameter space

The space $\mathrm{ITM}(d)$ of d intervals' translations is a convex polytope in $\mathbb{R}^{2d-1}$ with the Euclidean metric and the Lebesgue measure.

For every $n \geq 0$, $\Omega_{n+1} = \Omega_n$ is finitely many linear inequalities. Thus the set of finite type ITMs is at least a union of countably many open cells.

FINITENESS PROBLEM

How big is the set $\mathcal{S}$ of ITMs of infinite type?

THEOREM (2012)

arXiv:1203.3405

In the 5-dim space $\mathrm{ITM}(3)$, *the set $\mathcal{S}$ has zero Lebesgue measure. Moreover, from numerics (Bruin, Clack, 2011) follows*

$$4 \leq \dim_H(\mathcal{S} \cap \mathrm{ITM}(3)) \leq 4.88.$$

Main tool: induction (renormalization)

Global idea of renormalization

- We investigate some class of maps
- Renormalizable maps: there exists a proper region s.t. the rescaled first-return map is of the same class
- Study the dynamics of some renormalization operator in the parameter space:
  - Fixed points ~ self-similar maps (like BK example)
  - Invariant sets
- Learn about the structure of the parameter space

Why the induction is useful here

$\Delta \subset \Omega$ is *regular* if $\forall x \in \Omega$ some $T^n x \in \Delta$, n uniformly bounded.
T_Δ is the induced map.

$\Delta \subset \Omega$ is a *trap* if it is regular and $T\Delta \subset \Delta$. Then $T_\Delta = T|_\Delta$.

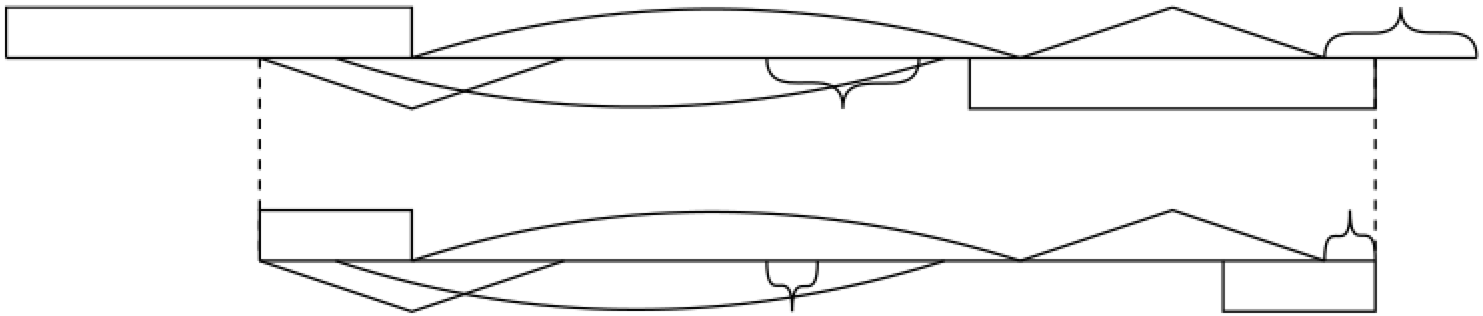
Lemma

Assume X is transitive.

- *Let T have finite type. Then for any regular Δ the map T_Δ has finite type.*
- *Let T_Δ have finite type for some regular Δ. Then T has finite type.*

So, we renormalize until we see it's already finite type.
Otherwise it's infinite type.

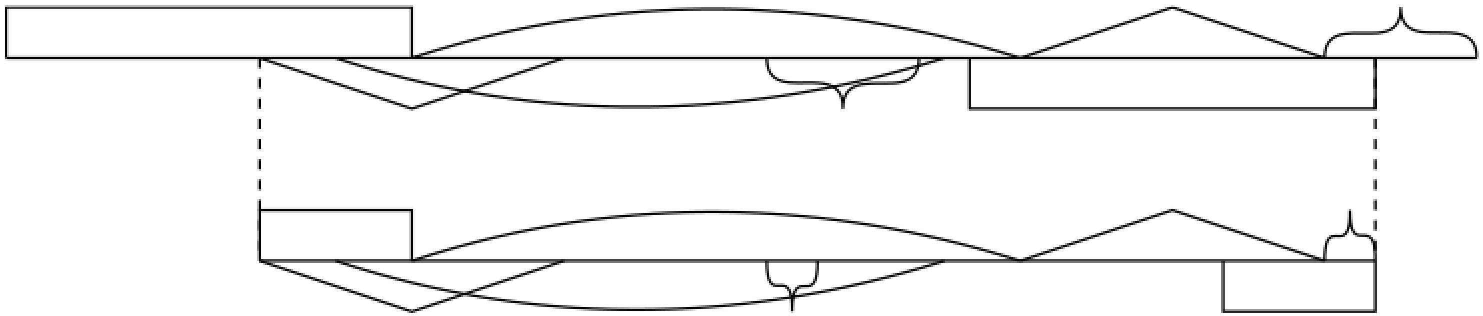
First step: dimension reduction



$T : \Omega \to \Omega$ is *tight* if $[\inf T\Omega, \sup T\Omega) = \Omega$.
$\mathrm{TITM}(d)$ is the space of tight ITMs of d intervals.

$\dim \mathrm{TITM}(d) = \dim \mathrm{ITM}(d) - 2 = 2d - 3.$

First step: dimension reduction



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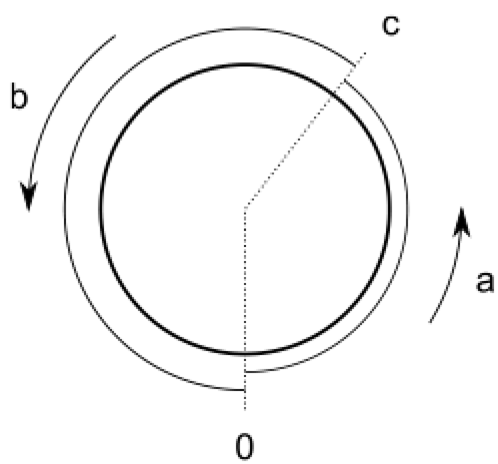
LEMMA

For any $T \in \mathrm{ITM}(d)$ there exists a trap Δ such that the map T_Δ is a tight interval translation map of r intervals, $r \leq d$.

Special class: double rotations



Suzuki, Ito, Aihara, 2005

A *double rotation* is

$$f_{(a,b,c)}(x) = \begin{cases} \{x+a\}, & \text{if } x \in [0,c), \\ \{x+b\}, & \text{if } x \in [c,1). \end{cases}$$

Independent rotations of two complementary arcs of S^1.

$$\dim \mathrm{Rot}(2) = 3.$$

Any double rotation is an ITM of 2–4 intervals.

Tight ITMs reduce to double rotations

THEOREM (2012)

$\mathrm{TITM}(3) = A \cup B \cup C \ \cup \ K$:

- $A \cup B \cup C$ *is open and dense.*
- K *is a union of countably many hyperplanes.*

Moreover,

- *any* $T \in A$ *is a double rotation,*
- *any* $T \in B$ *is reduced to a* $\mathrm{Rot}(2)$ *via Type 1 induction,*
- *any* $T \in C$ *is reduced to a* $\mathrm{Rot}(2)$ *via Type 2 induction.*

The inductions are piecewise-invertible rational maps.

Why double rotations

Theorem (Bruin, Clack, 2011)

The set $\mathcal{S} \cap \mathrm{Rot}(2)$ *has zero Lebesgue measure.*
Moreover, numerically

$$2 \leq \dim_H(\mathcal{S} \cap \mathrm{Rot}(2)) \leq 2.88.$$

Proof by Suzuki, Ito, Aihara's renormalization in the parameter space.

Whole strategy

THEOREM (2012)

In the 5-dim space ITM(3), *the set* $\mathcal{S}$ *has zero Lebesgue measure. Moreover, from numerics (Bruin, Clack, 2011) follows*

$$4 \le \dim_H(\mathcal{S} \cap \mathrm{ITM}(3)) \le 4.88.$$

1. Reduce to the tight ITMs (dim-2)
2. Reduce to the double rotations
3. Run the renormalization for the double rotations
4. Infinite type maps is an $\mathcal{R}$ -invariant subset
5. $\mathcal{R}$ has an ergodic a.c.i.m

Open problems

- Validate the dim numerics of Bruin, Clack.
- Is it true that for any d, infinite types are zero measure?
- Can X be a Cantor set of positive measure?
- Is X(T) continuous in the Hausdorff topology?

- Is it true that every ITM has an SRB measure constructed as a limit of the renormalized Lebesgue measures? If so, is it continuous in T?

And for piecewise translations in $\dim > 1$, almost nothing is known.

Renormalization for double rotations

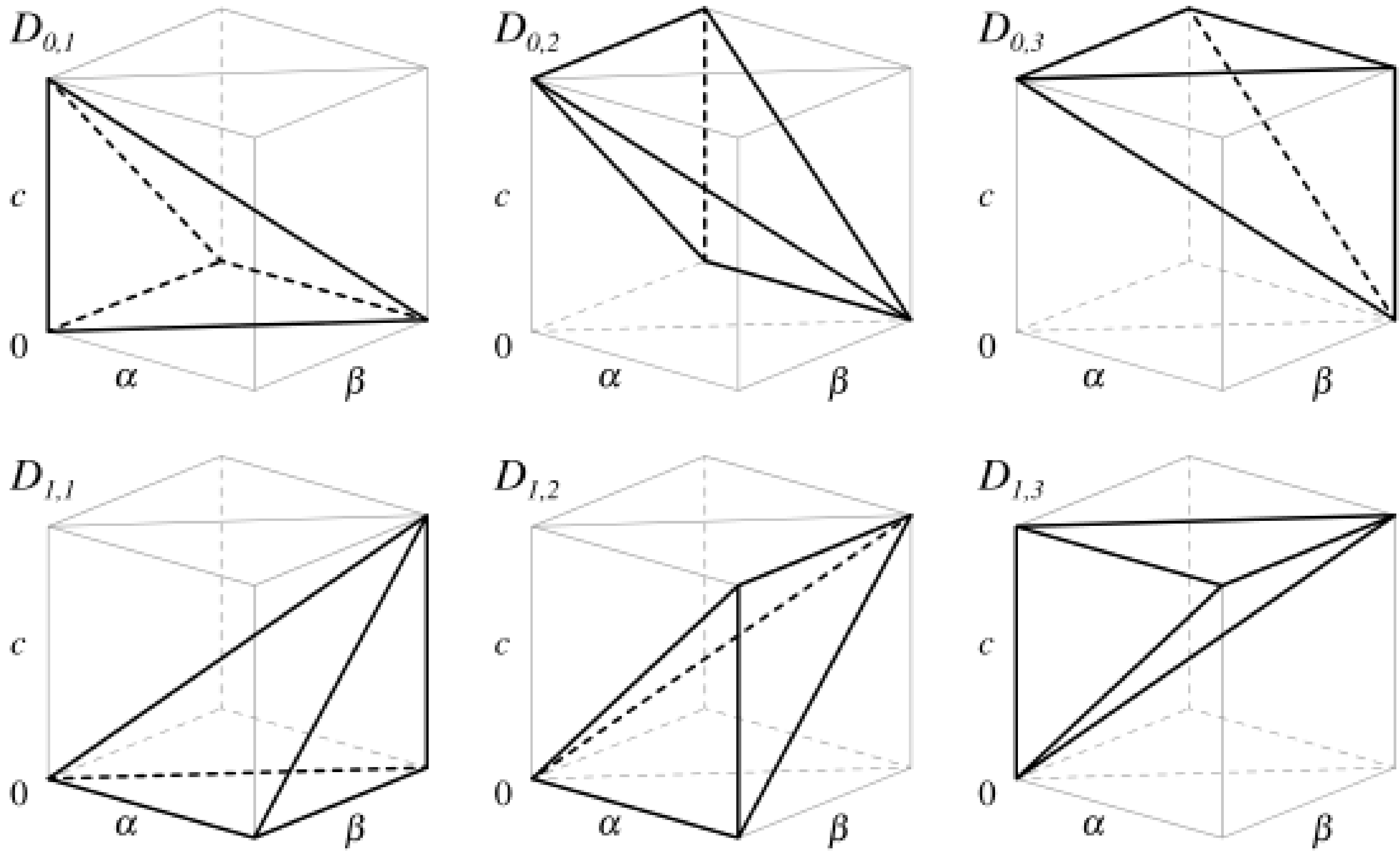
Let $D = [0,1) \times [0,1) \times [0,1]$ be the parameter space of double rotations.

$$D_{0,1} = \{(\alpha,\beta,c) \in D_0 \,|\, c \le 1-\beta\}, \qquad D_{1,1} = \{(\alpha,\beta,c) \in D_1 \,|\, c \le \beta\},$$
$$D_{0,2} = \{(\alpha,\beta,c) \in D_0 \,|\, 1-\beta < c < 1-\alpha\}, \qquad D_{1,2} = \{(\alpha,\beta,c) \in D_1 \,|\, \beta < c < \alpha\},$$
$$D_{0,3} = \{(\alpha,\beta,c) \in D_0 \,|\, 1-\alpha \le c\}, \qquad D_{1,3} = \{(\alpha,\beta,c) \in D_1 \,|\, \alpha \le c\}.$$

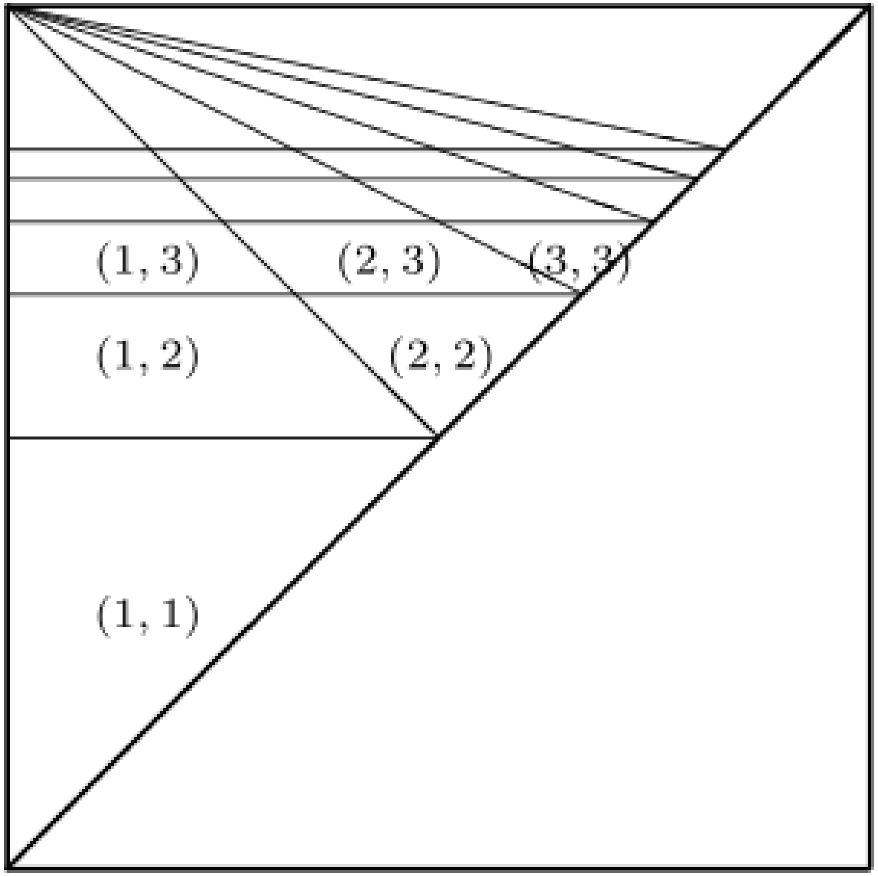
$$\mathcal{R}(\alpha,\beta,c) = \begin{cases} \left(\left\{\frac{\alpha}{1-\beta}\right\}, \left\{\frac{\beta}{1-\beta}\right\}, \frac{c}{1-\beta}\right) & \text{if } (\alpha,\beta,c) \in D_{0,1}, \\ \left(\left\{\frac{\alpha-1}{\alpha}\right\}, \left\{\frac{\beta-1}{\alpha}\right\}, \frac{c+\alpha-1}{\alpha}\right) & \text{if } (\alpha,\beta,c) \in D_{0,3}, \\ \left(\left\{\frac{\alpha-1}{\beta}\right\}, \left\{\frac{\beta-1}{\beta}\right\}, \frac{c}{\beta}\right) & \text{if } (\alpha,\beta,c) \in D_{1,1}, \\ \left(\left\{\frac{\alpha}{1-\alpha}\right\}, \left\{\frac{\beta}{1-\alpha}\right\}, \frac{c-\alpha}{1-\alpha}\right) & \text{if } (\alpha,\beta,c) \in D_{1,3}. \end{cases}$$

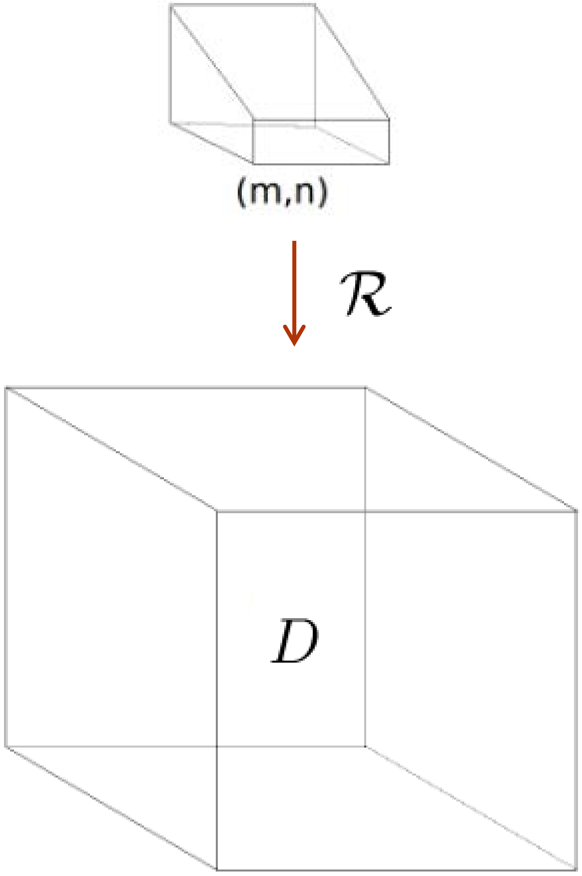
For other pieces, $f_{(\alpha,\beta,c)}$ can already be shown to be finite type.

Parameter space partition



Parameter space subpartition





Theorem (Mañé)

Let $\mathcal{R}: D \to D$ be a C^1 piecewise expanding map of a compact manifold D with a bounded distortion. Assume also $\mathcal{R}$ is topologically mixing and preserves a Markov partition with finite image partition. Then $\mathcal{R}$ has an absolutely continuous invariant probability measure μ. Moreover, μ is ergodic, its density is bounded and bounded away from zero.

Thus any invariant set has either full or zero Lebesgue measure.

Main theorem is proven.

Thank you!

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