Totally and partially ordered Non-singular Morse-Smale Flows on S^3

Beatriz Campos and Pura Vindel
Departament de Matemàtiques. IMAC. Universitat Jaume I de Castelló. Spain.
e-mail: campos@mat.uji.es, vindel@mat.uji.es

ABSTRACT

Non-singular Morse-Smale flows are characterized by the round handle decomposition of the manifold where they are defined [1], [4]. For NMS flows on the 3-dimensional sphere S^3, M. Wada obtains a characterization of the flows in terms of links of periodic orbits [5]. From the round handle decomposition of NMS flows on S^3 we determine which flows have heteroclinic trajectories connecting saddle orbits due to transversal intersections of invariant manifolds [2]. In this paper we show that the presence of heteroclinic trajectories imposes an order in the round handle decomposition of a Non-singular Morse-Smale flow on S^3. We also obtain that this order is total for NMS flows composed of one repulsive, one attractive and *n* unknotted saddle orbits.

1. DEFINITIONS AND PREVIOUS RESULTS

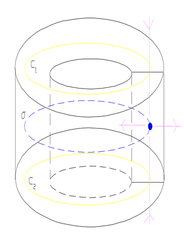
Non-Singular Morse-Smale flows are characterized by:

- Finite number of hyperbolic periodic orbits
- No singular points
- Invariant manifold of a saddle orbits intersects transversally invariant manifolds of the others.

Round Handles

A pair $(M,\partial_- M)$ is called:

- A round 0-handle if $(M,\partial_- M)\cong(D^2\times S^1,\emptyset)$.
- a round 2-handle if $(M,\partial_- M)\cong(D^2\times S^1,\partial D^2\times S^1)$.
- a round 1-handle if $(M,\partial_- M)\cong(D^1\times D^1\times S^1, D^1\times\partial D^1\times S^1)$:

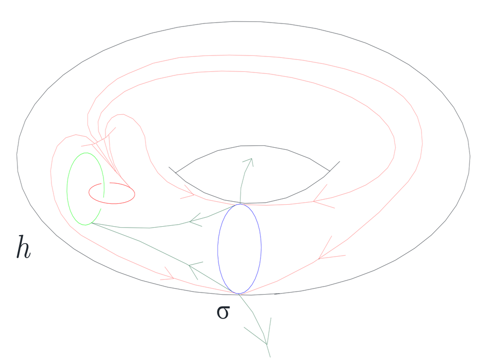


A Round Handle decomposition

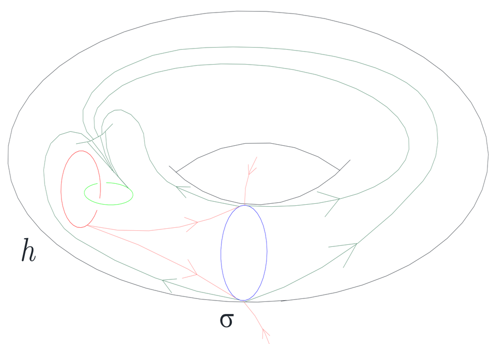
for $(X,\partial_- X)$ is a filtration $\partial_- X\times I=X_0\subset X_1\subset X_2\subset\ldots\subset X$ where each X_i is obtained from X_{i-1} by attaching a round handle.

Theorem (Asimov, Morgan): Let $(X, \partial_- X)$ have a NMS flow. Then $(X,\partial_- X)$ has a round handle decomposition whose core circles are the close orbits of the flow.

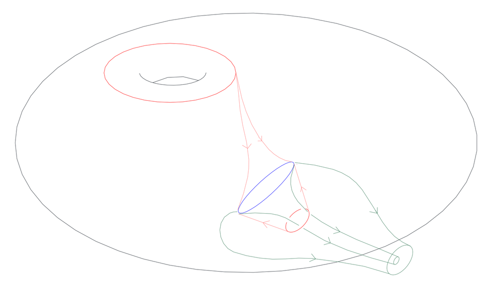
Fat Round Handles are obtained from a manifold X_i by attaching a round 1-handle:



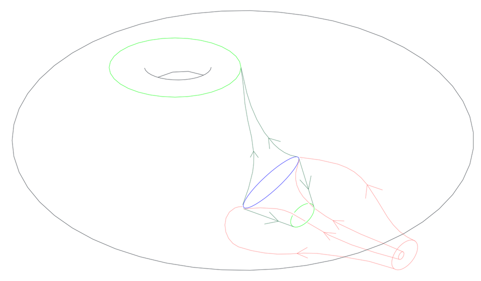
repulsive fat round handle: (h.u)



attractive fat round handle: (h.u)



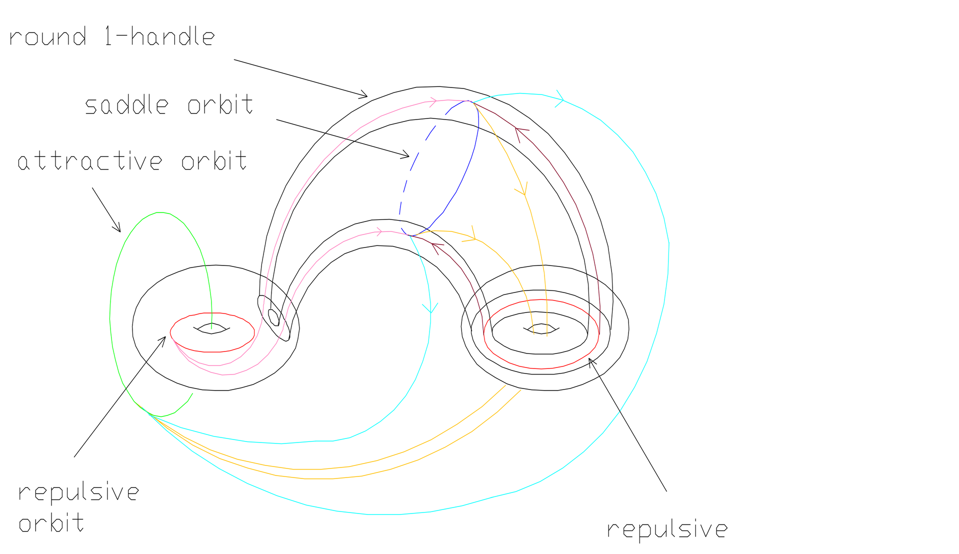
repulsive fat round handle: (d.d.u)



attractive fat round handle: (d.d.u)

Fat Round Handle Decomposition

for a manifold M is a filtration $\emptyset = M_0 \subset M_1 \subset \cdots \subset M_N = M$ where each $M_i = \cup_{j=1}^{i} C_j$ and C_j is a 0 or a 2-handle, or a fat round handle.



Attachment to two tori by means of one essential and one inessential circles

Theorem (Wada): characterizes the set of the periodic orbits of NMS flows on S^3 in terms of knots and links, using a generator, the hopf link, and six operations, basically, split sums and cabling.

- **The link of periodic orbits of a NMS system on S^3 is not in 1-1 correspondence with the class of topological equivalence of the associated flow.**
- **We associate dual graphs to flows in order to obtain a topological invariant in 1-1 correspondence with the class of flows [3].**
- **We build the primitive dual graphs from the picture of the flow of the six basic flows and we define operations of graphs in order to build graphs for flows obtained by attaching more round handles.**
- **Heteroclinic trajectories connecting saddle orbits appear when there are transversal intersections of the invariant manifolds of the saddle orbits [2].**
- **Transversal intersections occur when fat round handles corresponding to solid tori are identified along their boundaries.**

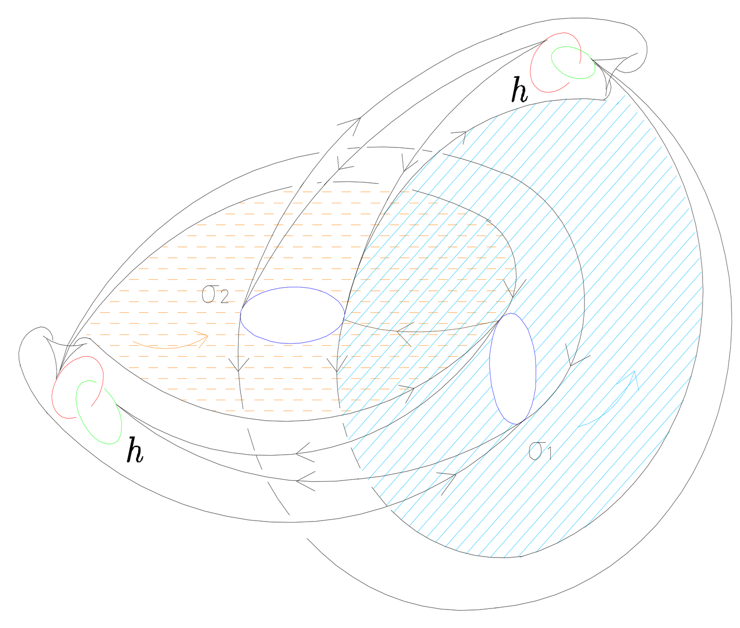
2. PARTIALLY ORDERED NMS FLOWS ON S^3

This flow is obtained by identifying two fat round handles of type h.u along their boundaries

Its link of periodic orbits corresponds to apply twice operation II of Wada on hopf links h.

One heteroclinic trajectory connecting two saddles appears.

$$\sigma_1 < \sigma_2$$



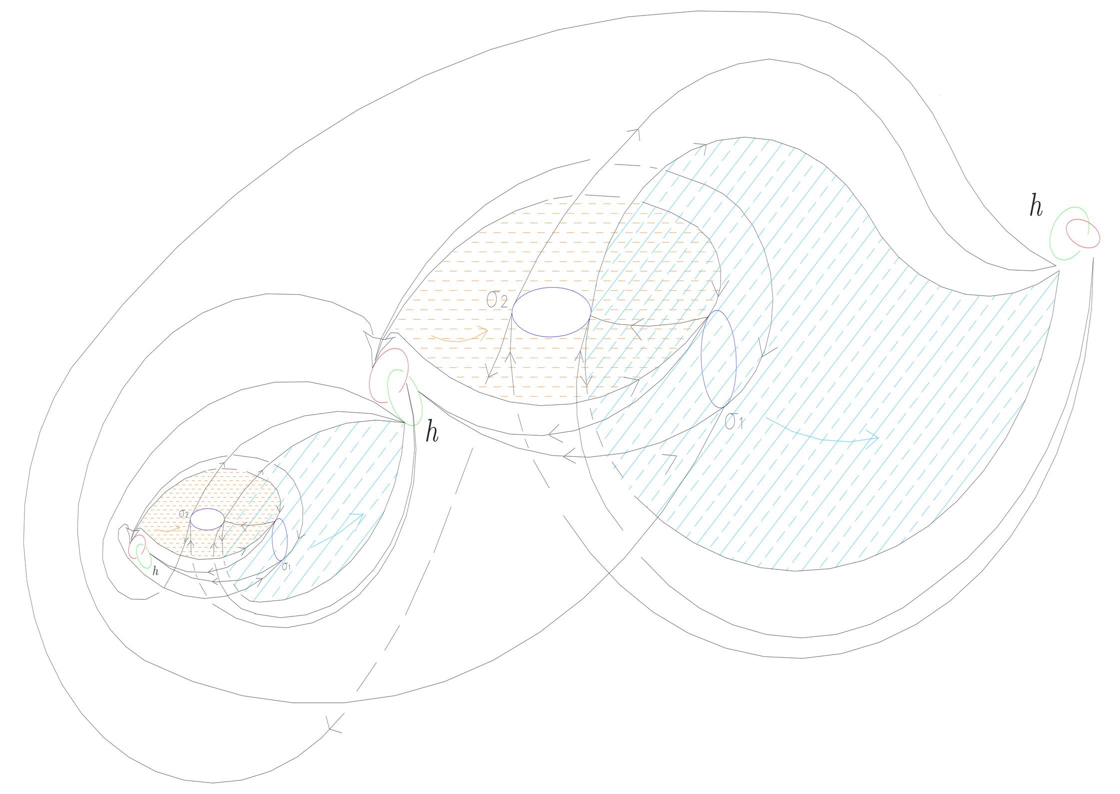
II(II(h,h),h)=h·h·u·u

This flow is obtained by iterating attachments of fat round handles.

Its link of periodic orbits corresponds to apply four times operation II of Wada on hopf links.

Two heteroclinic trajectories connecting saddles appear.

$$\sigma_1 < \sigma_2 \text{ and } \sigma_3 < \sigma_4$$



II(II(II(II(h,h),h),h),h)=h·h·h·u·u·u·u

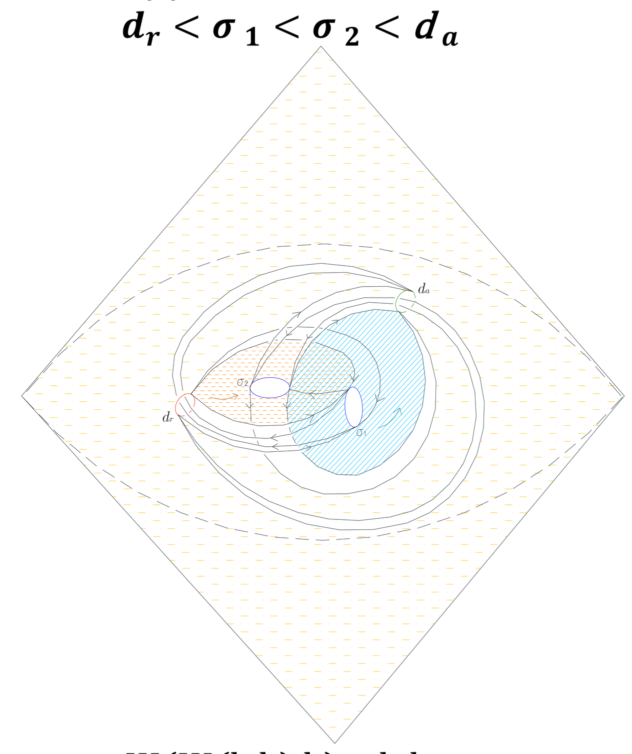
3. TOTALLY ORDERED NMS FLOWS ON S^3

This flow is obtained by identifying two fat round handles of type d.u along their boundaries.

Its link of periodic orbits corresponds to apply twice operation III of Wada on hopf links.

One heteroclinic trajectory connecting two saddles appears.

$$d_r < \sigma_1 < \sigma_2 < d_a$$



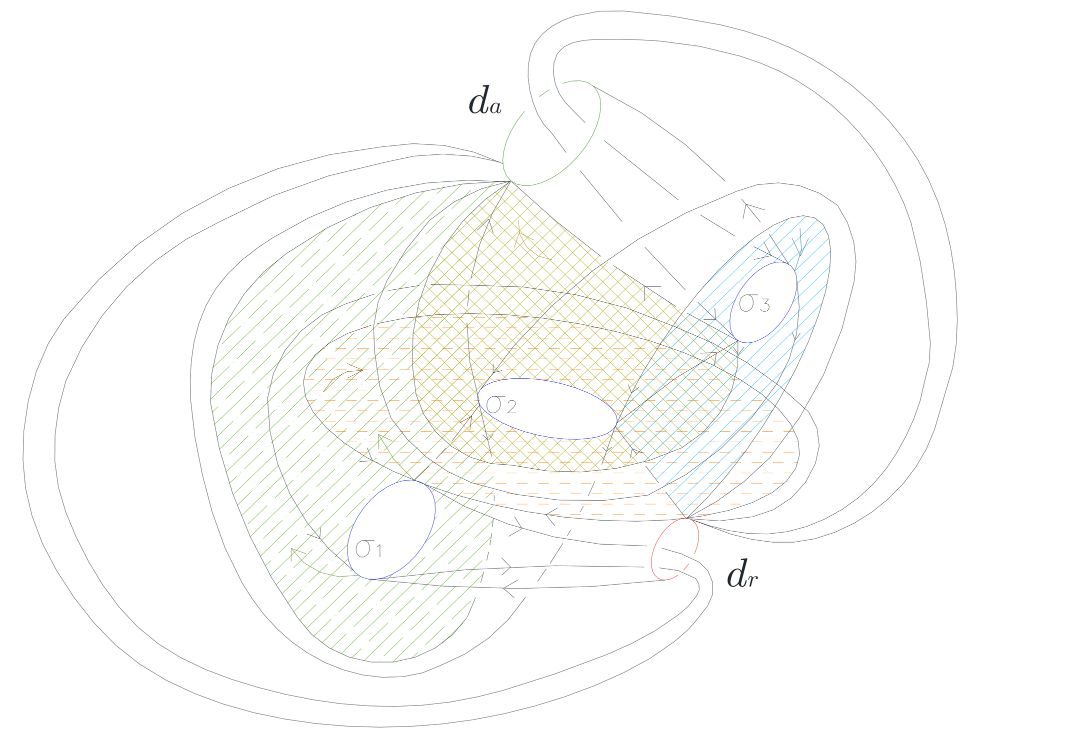
III(III(h,h),h)=d·d·u·u

This flow is obtained by iterating attachments of fat round handles.

Its link of periodic orbits corresponds to apply thrice operation III of Wada on hopf links.

Two heteroclinic trajectories connecting saddles appear.

$$d_r < \sigma_1 < \sigma_2 < \sigma_3 < d_a$$



III(III(III(h,h),h),h)=d·d·u·u·u

4. MAIN RESULTS

- ***Theorem*. The heteroclinic trajectories induce an order in a NMS flow on S^3 with unknotted and unlinked saddle periodic orbits.**
- ***Corollary.* The set of unknotted and unlinked saddle orbits is totally ordered when these type of NMS flows on S^3 come from operation III**

The link can be written as:

$$d_r \cdot d_a \cdot u \cdot \ldots \cdot u \cdot u$$

and the order is

$$d_r < \sigma_1 < \sigma_2 < \ldots < \sigma_n < d_a$$

where $\sigma_1 < \sigma_2$ means that the heteroclinic trajectory goes from σ_1 to σ_2.

REFERENCES

[1] D. Asimov. *Round handles and non-singular Morse-Smale flows.* Annals of Mathematics, 102 (1975), 41-54.

[2] B. Campos and P. Vindel. *Transversal intersections of invariant manifolds of NMS flows on S^3.* Discrete and Continous Dynamical Systems 32 (2012), 41-56.

[3] B. Campos and P. Vindel. *Dual graphs of Non-singular Morse-Smale flows on S^3 with unknotted saddle periodic orbits.* To be published.

[4] Morgan, J.W. *Non-singular Morse-Smale flows on 3-dimensional manifolds.* Topology 18 (1978), 41-53.

[5] Wada, M. *Closed orbits of non-singular Morse-Smale flows on S^3.* J. Math. Soc. Japan 41, nº 3 (1989), 405-413.