Simultaneous Linearization of Involutions

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Goal: Discuss the simultaneous linearization of pairs of involutions whose composition is normally hyperbolic in R^n.

Historical Facts

The problem of simultaneous behavior of diffeomorphisms has been treated by many authors and several interesting results have been obtained in different contexts.
Among such results, we mention the Bochner-Montgomery Theorem , which is a well known and useful result about linearization of a compact group of transformations around a fixed point. This theorem is preceded by a remarkable work by Cartan. We treat the problem of simultaneous $C^0 - conjugacy$ of involutions.

Voronin presents a list of problems including the analytic classification of divergent diagrams of pairs of folds and pairs of associated involutions on $(\mathbb{C}, 0)$.

1i- Cartan H. : *If a compact group of authomorphisms has a fixed point then in a suitable chosen local coordinates around a fixed point all transformations are linear*.

1ii- In our approach we deal with a group generated by two involutions, that in general is not compact.

2i- A classical theorem due to Bochner and Montgomery implies that an s-tuple of involutions generating an Abelian group is simultaneously linearizable.

2ii- Two involutions with normally hyperbolic composition generates a non-Abelian group.

– An involution is a germ of diffeomorphism $\varphi : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ satisfying $\varphi \circ \varphi = Id$.

– For any involution φ on $(\mathbf{R}^n, 0)$, the germ of diffeomorphism $h = \frac{1}{2}(I + d\varphi(0) \circ \varphi)$ of $(\mathbf{R}^n, 0)$ is a conjugacy between φ and the germ of its linear part $d\varphi(0)$ at zero, namely $d\varphi(0) = h \circ \varphi \circ h^{-1}$.

– Let $\phi : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and $\mathcal{F}(f)$ denote the fixed-point set of ϕ.

– Let $\phi : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ be a germ of diffeomorphism, $\phi \neq Id$. Suppose that $\mathcal{F}(\phi)$ is a submanifold in $(\mathbf{R}^n, 0)$ and that $\dim \mathcal{F}(\phi) = k$. We say that ϕ is normally hyperbolic if the spectrum of $d\phi(0)$ has, counting multiplicity, $n - k$ elements out of the unit circle $S^1 \subset \mathbf{C}$.

– Given two involutions $\varphi_1, \varphi_2 : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$, we say that φ_1 and φ_2 are transversal if $\mathcal{F}(\varphi_1)$ and $\mathcal{F}(\varphi_2)$ are in general position at 0, i.e.,

$$\mathbf{R}^n = T_0\mathcal{F}(\varphi_1) + T_0\mathcal{F}(\varphi_2), \tag{1}$$

where $T_0 S$ denotes the tangent space to S at 0.

Example

Let $f_0 : R^2, 0 -> R^2, 0$ be a fold mapping represented by

$$f_0(x, y, z) = (x + x^3 + y^2 + y^6, y^2)$$

Singular set of this mapping is $S_0 = \{y = 0\}$ and its symmetric ϕ_0 ($f_0 \circ \phi_0 = f_0$) is an involution where:

$$\phi_0(x, y, z) = (x, -y) + h.o.t$$

with $Fix\{\phi_0\} = S_0$.
Consider any involution in $R^2, 0$ given by

$$\phi_1(x, y, z) = (2x + 3y, -x - 2y) + o(2)$$

So $Fix\{\phi_1\} = S_1 = \{x + 3y + h.o.t = 0\}$.
Observe that S_0 meets S_1 transversally at 0. Moreover, 0 is a hyperbolic (saddle) fixed point of the composition $\phi_0 \circ \phi_1$.
Our main result ensures that the pair (ϕ_0, ϕ_1) **is simultaneously** C^0 − *linearizable*.

main result

Remark that, generically speaking it is no loss of generality to assume that the pair (φ_1, φ_2) satisfies

$$\mathcal{F}(\varphi_i) = \mathcal{F}(d\varphi_i(0)), \; i = 1, 2 \tag{2}$$

main result

Consider $L : \mathbf{R}^n \to \mathbf{R}^n$ a linear normally hyperbolic isomorphism and

$$\mathbf{R}^n = E^s \oplus E^u \oplus \mathcal{F}(L), \tag{3}$$

where E^s and E^u are respectively the stable and unstable subspaces of L. Let

$$\begin{array}{ccc} \mathbf{R}^n & \xrightarrow{L} & \mathbf{R}^n \\ \downarrow & & \downarrow \\ \mathcal{F}(L) & \xrightarrow{I} & \mathcal{F}(L) \end{array} \tag{4}$$

be the hyperbolic bundle automorphism covering the identity I, whose fibres are all equal to $E^s \oplus E^u$.

main result

Theorem A:
Let (φ_1, φ_2) be a pair of transversal involutions on $(\mathbf{R}^n, 0)$ such that $\mathcal{F}(\varphi_i) = \mathcal{F}(d\varphi_i(0))$, $i = 1, 2$, $\varphi_1 \circ \varphi_2$ is normally hyperbolic and each φ_i respects the fibre bundle for $L = d(\varphi_1 \circ \varphi_2)(0)$.

Then

the pair is C^0-conjugated to (L_1, L_2), where $L_i = d\varphi_i(0)$, $i = 1, 2$.

Proposition

Proposition

Let L_1 and L_2 be linear involutions on $\mathbf{R}^n$ with $L_1 \circ L_2$ normally hyperbolic. Then $L_1 \circ L_2$ is hyperbolic **iff,** L_1 and L_2 are transversal, n is even and $\dim \mathcal{F}(L_1) = \dim \mathcal{F}(L_2) = n/2$.

Theorem:

Let L_1, L_2 be transversal linear involutions on $\mathbf{R}^n$, $n \geq 2$, with

i) $L_1 \circ L_2$ normally hyperbolic

and

ii) $\dim \mathcal{F}(\varphi_1) = \dim \mathcal{F}(\varphi_2) = r$.

Then

$n/2 \leq r \leq n-1$ and the pair (L_1, L_2) is linearly equivalent to a pair (ψ_1, ψ_2) such that ψ_1 and ψ_2 have matrices

$$\psi_1 = \left(\begin{array}{c|c|c} -I_{n-r} & 0 & \\ \cline{1-2} A & I_{n-r} & 0 \\ \hline 0 & 0 & I_{2r-n} \end{array}\right), \quad \psi_2 = \left(\begin{array}{c|c|c} I_{n-r} & I_{n-r} & \\ \cline{1-2} 0 & -I_{n-s} & 0 \\ \hline 0 & 0 & I_{2r-n} \end{array}\right),$$

with the submatrices

$$\phi_1 = \left(\begin{array}{c|c} -I_r & 0 \\ \hline A & I_r \end{array}\right), \quad \phi_2 = \left(\begin{array}{c|c} I_r & I_r \\ \hline 0 & -I_r \end{array}\right)$$

(A possessing some suitable form)

example

Consider $\varphi_0(x, y) = (x, -y)$,
$\varphi_1(x, y) = (ax + by, cx - ay)$, with $a^2 + bc = 1$ and
$\bar{\varphi}_1(x, y) = (\bar{a}x + \bar{b}y, \bar{c}x - \bar{a}y)$, with $\bar{a}^2 + \bar{b}\bar{c} = 1$

Then (φ_0, φ_1) is linearly equivalent to $(\varphi_0, \bar{\varphi}_1)$ **iff** $a = \bar{a}$ and $bc = \bar{b}\bar{c}$

Linear Involutions in $2D$

Let φ_0 and φ_1 be a pair of transversal linear involutions in $R^2, 0$.

We may choose coordinates such that

$\varphi_0(x, y) = (x, -y)$ and $\varphi_1(x, y) = (ax + by, cx - ay)$, with $a^2 + bc = 1$

0 is a hyperbolic fixed point provided that $a^2 > 1$ (so $bc < 0$).

Let J be the set of all homeomorphisms $h : R^2, 0 \to R^2, 0$ with $h\varphi_0 = \varphi_0 h$.

Consider:

1- Via $h_0 \in J$ where $h_0(x, y) = (x, Bb^{-1}y)$ and $B = (-bc)^{1/2}$ we may take $\varphi_1(x, y) = (ax + By, Bx + ay)$, with $a^2 - B^2 = 1$

Denote
$\lambda = (a + B)$,
$r(x, y) = (x + y, x - y)$ and
$s(x, y) = (\lambda x, \lambda^{-1} y)$

We have the following relation: $r^{-1}sr = \varphi_1$

Corresponding to another involution $\bar{\varphi}_1(x,y) = (\bar{a}x + \bar{b}y, \bar{c}x - \bar{a}y)$, with $\bar{a}^2 + \bar{b}\bar{c} = 1$ there are the objects:

$\bar{\varphi}_1(x,y) = (\bar{a}x + \bar{B}y, \bar{B}x - \bar{a}y)$, with $\bar{a}^2 - \bar{B} = 1$, $\bar{\lambda}$ and $\bar{s}$ satisfying $r^{-1}\bar{s}r = \bar{\varphi}_1$

2- Define
$\rho : R_+ \to R_+$ by $\rho(x) = x^k$, where $k = \frac{log\lambda}{log\bar{\lambda}}$,

$P : R, 0 \to R, 0$ by $P(x) = \rho(x)$ for $x \geq 0$ and $P(x) = -\rho(-x)$ for $x < 0$.

$K : R^2 \to R^2, 0$ by $K(x, y) = (P(x), P(y))$ that satisfies :

$$K\bar{s} = s\bar{K}.$$

Let $h = r^{-1}Kr$

3- Finally

$$h \in J \text{ and } h \circ \varphi_1 = \bar{\varphi_1} \circ h$$

Proof of Th. A

Lemma

Let φ be an involution. Given $\epsilon > 0$ there exists neighb. $U(0)$ and a $C^r - extension\ \theta : R^2 \to R^2$ of $\varphi \mid U$ of the form $\varphi'(0) + \alpha$ with $\alpha \in C_b^0(R^2)$ (space of bounded continuous mappings) is Lipschitz with bounded constant ϵ.; moreover $\theta \circ \theta = Id$

Theorem A

If 0 is a hyperbolic fixed point of $\phi = \varphi_0 \circ \varphi_1$ then (φ_0, φ_1) is $C^0 -$ *conjugated* to (L_0, L_1) , where $L_i = D\varphi_i(0)$.

From Lemma, let ϕ_i be the extension of φ_i.

So from Hartman's Th: there exists a *unique homeomorphism* h, of the form $h = Id + g$ with g–bounded satisfying:

$$h\phi_0\phi_1 h{-}1 = L_1 L_0$$

By a straightforward caculation we show that $L_0 h\phi_0$ is also a conjugacy between $\phi_0\phi_1$ and $L_1 L_0$

So $h = L_0 h\phi_0$.

Similarly one proves that $h = L_1 h\phi_1$.

Some References

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