An empirical stability analysis of the Caledonian Symmetric Four-Body model

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Introduction

The Caledonian Symmetric Four Body Problem (CSFBP) is a restricted four body system with a symmetrically reduced phase space. The study of the dynamics and stability of four-body systems like CSFBP is relevant in order to determine stable hierarchical arrangements which will be capable of hosting exoplanetary systems. The CSFBP was developed by Steves and Roy in [4]. They have shown that similar to the c^2H stability criterion in the three-body problem, global stability of the CSFBP system depends on a parameter called the Szebehely constant C_0. The Szebehely constant $C_0 = -\frac{c^2E}{G^2M^5}$ is a dimensionless function of the total energy (E) and the magnitude of the angular momentum of the system (c), where G is the gravitational constant, and M is the total mass.

A fundamental limitation in the published studies of the CSFBP model is that the existing numerical integration algorithms are inadequate to study orbits with close encounters, as the numerical integration fails due to collisional singularities. We have recently developed a global regularization scheme that consists of adapted versions of several known regularisation transformations such as the Levi-Civita-type coordinate transformations; that together with a time transformation, removes all the singularities due to colliding pairs of masses [1]. Using this newly developed numerical algorithm, we numerically investigate the relationship between the hierarchical stability of the system and the analytical stability parameter characterised by the Szebehely constant C_0.

1. Regularisation Method and Numerical Integration Scheme

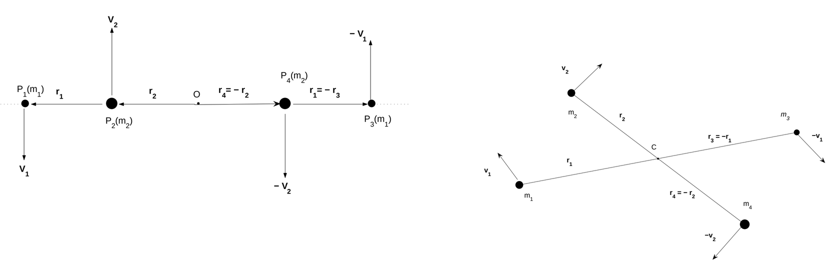

Figure 1: The CSFBP configuration for $t = 0$ and $t > 0$.

The coplanar CSFBP involves two pairs of distinct masses moving in coplanar, initially circular orbits, starting in a collinear arrangement with past-future symmetry and dynamical symmetry (See Fig. 1). The potential function in the Hamiltonian equations of motion of the CSFBP contains singular terms. We have adapted the global regularisation scheme of [5] to incorporate symmetries in the CSFBP. In order to remove collisional singularities in the model, we applied a Levi-Civita type transformation to all inter-body distance vectors. We have derived the regularized Hamiltonian (Γ) from the original Hamiltonian (H) using a time transformation function g which rescales the physical time t to the regularized time τ by

$$\Gamma = g(H - h_0), \tag{1}$$

$$g = \frac{(Q_1^2+Q_2^2)(Q_3^2+Q_4^2)(Q_5^2+Q_6^2)(Q_7^2+Q_8^2)}{(Q_1^2+Q_2^2+Q_3^2+Q_4^2+Q_5^2+Q_6^2+Q_7^2+Q_8^2)^{5/2}}, \tag{2}$$

to derive the regularized Hamiltonian equations

$$\frac{dQ_i}{d\tau} = \frac{\partial\Gamma}{\partial P_i}, \quad \frac{dP_i}{d\tau} = -\frac{\partial\Gamma}{\partial Q_i}, \tag{3}$$

where h_0 is the initial value of H, Q_i are the regularized position coordinates and P_i are the regularized momenta coordinates.

The resulting regularized Hamiltonian equations of motion are free from collisional singularities. Since the symbolic differentiation to derive the gradient of $\Gamma(Q_i, P_i)$ produces a large number of additive and multiplicative terms, we have adapted an algebraic optimisation algorithm in numerically implementing the regularisation scheme. The newly proposed regularisation algorithm is numerically and computationally very efficient in handling all types of two-body close encounters appearing in the CSFBP (See Figure 2)

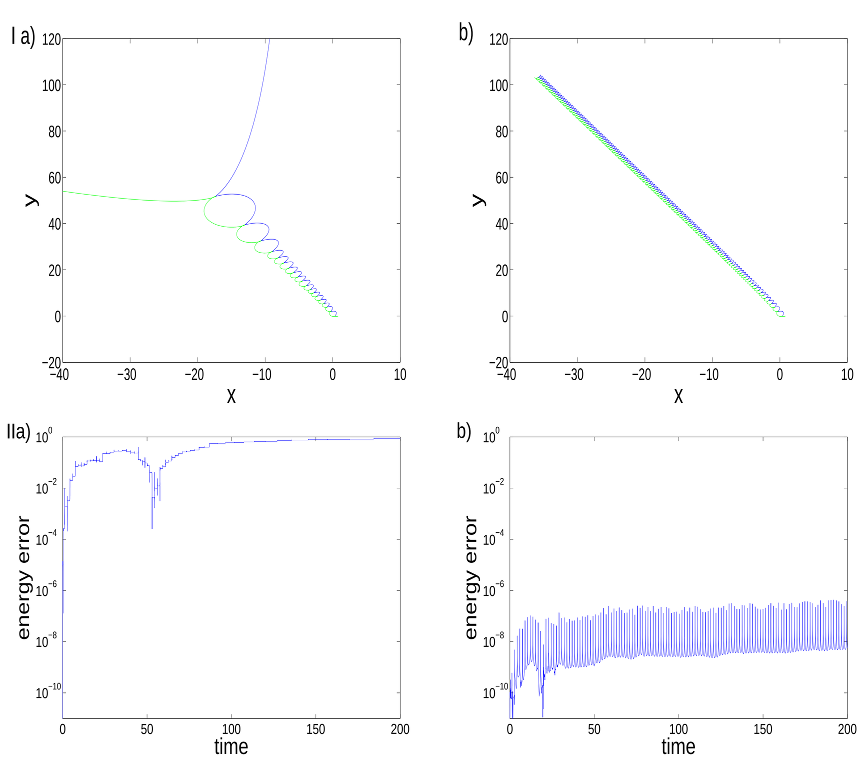

Figure 2: Irregular numerical orbits starting with the same initial conditions over the time period $[0, 200]$ a)non-regularised b) regularised.

2. Hierarchical stability

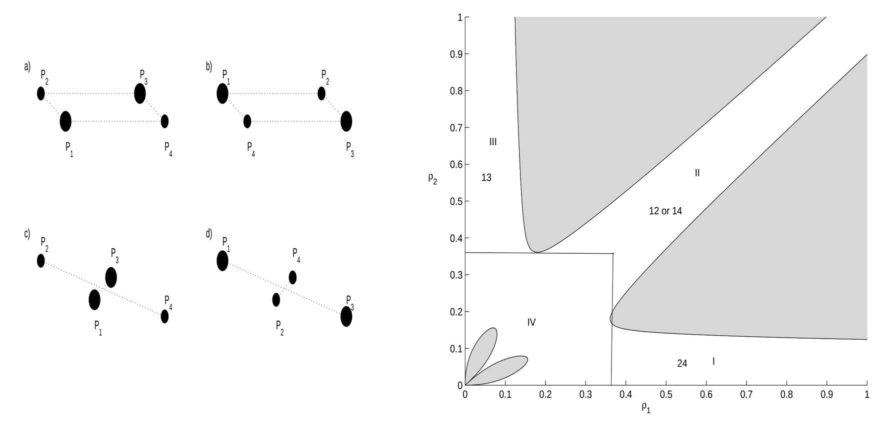

Figure 3: The four possible hierarchies in the CSFBP.

There are four different types of hierarchy states present in the CSFBP (see Fig. 3); two double binaries ('12' and '14') and two single binaries ('13' and '24'). A particular hierarchy state is determined by comparing their relative inter-body distances r_{12}, r_{13}, r_{14} and r_{24} and their radius vectors r_1 and r_2. The CSFBP system is considered to be hierarchically stable if it doesn't change its initial hierarchical arrangement.

According to [4], when the stability parameter C_0 reaches a critical value C_{crit}, the phase space becomes topologically disconnected and hierarchy changes are not possible after that. Szell et.al [2] numerically verified this, however numerical integrations using non-regularised schemes ran into difficulties, whenever they came across orbits with two-body close encounters and they had to avoid all types of collision orbits completely from their analysis.

	Type of changes	$C_0 = 10$	$C_0 = 20$	$C_0 = 30$	$C_0 = 40$	$C_0 = 47$
1213	DB1SB2	446	390	367	314	0
1214	DB1DB2	4	0	0	0	0
1224	DB1SB1	572	483	212	112	0
1312	SB2DB1	430	78	244	211	0
1314	SB2DB2	430	78	244	211	0
1324	SB2SB1	546	130	236	122	0
1412	DB2DB2	4	0	0	0	0
1413	DB2SB2	446	390	367	314	0
1424	DB2SB1	572	483	212	112	0
2412	SB1DB1	369	309	91	33	0
2413	SB1SB2	590	292	102	116	0
2414	SB1DB2	369	309	91	33	0
Total		4778	2942	2166	1578	0

Table 1: Columns represent fixed C_0 values while rows represent the number of each type of hierarchy changes for $\mu = 1$

We numerically investigate the hierarchical stability of the CSFBP using the newly developed regularised codes for a wide range of Szebehely's constant and initial conditions, recording the number and type of hierarchical changes for a comprehensive set of CSFBP orbits, including those which pass through close encounters (See Table 1 and Fig. 4).

Different from [2], we also remove the bias in the initial configurations by including the initial configurations for $r_1 < r_2$ enabling a start with a 13 hierarchy and the initial configurations for the collinear arrangement of bodies in order 1-4-2-3 or 4-1-3-2 enabling a start with a 14 hierarchy. Even with the presence of orbits with close encounters, the global hierarchical features of the model remained unchanged. Beyond the critical value of Szebehely constant C_{crit}, hierarchy changes are not possible for all time, for any values of mass ratio μ.

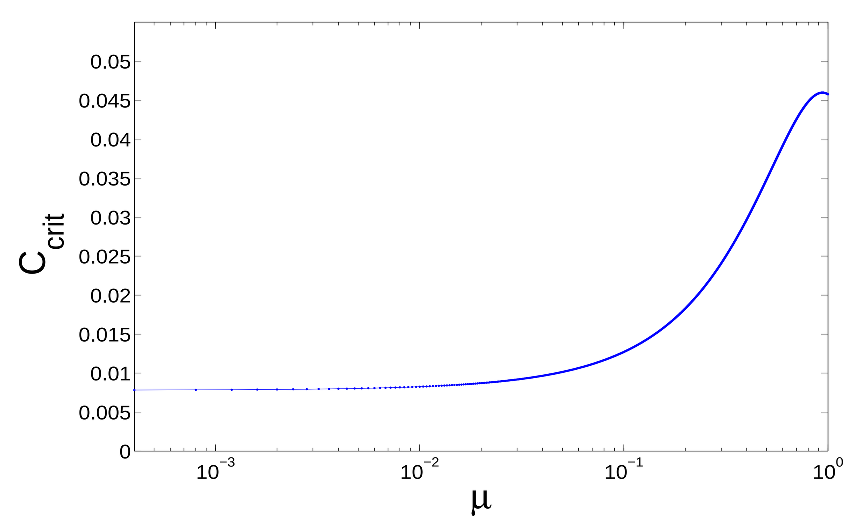

Figure 4: Critical values of Szebehely constant C_{crit} as a function of mass ratio μ

3. Empirical Stability Analysis

We empirically analyse the hierarchical stability of the CSFBP by studying hierarchically stable regions in the phase space of the CSFBP for any value of $C_0 < C_{crit}$ using the newly developed regularised codes (See Fig. 5). We are able to identify regions of phase space, with initial conditions which lead to hierarchically stable orbits for a finite integration time (empirical hierarchical stability). The regions of 'empirical hierarchical stability' are closely related to the regular and chaotic behaviour of the phase space studied using fast chaos detection methods [3]. We also empirically define a region in the phase space, in which a cluster of orbits remain hierarchically stable for a given integration time that is taken to be of long time duration (10^6 in Fig. 5).

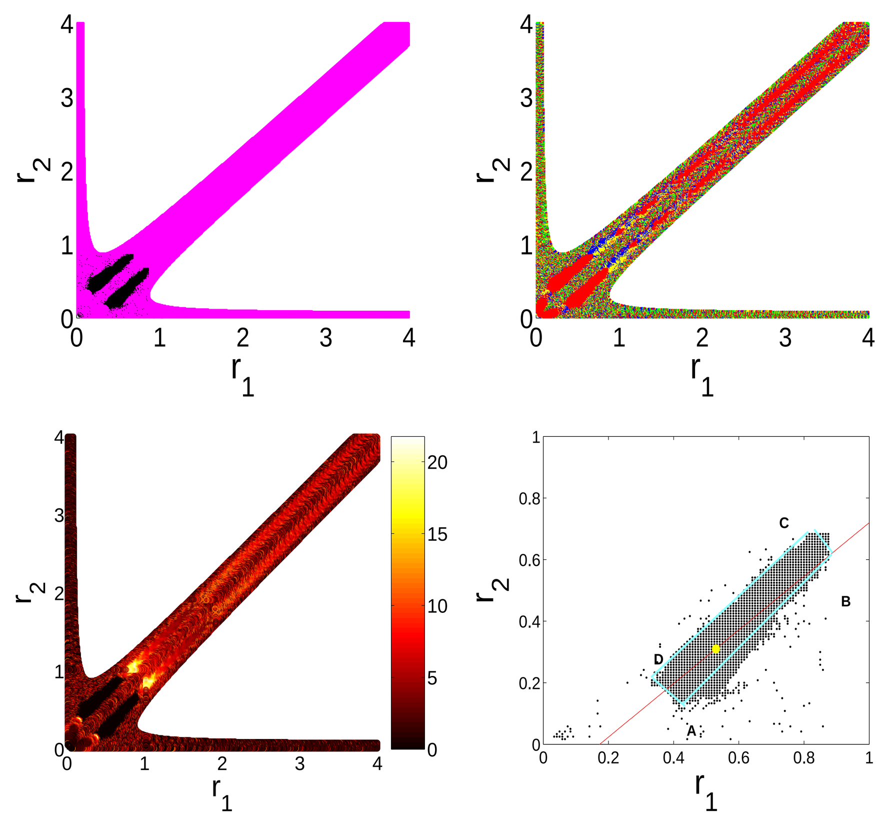

Figure 5: Empirical hierarchical stability of the model with $\mu = 1$, $E_0 = -7$, $C_0 = 10$, for integration time $t = 10^6$..

For the equal mass case, we have found islands of hierarchically stable clusters in the phase space, in which families of periodic and quasi-periodic orbits appears (See Fig. 6). It is also possible to reach hierarchical stability for a given integration time; if we initially place the orbits at a particular distance away from the origin (See Fig. 7).

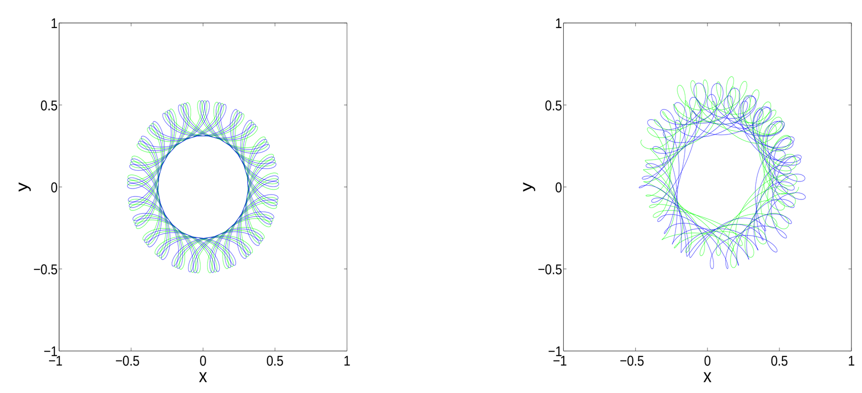

Figure 6: Periodic and quasi-periodic orbits appearing in the hierarchically stable blocks.

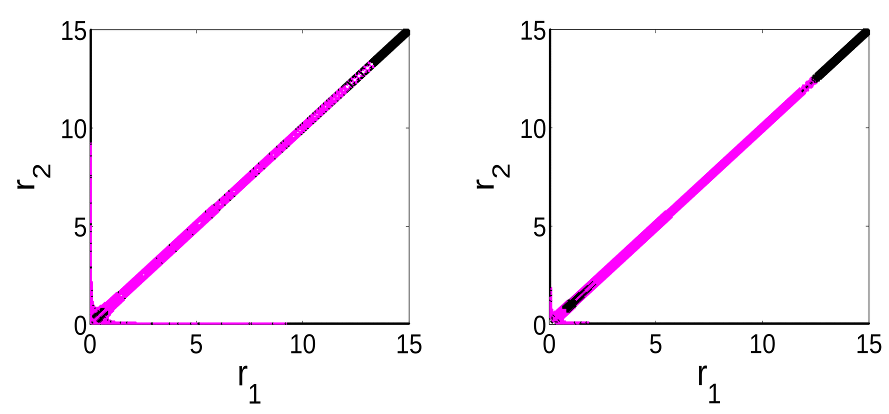

Figure 7: Global hierarchical stability of the model shown over an integration time $t = 10^8$ for $C_0 = 10$ and $C_0 = 40$.

Conclusion

We numerically explored the hierarchical evolution of the orbits in the CSFBP model using the newly proposed global regularisation scheme for a wide range of initial conditions, recording the number and type of hierarchical changes in the dynamics for a comprehensive set of CSFBP orbits, including those which pass through close encounters. It is confirmed that the hierarchical stability of the system solely depended upon the Szebehely constant C_0. We introduced an 'empirical hierarchical stability' concept to study hierarchically stable regions for $C_0 \ll C_{crit}$, which is a 'local' hierarchical stability with respect to a given numerical integration time. Regions of the hierarchically stable and unstable orbits in the phase space of the CSFBP are determined and their connection with the chaotic and regular regions of the phase space is explored.

References

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