# Regularization and Singular Perturbation Techniques for Non-Smooth Systems


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## Introduction

This poster is concerned with some aspects of the qualitative-geometric theory of Non-smooth Systems. We study the connection between the regularization process of non-smooth vector fields and the singular perturbation problems. As a matter of fact, the main results in our setting fill a gap between these areas. Exploiting such results, applications to singular perturbation problems with non-smooth reduced system are discussed.

## 1. The Regularization of Non-Smooth Systems

First we consider non-smooth differential equations $\dot{p} = X_0(p)$ around $p_0 \in \mathbb{R}^\ell$ having a codimension–one submanifold $\Sigma$ as its discontinuity set. More precisely let $F : (\mathbb{R}^\ell, p_0) \longrightarrow (\mathbb{R}, 0)$ be a $\mathcal{C}^k$ function having $0 \in \mathbb{R}$ as a regular value with $k$ big enough for our purposes. We denote $F^{-1}(0)$ by $\Sigma$. We write $\Omega^k = \Omega^k(\mathbb{R}^\ell, F)$ the space of vector fields $X_0$ such that

$$X_0(q) = \begin{cases} X_0^1(q) & \text{if} \quad q \in \Sigma_+ = \{F(q) \geqslant 0\}, \\ X_0^2(q) & \text{if} \quad q \in \Sigma_- = \{F(q) \leqslant 0\}, \end{cases} \tag{1}$$

where $X_0^1 = (f_1, ..., f_\ell), X_0^2 = (g_1, ..., g_\ell) \in \mathcal{X}^k$ and $\mathcal{X}^k$, $k \geq 1$, denotes the set of $\mathcal{C}^k$ vector fields defined on $\mathbb{R}^\ell$. The topology on $\mathcal{X}^k$ is the usual $\mathcal{C}^k$ topology. We write $X_0 = (X_0^1, X_0^2) \in \Omega^k(\mathbb{R}^\ell, F) \subseteq \mathcal{X}^k \times \mathcal{X}^k$ which we will accept to be multivalued at the points of $\Sigma$. We consider the product topology on $\Omega^k(\mathbb{R}^\ell, F)$.

Filippov has considered differential systems with discontinuities in the right–hand sides. We are allowing the case where such discontinuities occur on an algebraic variety $\Sigma$. The regions in $\Sigma$ are classified as:

- *Sliding Region*: $\Sigma^{sl} = \{p \in \sigma : X_0^1F(p) < 0, X_0^2F(p) > 0\} \subset \Sigma$. In this case any orbit which meets $\Sigma^{sl}$ remains tangent to $\Sigma$ for positive time. This region is the part of $\Sigma$ on which $X_0^1$ and $X_0^2$ point inward to $\Sigma$.
- *Escaping Region*: $\Sigma^{es} = \{p \in \Sigma : X_0^1F(p) > 0, X_0^2F(p) < 0\} \subset \Sigma$. In this case any orbit which meets $\Sigma^{es}$ remains tangent to $\Sigma$ for negative time.
- *Sewing Region*: $\Sigma^{sw} = \{p \in \Sigma : (X_0^1F(p)).(X_0^2F(p)) > 0\} \subset \Sigma$. In general a point in phase space which moves on an orbit of $X_0$ reaches a point in $\Sigma^{sw}$ crosses $\Sigma$.

On $\Sigma^S = \Sigma^{sl} \cup \Sigma^{es}$ the flow slides on $\Sigma$; the flow follows a well defined smooth vector field $X^S$ called *sliding vector field*. It is tangent to $\Sigma$ and defined at $q \in \Sigma^S$ by $X^S(q) = m - q$ with $m$ being the point where the segment joining $q + X_0^1(q)$ and $q + X_0^2(q)$ is tangent to $\Sigma$.

An approximation of the discontinuous vector field $X_0 = (X_0^1, X_0^2) \in \Omega^k(\mathbb{R}^\ell, F)$ by a one-parameter family of continuous vector fields will be called a regularization of $X_0$. The main problem then is to translate certain dynamical properties of the original one to the regularized system. What is familiar may or may not be a matter of taste, at least it depends a lot on the dynamical properties of one's interest.

Let $X_0 = (X_0^1, X_0^2) \in \Omega^k(\mathbb{R}^\ell, F)$. Sotomayor and Teixeira, introduced the $\varepsilon$-regularization process. More precisely, we consider a one–parameter family of smooth vector fields $X_\varepsilon, \varepsilon > 0$, such that:

- $X_\varepsilon$ is equal to $X_0^1$ in all points of $\Sigma_+$ whose distance to $\Sigma$ is bigger than $\varepsilon$;
- $X_\varepsilon$ is equal to $X_0^2$ in all points of $\Sigma_-$ whose distance to $\Sigma$ is bigger than $\varepsilon$.

A $C^\infty$ function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ is a transition function if $\lim_{x \to -\infty} \varphi(x) = -1$, $\lim_{x \to \infty} \varphi(x) = 1$ and $\varphi'(x) > 0$ for all $x \in \mathbb{R}$. The $\varphi$–regularization of $X_0 = (X_0^1, X_0^2)$ is the 1–parameter family $X_\varepsilon \in C^r$ given by

$$X_\varepsilon(q) = \left(\frac{1}{2} + \frac{\varphi_\varepsilon(F(q))}{2}\right) X_0^1(q) + \left(\frac{1}{2} - \frac{\varphi_\varepsilon(F(q))}{2}\right) X_0^2(q). \tag{2}$$

We assume that $\varphi_\varepsilon(x) = \varphi(x/\varepsilon)$, for $\varepsilon > 0$. Assuming that $F^{-1}(0)$ is represented, locally around a point $p \in \Sigma$, by the function $F(x_1, ..., x_\ell) = x_1$ and denoting the vector fields $X_0^1$ and $X_0^2$ by $X_0^1 = (f_1, ..., f_\ell)$ and $X_0^2 = (g_1, ..., g_\ell)$ we have that the trajectories of the regularized vector field $X_\varepsilon$ are the solutions of the differential system

$$\dot{x}_i = \frac{f_i + g_i}{2} + \varphi(\frac{x_1}{\varepsilon})\frac{f_i - g_i}{2}; \quad i = 1, ..., \ell; \quad \dot{\varepsilon} = 0 \tag{3}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is a transition function. We transform this system into a singular perturbation problem $H_\eta$ by considering $x_1 = \eta \cos\psi$, and $\varepsilon = \eta \sin\psi$, with $\eta \geq 0$ and $\psi \in [0, \pi]$

$$(H_\eta) : \begin{cases} \eta\psi' &= \alpha_1(\eta, \psi, x_2, ..., x_\ell) \\ x_i' &= \alpha_i(\eta, \psi, x_2, ..., x_\ell), \quad i = 2, ..., \ell \end{cases}. \tag{4}$$

**Theorem 0.1.** *(* **Regular Case** *) The sliding region $\Sigma^S \subset \Sigma$ is homeomorphic to the slow manifold $\alpha_1(0, \psi, x_2, ..., x_\ell) = 0$ and the sliding vector field $X^S$ is topologically equivalent to the so called reduced problem*

$$0 = \alpha_1(0, \psi, x_2, ..., x_\ell), \quad x_i' = \alpha_i(0, \psi, x_2, ..., x_\ell), \quad i = 2, ..., \ell.$$

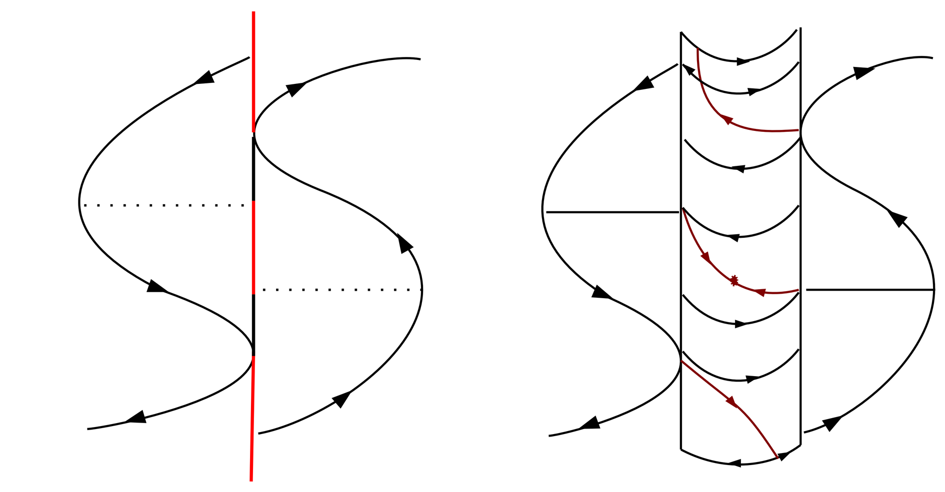
**Example** Consider $X_0^1(x_1, x_2) = (3x_2^2 - x_2 - 2, 1)$, and $X_0^2(x_1, x_2) = (-3x_2^2 - x_2 + 2, -1)$. We assume that $F(x_1, x_2) = x_1$. The regularized vector field is

$$X_\varepsilon = \left(\frac{1}{2} + \frac{1}{2}\varphi(\frac{x_1}{\varepsilon})\right)(3x_2^2 - x_2 - 2, 1) + \left(\frac{1}{2} - \frac{1}{2}\varphi(\frac{x_1}{\varepsilon})\right)(-3x_2^2 - x_2 + 2, -1).$$

Applying our technique we get the singular perturbation problem

$$\eta\dot{\psi} = -\sin\psi(-x_2 + \varphi(\cot\psi)(3x_2^2 - 2)), \quad \dot{x}_2 = \varphi(\cot\psi).$$

The slow manifold is given implicitly by $\varphi(\cot\psi) = \frac{x_2}{3x_2^2 - 2}$ which defines two functions $s_1(\psi) = \dfrac{1 + \sqrt{1 + 24\varphi^2(\cot\psi)}}{6\varphi(\cot\psi)}$ and $s_2(\psi) = \dfrac{1 - \sqrt{1 + 24\varphi^2(\cot\psi)}}{6\varphi(\cot\psi)}$. The function $s_1(\psi)$ is increasing, $s_1(0) = 1$, $\lim_{\psi \to \frac{\pi}{2}^-} s_1(\psi) = +\infty$, $\lim_{\psi \to \frac{\pi}{2}^+} s_1(\psi) = -\infty$ and $s_1(\pi) = -1$. The function $s_2(\psi)$ is increasing, $s_2(0) = -\dfrac{2}{3}$, $\lim_{\psi \to \frac{\pi}{2}} s_2(\psi) = 0$ and $s_2(\pi) = \dfrac{2}{3}$. The slow manifold is the union of three simply connected pieces, one of them connecting two fold points and the others having only one fold point on the boundary.



Red Line= Sliding Line

**The usual definition of sliding vector field can not be used in the case when $0$ is not a regular value of $F$. We propose an alternative way by using the blow up techniques and the regularization.**

## 2. Slow fast systems on algebraic varieties

Let $(0, 0, 0) \in \mathcal{U} \subset \mathbb{R}^3$ be an open set and $F : \mathcal{U} \to \mathbb{R}$ a polynomial function. Suppose $(0, 0, 0) \in \Sigma = F^{-1}(0)$, $\Sigma^+ = F^{-1}(0, \infty)$ and $\Sigma^- = F^{-1}(-\infty, 0)$. We suppose that there exists $C^\infty$–diffeomorphism $\Psi : \mathcal{S} \to F^{-1}(0)$ where $\mathcal{S}$ is one of the following subsets of $\mathbb{R}^3$:

(a) $\mathcal{R} = \{(x, y, z) \in \mathbb{R}^3; z = 0\}$ (regular case);

(b) $\mathcal{D} = \{(x, y, z) \in \mathbb{R}^3; xy = 0\}$ (double crossing);

(c) $\mathcal{T} = \{(x, y, z) \in \mathbb{R}^3; xyz = 0\}$ (triple crossing) ;

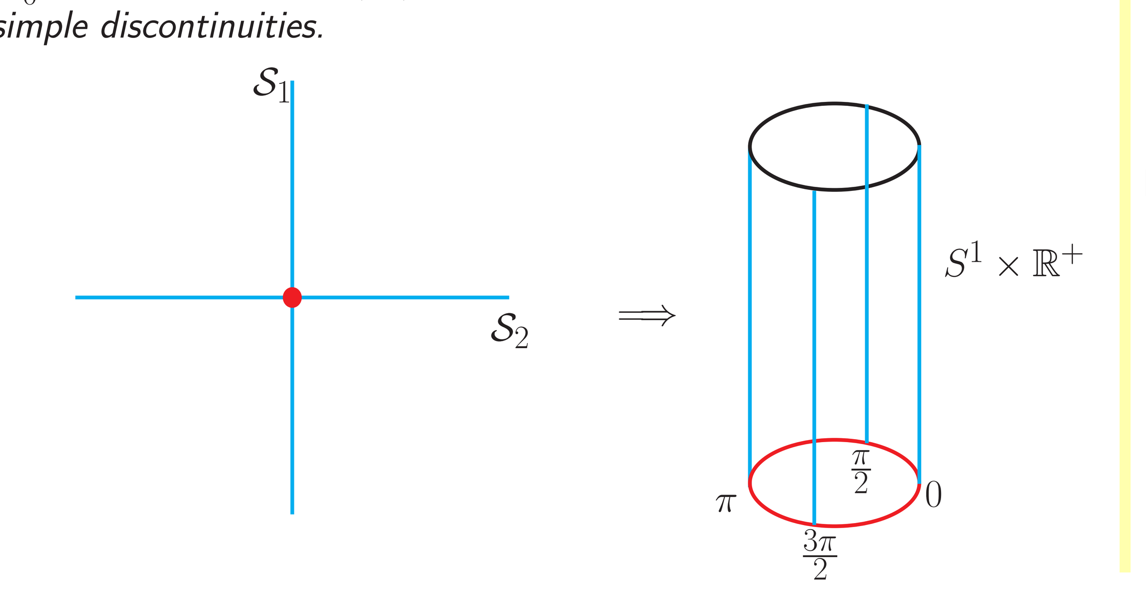
(d) $\mathcal{C} = \{(x, y, z) \in \mathbb{R}^3; z^2 - x^2 - y^2 = 0\}$ (cone);

(e) $\mathcal{W} = \{(x, y, z) \in \mathbb{R}^3; zx^2 - y^2 = 0\}$ (Whitney's umbrella).

**Definition 0.1.** *We say that $p \in \Sigma$ is a simple discontinuity if there exists an open set $U \subseteq \mathbb{R}^\ell$ with $p \in U$ and a differential function $F : U \to \mathbb{R}$ such that $0$ is a regular value of $F$ and $F^{-1}(0) = \Sigma \cap U$.*

We denote $X_0 \in \Omega(\mathcal{U}, \mathcal{R})$ if the discontinuous differential system $\dot{p} = X_0(p), p \in \mathcal{U} \setminus \Sigma$, has switching manifold $\Sigma$ diffeomorphic to $\mathcal{R}$. Analogously we denote $X_0 \in \Omega(\mathcal{U}, \mathcal{D})$, $X_0 \in \Omega(\mathcal{U}, \mathcal{T})$, $X_0 \in \Omega(\mathcal{U}, \mathcal{C})$ and $X_0 \in \Omega(\mathcal{U}, \mathcal{W})$.

**Theorem 0.2.** *(* **Double Crossing** *) Consider a discontinuous vector field $X_0(x_1, x_2, x_3)$ which is determined by 4 smooth vector fields: $X_0^1$ on $U_1 = \{x_2 > 0, x_3 > 0\}$; $X_0^2$, on $U_2 = \{x_2 < 0, x_3 > 0\}$; $X_0^3$ on $U_3 = \{x_2 < 0, x_3 < 0\}$; and $X_0^4$ on $U_4 = \{x_2 > 0, x_3 < 0\}$. Consider the map $\phi : \mathbb{R} \times S^1 \times \mathbb{R}^+ \to \mathbb{R}^3$ given by $\phi(x_1, \theta, r) = (x_1, r\cos\theta, r\sin\theta)$. The vector field $\overline{X_0}$ determined by $\overline{X_0^i}, i = 1, ..., 4$ on $\phi^{-1}(U_i)$ induced by $\phi$ on $\mathbb{R} \times S^1 \times \mathbb{R}^+$ has only simple discontinuities.*




The sliding vector field idealized by Filippov can not be uniquely extended for intersecting switching discontinuous manifold. However, the blow up method developed by us can be applied for this kind of the surface. Since the method produces a differential system which is equivalent to the sliding vector field for the regular case, our method can be considered like a generalization of the Filippov convention. The number of blow ups necessary to get a sliding vector field like the idealized by Filippov:

- $\Omega(\mathcal{U}, \mathcal{F})$ :1
- $\Omega(\mathcal{U}, \mathcal{D})$ : 2
- $\Omega(\mathcal{U}, \mathcal{C})$ : 2
- $\Omega(\mathcal{U}, \mathcal{T})$ : 3
- $\Omega(\mathcal{U}, \mathcal{W})$ : 3

## 3. Slow flow and sliding mode

Consider the following system.

$$\dot{x_1} = A(x_1, x_2, x_3) \quad \dot{x_2} = B(x_1, x_2, x_3) \quad \varepsilon\dot{x_3} = -x_3 \tag{5}$$

where

$$A(x_1, x_2, x_3) = \begin{cases} f_1(x_1, x_2, x_3), & \text{if} \quad x_1 \geq 0 \\ f_2(x_1, x_2, x_3), & \text{if} \quad x_1 \leq 0 \end{cases}$$

$$B(x_1, x_2, x_3) = \begin{cases} g_1(x_1, x_2, x_3), & \text{if} \quad x_1 \geq 0 \\ g_2(x_1, x_2, x_3), & \text{if} \quad x_1 \leq 0 \end{cases}$$

and $\varepsilon \geq 0$. Suppose $f_i, g_i, i = 1, 2$, are of class $C^k$ with $k \geq 1$, on the open set $U \subseteq \mathbb{R}^3$ and that $0 \in U$. We will denote $X_0^i = (f_i, g_i, -\frac{x_3}{\varepsilon})$ and $X_0 = (X_0^1, X_0^2) \in \Omega_\varepsilon^k(U)$.

The switching manifold is given by $\Sigma = \{(x_1, x_2, x_3) \in U; x_1 = 0\}$ and the slow manifold is $\mathcal{M} = \{(x_1, x_2, x_3) \in U; x_3 = 0\}$. We have that $\Sigma$ and $\mathcal{M}$ are 2–dimensional orientate manifolds.

The trajectories of the reduced problem are the trajectories of the discontinuous system

$$(x_1, x_2, x_3) \in \mathcal{M}, \quad (\dot{x_1}, \dot{x_2}) = \begin{cases} \Pi \circ X_0^1 = (f_1, g_1) & \text{if} \quad x_1 \geq 0 \\ \Pi \circ X_0^2 = (f_2, g_2) & \text{if} \quad x_1 \leq 0 \end{cases} \tag{6}$$

where $\Pi$ denotes the projection on the $(x_1, x_2)$-plane $\Pi(x_1, x_2, x_3) = (x_1, x_2)$.

There exists a two dimensional singular perturbation problem

$$\psi' = \alpha(\eta, \psi, x_2), \quad x_2{}' = \eta\beta(\eta, \psi, x_2), \tag{7}$$

with $\eta \geq 0$, $\psi \in (0, \pi)$, $x_2 \in \Sigma \cap \mathcal{M}$ and $\alpha$ and $\beta$ of class $C^r$ such that the sliding region $(\Sigma \cap \mathcal{M})^S$is homeomorphic to the slow manifold $\alpha(0, \psi, x_2) = 0$ of (7) and the sliding vector field $X^{\Sigma \cap \mathcal{M}}$ and the reduced problem are topologically equivalent.

**Proposition 0.1.** *Let $X_0 = (X_0^1, X_0^2) \in \Omega_\varepsilon^k(U)$ be a non-smooth system. The sliding region $\Sigma_r^{sl}$(resp.escaping, sewing) of the reduced problem* (6) *($\varepsilon = 0$) is contained in the sliding region $\Sigma^{sl}$ (resp.escaping, sewing) of system* (5) *($\varepsilon \neq 0$). Moreover, if $p \in \Sigma \cap \mathcal{M} \cap \Sigma_r^S$ is on the sliding region of the reduced problem* (6) *then there exists an open set $\mathcal{V} \subseteq \Sigma^S$ such that $p \in \mathcal{V} \supseteq \Sigma_r^S$.*

Our last theorem says that the sliding mode associated to the system $X_i = (f_i, g_i, -\frac{x_3}{\varepsilon}), i = 1, 2$ is a smooth slow-fast system. Moreover, the reduced problem associated to this smooth slow-fast system coincides with the sliding vector field associated to the reduced problem of (6). It implies, via the classical Fenichel's Theorem that any structure of the sliding vector field associated to the reduced problem of (6) which persists under regular perturbation persists under singular perturbation.

**Proposition 0.2.** *The sliding vector field associated to $X_0^i = (f_i, g_i, -\dfrac{x_3}{\varepsilon}), i = 1, 2,$ is a smooth slow-fast system of the form $\dot{x_2} = \gamma(x_2, x_3), \varepsilon\dot{x_3} = -x_3$,where $\gamma$ is a $C^k$ map. Moreover the dynamics of the reduced problem is given by $\dot{x_2} = \gamma(x_2, x_3), \quad 0 = -x_3$, and coincides with the dynamics of the sliding vector field associated to the reduced problem* (6) .

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