The complete topological classification of Lotka-Volterra systems

by

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A Lotka-Volterra differential system can be written in the form:

$$\text{L-V} \quad \begin{aligned} \frac{dx}{dt} &= x(a_0 + a_1 x + a_2 y) \\ \frac{dy}{dt} &= y(b_0 + b_1 x + b_2 y) \end{aligned} \qquad a_i, b_j \in \mathbb{R}$$

Observation: These systems look simple: they are quadratic and they have 2 invariant lines:

$$x = 0, \quad y = 0$$

Problem: Classify topologically the whole class of L-V systems.

Remark: This problem is a part of a much larger problem of classifying the whole class of quadratic differential systems

The following phrase appears in the literature in the context of Hilbert's 16-th problem:

"Quadratic vector fields are relatively simple amids other polynomial vector fields."

This is a trivial statement but as it is mentioned in the context of Hilbert's 16th problem we need to point out that this problem is GLOBAL. It involves the whole class of polynomial systems for <u>a fixed degree n</u>.

In particular it is a GLOBAL problem for quadratic systems i.e. for $n=2$.

It is completely besides the point that quadratic systems are simpler than higher degree systems.
What actually counts here is the whole class of QS.

Question: Is the class QS of quadratic systems a simple object or not?

Answer: It is not!

While other problems opened for more than a century were solved, Hilbert's 16th problem is still open even in the quadratic case.

Hilbert's 16th problem: For a fixed natural number n, determine the maximum

$$\max_n \# LC$$

of the number of limit cycles of degree n polynomial differential systems.

$$QS = \left\{ (S) \begin{cases} \frac{dx}{dt} = P(x,y) \\ \frac{dy}{dt} = Q(x,y) \end{cases} \middle| \begin{array}{l} P, Q \in \mathbb{R}[x,y] \\ \deg(S) = \max(\deg P, \deg Q) \\ = 2 \end{array} \right\}$$

On QS we have the action of the group

$$G = \mathrm{Aff}(2,\mathbb{R}) \times \mathbb{R}^*$$

of affine transformations and time rescaling

$$\dim QS/_G = 5$$

The exploration of the class QS began with studies of some subclasses of lower dimensions modulo G.

So far only subclasses of dimension 2 and 3, modulo G have been studied. A 4-dimensional subclass is in the process of being explored.

The following 4 classes have been studied together with Nicolae Vulpe

$$QSL_i = \left\{ S \in QS \;\middle|\; \begin{array}{l} S \text{ has invariant} \\ \text{lines of total} \\ \text{multiplicity } i \end{array} \right\}$$

for $i = 6, 5, 4$ (6 is the maximum possible)

This study is completely algebraic.

These systems are Darboux or Liouvillian integrables.

QSL_6/G contains only 11 orbits

$\dim QSL_5/G = 1$

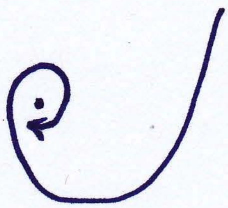
$\dim QSL_4/G = 2$

$$QS_{C_2 \equiv 0} = \left\{ S \in QS \;\middle|\; \begin{array}{l} \text{all points at infinity} \\ \text{of } S \text{ are singularities} \end{array} \right\}$$

$QS_{C_2 \equiv 0}/G$ is formed by 5 orbits and four 1-parameter families of orbits

The following classes have been studied by Artés, Llibre & Schlomiuk:

$$QW3 = \left\{ s \in QS \;\middle|\; \begin{array}{c} s \text{ has a weak focus} \\ \text{of order 3} \end{array} \right\}$$



strong focus

weak focus

$$QW2 = \left\{ s \in QS \;\middle|\; \begin{array}{c} s \text{ has a weak focus} \\ \text{of order 2} \end{array} \right\}$$

$$QW1_{IL} = \left\{ s \in QS \;\middle|\; \begin{array}{c} s \text{ has a weak focus} \\ \text{of order 1 and an} \\ \text{invariant line} \end{array} \right\}$$

$$\dim QW3/G = 2$$

$$\dim QW2/G = 3$$

$$\dim QW1/G = 3$$

The subclass which produced the deepest insights into QS is QW2.

Ongoing work on QW1 are producing even deeper insights.

$$\dim QW1/G = 4$$

In QW1 we also have, apart from bifurcations of graphics and singularities, bifurcations of multiple limit cycles.

The study of the class of Lotka-Volterra systems is part of this much vaster program of exploring the quadratic class

Together with Vulpe we wrote two paper on the classification of L-V systems:

1) Vulpe & D.S, Global classification of planar L-V differential systems according to their configurations of their invariant straight lines.
J. of Fixed Point Theory & Appl. 8 (2010), 177-245 (68 pages)

2) Vulpe & D.S. The global topological classification of L-V quadratic differential systems.
EJDE, Vol. 2012 (2012) No 64, pp 1-70 (70 pages)

Previous work on L-V systems:

Reyn 1987
Wörz-Buserkros 1993
Georgescu 2007
Cao & Jiang 2008

All claimed to have a complete "classification" of the L-V systems.

We confronted these papers and found that:

- The results are not in agreement
- All 4 articles contain errors:
  - Some Distinct phase portraits are claimed to be topologically equivalent
  - Topologically equivalent phase portraits (some) are claimed to be distinct
- No proofs that the portraits listed are distinct.

Definition We call configuration of invariant lines of a differential system (S), the set of all invariant lines of (S), each one of those not filled up with singularities endowed with its multiplicity and together with all the real singularities of (S) situated on the invariant lines, each isolated one endowed with its own multiplicity.

L-V systems with a finite number of singularities

The L-V systems have a total of 25 such configurations for systems having 3 singular points at infinity and 22 configurations with 2 singular point at infinity.

12 L-V systems with an ∞ number of singularities

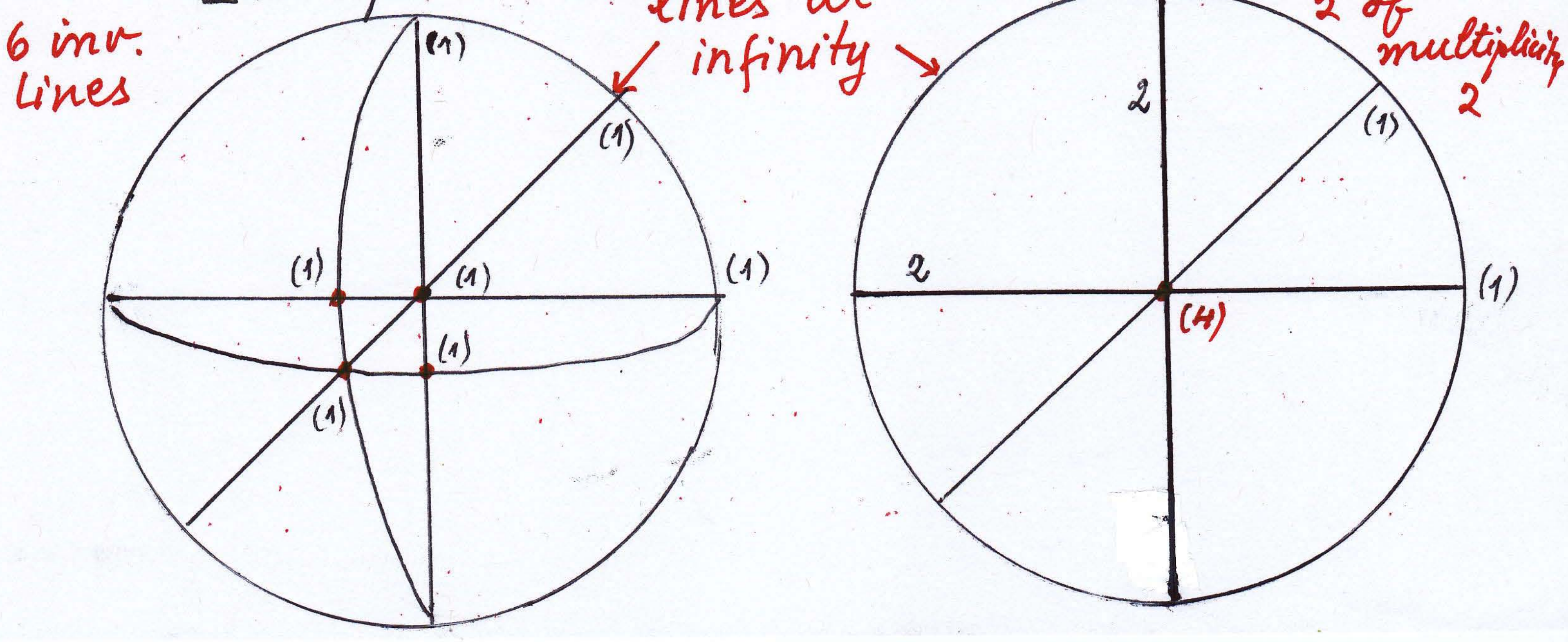
Our basic observation:

The L-V systems have an algebro-geometric structure: they possess invariant lines, at least 2 and sometimes more.

We introduced the basic classifying concept of "configuration of invariant lines".

$x = 0$, $y = 0$ are always invariant lines but we may have more (up to 6).

Examples:



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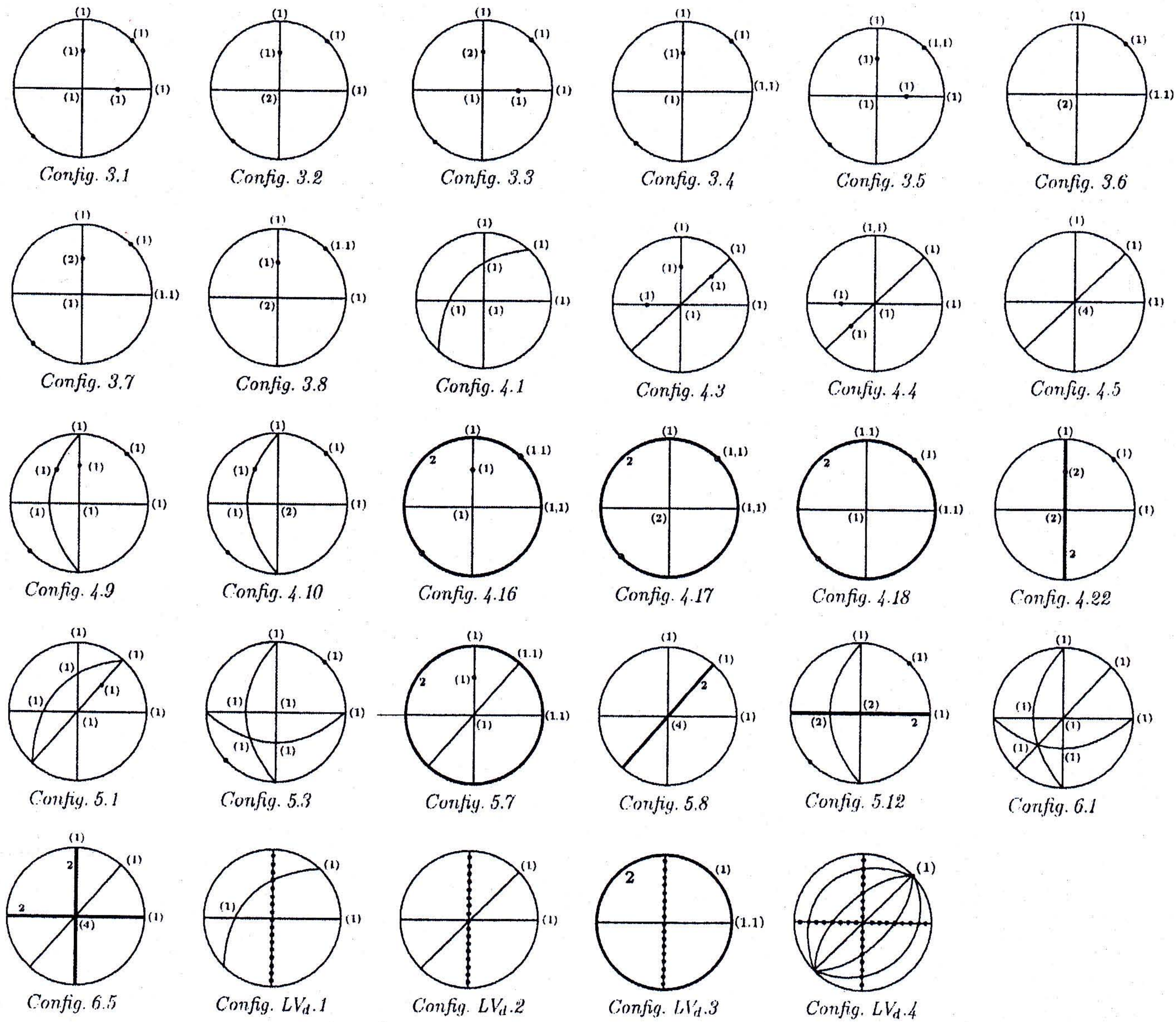

FIG. 1. **The case $\eta \neq 0$**

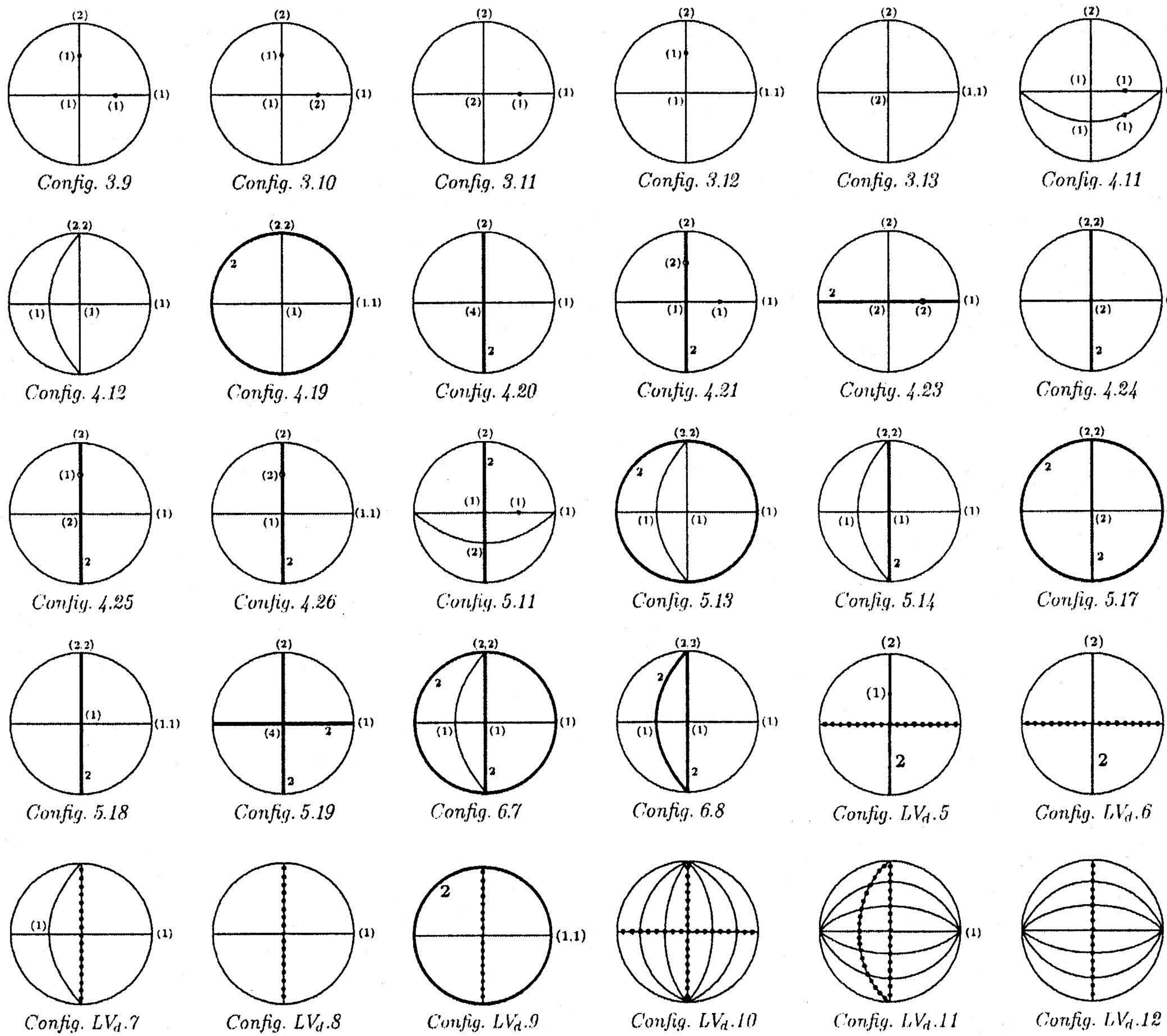

FIG. 2. **The case $\eta = 0, C_2 \neq 0$**

For L-V systems with more than 3 invariant lines we could use the classifications of QSL_i for $i \geq 4$.

So it remained to do the classification for L-V systems with exactly 3 invariant lines.

There are exactly 13 configurations for such systems.

Hence for each one of the 13 configurations we studied the phase portraits. For this we relied on Bautin's theorem:

Each L-V system has no limit cycles

Theorem 1.3. *The class of all Lotka-Volterra quadratic differential systems has a total of 112 topologically distinct phase portraits. Among these, 60 portraits are for systems with three simple invariant lines; 27 are portraits of systems with invariant lines of total multiplicity at least four; 5 phase portraits are for Lotka-Volterra systems which have the line at infinity filled up with singularities; 20 phase portraits are for the degenerate systems.*

(i) Consider the 13 configurations Config. 3.j, $j \in \{1, \ldots, 13\}$ (see Definition 2.2) with three simple invariant lines given in Fig. 4. For each configuration Config. 3.j we have a number n_j of topologically distinct phase portraits. Then $\sum_{j=1}^{13} n_j = 65$ and the 65 phase portraits (not necessarily topologically distinct) are given in Fig. 5. The necessary and sufficient affine invariant conditions for the realization of each one of these portraits are given in Table 5.

(ii) Consider the 34 configurations of Lotka-Volterra systems Config. 4.1, ..., Config. 6.8 with invariant lines of total multiplicity at least four given in Fig. 4. For each one of these 34 configurations we have a number m_i, $i \in \{1, \ldots, 34\}$ of topologically distinct phase portraits. Then $\sum_{i=1}^{34} m_i = 59$ and the 59 phase portraits (not necessarily topologically distinct) are given in Fig. 3. The necessary and sufficient affine invariant conditions for the realization of each one of these portraits are given in Table 3.

(iii) Consider the 4 configurations of Lotka-Volterra systems Config. C_2.j, $j \in \{1, 2, 5, 7\}$ with the line at infinity filled up with singularities given in Fig. 2. For each one of these 4 configurations we have a unique phase portrait, except for the configuration Config. C_2.j for which we have two phase portraits. The 5 phase portraits are topologically distinct and they are given in Fig. 1. The necessary and sufficient affine invariant conditions for the realization of each one of these portraits are given in Table 3.

(iv) Consider the 14 configurations Config. LV_d.j, $j \in \{1, \ldots, 14\}$ given in Fig. 2, of the degenerate quadratic Lotka-Volterra systems. For each configuration Config. LV_d.j we have a number s_j of topologically distinct phase portraits. Then $\sum_{j=1}^{14} s_j = 20$ and the 20 phase portraits given in Fig. 6 are topologically distinct. The necessary and sufficient affine invariant conditions for the realization of each one of these portraits are given in Table 6.

(v) Of the 149 phase portraits obtained by listing those occurring in the classes (i)–(iv), only 112 are topologically distinct (see Diagrams 1–6).

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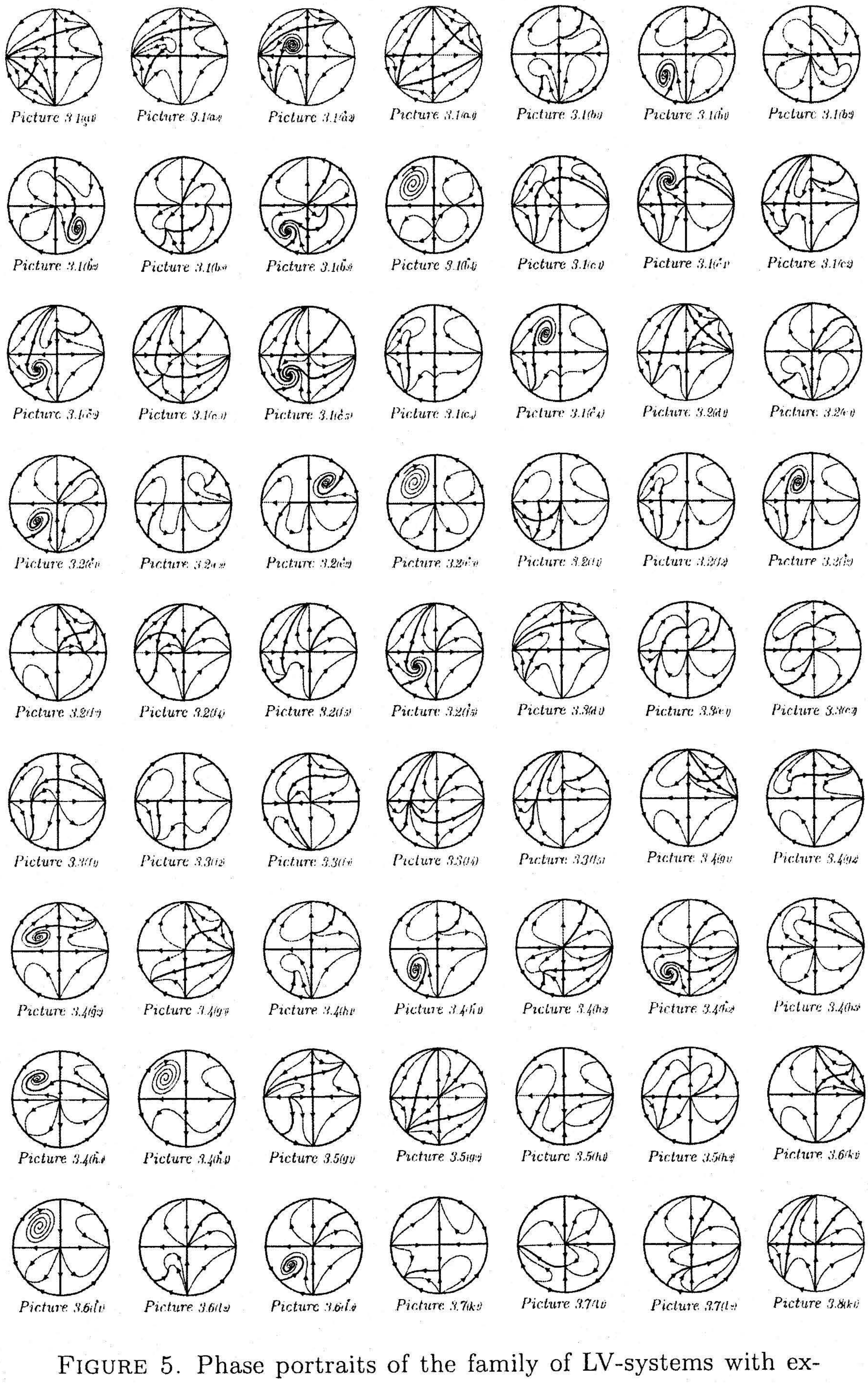

FIGURE 5. Phase portraits of the family of LV-systems with exactly three invariant lines

Theorem 3.15. *Consider an LV-system (S). I. Then (S)*

(1) *has only real invariant lines;*
(2) *has only real singularities, at least two of them at infinity;*
(3) *has no finite singularities of multiplicity three;*
(4) *has a focus only if (S) has exactly three invariant lines, all simple;*
(5) *has no weak foci;*
(6) *has no limit cycles.*

II. In the generic case when the system (S) has exactly three invariant lines all simple, (S) has no centers.

Theorem 3.16. *I. Of 112 topologically distinct phase portraits of LV-systems only 18 possess graphics and all of them occur in* QSL_i, $i \in \{3, 4\}$. *More precisely we have:*

(i) *8 distinct isolated graphics occur in systems with exactly three invariant lines, all simple. All of them are triangles with an infinite side and they surround a focus.*
(ii) *4 distinct isolated graphics occur in systems in* QSL_4 *all of them are triangles, one finite and three with an infinite side and they surround a center.*
(iii) *non-isolated graphics occur in 6 topological distinct phase portraits of systems in* QSL_4. *In each one of them we have two infinite families of graphics. These graphics are: (a) homoclinic loops with either a finite singularity or with an infinite singularity; (b) limiting triangles of families of homoclinic loops.*

II. Infinite families of degenerate graphics occur in: (a) LV-systems with all points at infinity singular, excepting the systems with the phase portrait Picture $C_2.5(a)$, *and (b) degenerate LV-systems.*

Proof. I. The proof of the points (i) and (ii) results from Fig. 5 and Fig. 3 respectively.

3.3. Topologically distinct phase portraits of LV-systems. To find the exact number of topologically distinct phase portraits of LV-systems, we use a number of topological invariants for distinguishing (or identifying) phase portraits. We list below the topological invariants we need and the notation we use.

I. *Singularities, invariant lines, multiplicities and indices:*

- $\mathcal{N}$ = total number of all singularities (they are all real) of the systems;
- $\binom{\mathcal{N}_f}{\mathcal{T}_m}$ = the number $\mathcal{N}_f$ of all distinct finite singularities having a total multiplicity $\mathcal{T}_m$;
- $\deg J$ = the sum of the indices of all finite singularities of the systems;
- $\mathcal{N}_{\mathrm{AIL}}^{\mathrm{sing}}$ = total number of affine invariant lines filled up with singularities;
- $\mathcal{N}_\infty$ = total number of infinite singularities;

II. *Connections of separatrices:*

- $\#SC_s^s$ = total number of connections of a finite saddle to a finite saddle;
- $\#SC_s^S$ = total number of connections of a finite saddle to an infinite saddle;
- $\#SC_s^{S\mathcal{N}}$ =total number of connections of a finite saddle to an infinite saddle-node;
- $\#SC_{sn}^s$ = total number of connections of a finite saddle-node to a finite saddle;
- $\#SC_{sn}^S$ = total number of connections of a finite saddle-node to an infinite saddle;
- $\#SC_{sn}^{S\mathcal{N}}$ = total number of connections of a finite saddle-node to an infinite saddle-node;
- $\#SC_{sn(hh)}^S$ = total number of separatrices dividing the two hyperbolic sectors of finite saddle-nodes, going to infinite saddles;
- $\#SC_{sn(hh)}^{S\mathcal{N}}$ = total number of separatrices dividing the two hyperbolic sectors of finite saddle-nodes connecting with separatrices of infinite saddle-nodes.
- $\#Sep_{(HH)}^{S\mathcal{N}}$ = total number of separatrices of infinite saddle-nodes located in the finite plane and dividing the two hyperbolic sectors.

III. *The number of separatrices or orbits leaving from or ending at a singular point:*

- $M_{\mathrm{sep}}^{\tilde{n}} = \max\{\mathrm{sep}(\tilde{n}) \mid \tilde{n}$ is a node$\}$, where $\mathrm{sep}(\tilde{n})$ is the number of separatrices leaving from or ending at a finite node $\tilde{n}$;
- $M_{\mathrm{sep}}^{\tilde{s}\tilde{n}} = \max\{\mathrm{sep}(\tilde{s}\tilde{n}) \mid \tilde{s}\tilde{n}$ is a node$\}$, where $\mathrm{sep}(\tilde{s}\tilde{n})$ is the number of separatrices leaving from or ending at a finite saddle-node $\tilde{s}\tilde{n}$;
- $M_{\mathrm{orb}} = \max\{\mathrm{orb}(p) \mid p$ is a finite singularity$\}$, where $\mathrm{orb}(p)$ is the number of orbits leaving from or arriving at p;
- $M_{\mathrm{ORB}} = \max\{\mathrm{orb}(p_1,p_2) \mid p_1,p_2$ are infinite singularities$\}$, where $\mathrm{orb}(p_1,p_2)$ is the number of orbits connecting p_1 with p_2.

Using the topological invariants listed above we construct the following global topological invariant $\mathcal{I} = (\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3)$, where

$$\mathcal{I}_1 = \Big(\mathcal{N}, \binom{\mathcal{N}_f}{\mathcal{T}_m}, \deg J, \mathcal{N}_{ILA}^{\mathrm{sing}}, \mathcal{N}_\infty\Big),$$

$$\mathcal{I}_2 = \Big(\#SC_s^s, \#SC_s^S, \#SC_s^{S\mathcal{N}}, \#SC_{sn}^s, \#SC_{sn}^S, \#SC_{sn}^{S\mathcal{N}}, \#SC_{sn(hh)}^S, \#Sep_{(HH)}^{S\mathcal{N}}\Big),$$

$$\mathcal{I}_3 = \Big(M_{\mathrm{sep}}^{\tilde{n}}, M_{\mathrm{sep}}^{\tilde{s}\tilde{n}}, M_{\mathrm{orb}}, M_{\mathrm{ORB}}\Big),$$

The classification obtained is independent of the normal forms in which the systems may be presented.

The classification is done in terms of polynomial invariants.

$$\frac{dx}{dt} = a_0 + \underbrace{a_{10}x + a_{01}y}_{P_1(x,y)} + \underbrace{a_{20}x^2 + a_{11}xy + a_{02}y^2}_{P_2(x,y)}$$

$$\frac{dy}{dt} = b_0 + \underbrace{b_{10}x + b_{01}y}_{Q_1(x,y)} + \underbrace{b_{20}x^2 + b_{11}xy + b_{02}y^2}_{Q_2(x,y)}.$$

Define

$$C_2(x,y) = y\,P_2(x,y) - x\,Q_2(x,y)$$

$$\eta = \text{Discrim}\,(C_2(x,y))$$

$$K(x,y) = \det(\text{Jacobian}\,(P_2, Q_2))$$

$$\mu_0 = \text{Discrim}\,(K(x,y))/16$$

These are basic invariant polynomials.

Table 5

<table>
<tr><th>Configu-ration</th><th>Necessary and sufficient conditions</th><th colspan="3">Additional conditions for phase portraits</th><th>Phase portrait</th></tr>
<tr><td rowspan="26">Config. 3.1</td><td rowspan="26">$\eta > 0, \mu_0 B_3 H_9 \neq 0$, $B_2 = 0$ and either $\theta \neq 0$ or $(\theta = 0 \ \&\ NH_7 \neq 0)$</td><td rowspan="4">$\mu_0 < 0$, $K < 0$</td><td rowspan="3">$W_4 \geq 0$</td><td>$B_3U_1 < 0, U_2 < 0$</td><td>Picture 3.1(a1)</td></tr>
<tr><td>$B_3U_1 < 0, U_2 > 0$</td><td>Picture 3.1(a2)</td></tr>
<tr><td>$B_3U_1 > 0$</td><td>Picture 3.1(a3)</td></tr>
<tr><td>$W_4 < 0$</td><td>–</td><td>Picture 3.1($\overset{*}{a}2$)</td></tr>
<tr><td rowspan="12">$\mu_0 < 0$, $K > 0$</td><td rowspan="6">$W_4 > 0$ or $W_4 = 0$ & $W_3 \geq 0$</td><td>$B_3U_1 < 0$</td><td rowspan="2">Picture 3.1(b1)</td></tr>
<tr><td>$B_3U_1 > 0, U_2 > 0$, $U_4 > 0, U_3 > 0$</td></tr>
<tr><td>$B_3U_1 > 0, U_2 < 0$, $B_3H_{14} > 0$</td><td rowspan="2">Picture 3.1(b2)</td></tr>
<tr><td>$B_3U_1 > 0, U_2 > 0$, $U_4 > 0, U_3 < 0$</td></tr>
<tr><td>$B_3U_1 > 0, U_2 < 0$, $B_3H_{14} < 0$</td><td rowspan="2">Picture 3.1(b3)</td></tr>
<tr><td>$B_3U_1 > 0, U_2 > 0$, $U_4 < 0$</td></tr>
<tr><td rowspan="6">$W_4 < 0$ or $W_4 = 0$ & $W_3 < 0$</td><td>$B_3U_1 < 0, U_2 < 0$</td><td rowspan="2">Picture 3.1($\overset{*}{b}1$)</td></tr>
<tr><td>$B_3U_1 > 0, U_2 > 0$, $U_4 > 0, U_3 > 0$</td></tr>
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<tr><td>$B_3U_1 > 0, U_2 > 0$, $U_4 < 0$</td><td>Picture 3.1($\overset{*}{b}3$)</td></tr>
<tr><td>$B_3U_1 < 0, U_2 > 0$</td><td>Picture 3.1($\overset{*}{b}4$)</td></tr>
<tr><td rowspan="10">$\mu_0 > 0$</td><td rowspan="6">$W_4 > 0$ or $W_4 = 0$ & $W_3 \geq 0$</td><td>$U_2 < 0, B_3H_{14} < 0$</td><td rowspan="2">Picture 3.1(c1)</td></tr>
<tr><td>$U_2 > 0, U_4 > 0$, $B_3U_1 > 0$</td></tr>
<tr><td>$U_2 < 0, B_3H_{14} > 0$ $B_3U_1 < 0$</td><td>Picture 3.1(c2)</td></tr>
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<tr><td>$U_2 > 0, U_4 < 0$</td></tr>
<tr><td>$U_2 > 0, U_4 > 0$, $B_3U_1 < 0$</td><td>Picture 3.1(c4)</td></tr>
<tr><td rowspan="4">$W_4 < 0$ or $W_4 = 0$ & $W_3 < 0$</td><td>$B_3U_1 > 0, U_2 < 0$</td><td>Picture 3.1($\overset{*}{c}1$)</td></tr>
<tr><td>$B_3U_1 < 0, U_2 < 0$</td><td>Picture 3.1($\overset{*}{c}2$)</td></tr>
<tr><td>$B_3U_1 > 0, U_2 > 0$</td><td>Picture 3.1($\overset{*}{c}3$)</td></tr>
<tr><td>$B_3U_1 < 0, U_2 > 0$</td><td>Picture 3.1($\overset{*}{c}4$)</td></tr>
</table>

Definition 1.1. *A polynomial $U(a\,,x,y)\in\mathbb{R}[a,x,y]$ is called a comitant with respect to $(\mathcal{A},G)$, where $\mathcal{A}$ is an affine invariant subset of polynomial systems (**PS**) and G is a subgroup of* $\mathrm{Aff}\,(2,\mathbb{R})$, *if there exists $\chi\in\mathbb{Z}$ such that for every $(g,\,\boldsymbol{a})\in G\times\mathbb{R}^m_{\mathcal{A}}$ the following identity holds in $\mathbb{R}[x,y]$:*

$$U(r_g(\boldsymbol{a}),\ g(x,y)\,)\equiv\ (\det\,g)^{-\chi}\,U(\boldsymbol{a},x,y),$$

where $\det g=\det M$. *If the polynomial U does not explicitly depend on x and y then it is called invariant. The number $\chi\in\mathbb{Z}$ is called the* weight *of the comitant $U(a,x,y)$. If $G=GL(2,\mathbb{R})$ (or $G=Aff(2,\mathbb{R})$) and $\mathcal{A}=$ **PS** then the comitant $U(a,x,y)$ is called GL-comitant (respectively, affine comitant).*

Definition 1.2. *A subset $X\subset\mathbb{R}^m$ will be called G-invariant, if for every $g\in G$ we have $r_g(X)\subseteq X$.*

Let $T(2,\mathbb{R})$ be the subgroup of $\mathrm{Aff}\,(2,\mathbb{R})$ formed by translations. Consider the linear representation of $T(2,\mathbb{R})$ into its corresponding subgroup $\mathcal{T}\subset GL(m,\mathbb{R})$, i.e. for every $\tau\in T(2,\mathbb{R})$, $\tau:\ x=\tilde{x}+\alpha, y=\tilde{y}+\beta$ we consider as above $r_\tau:\ \mathbb{R}^m\longrightarrow\mathbb{R}^m$.

Definition 1.3. *A comitant $U(a,x,y)$ with respect to $(\mathcal{A},G)$ is called a T-comitant if for every $(\tau,\,\boldsymbol{a})\in T(2,\mathbb{R})\times\mathbb{R}^m_{\mathcal{A}}$ the identity $U(r_\tau\cdot\boldsymbol{a},\ \tilde{x},\,\tilde{y})\ =\ U(\boldsymbol{a},\ \tilde{x},\,\tilde{y})$ holds in $\mathbb{R}[\tilde{x},\tilde{y}]$.*

Definition 1.4. *The polynomial $U(a,x,y)\in\mathbb{R}[a,x,y]$ has well determined sign on $V\subset\mathbb{R}^m$ with respect to $x,\ y$ if for every fixed $\boldsymbol{a}\in V$, the polynomial function $U(\boldsymbol{a},x,y)$ is not identically zero on V and has constant sign outside its set of zeroes on V.*

Observation 1.3. *We draw attention to the fact, that if a T-comitant $U(a,x,y)$ with respect to $(\mathcal{A},G)$ of even weight is a binary form in $x,\ y$, of even degree in the coefficients of the systems and has well determined sign on the affine invariant algebraic subset $\mathbb{R}^m_{\mathcal{A}}$ then this property is conserved by any affine transformation and the sign is conserved.*

Let us consider the polynomials

$$\begin{aligned}
&C_i(a,x,y)=yp_i(a,x,y)-xq_i(a,x,y)\in\mathbb{R}[a,x,y],\ \ i=0,1,2,\\
&D_i(a,x,y)=\frac{\partial}{\partial x}p_i(a,x,y)+\frac{\partial}{\partial y}q_i(a,x,y)\in\mathbb{R}[a,x,y],\ \ i=1,2.
\end{aligned}\tag{1.30}$$

As it was shown in [1] the polynomials

$$\left\{\ C_0(a,x,y),\quad C_1(a,x,y),\quad C_2(a,x,y),\quad D_1(a),\quad D_2(a,x,y)\ \right\}\tag{1.31}$$

of degree one in the coefficients of quadratic systems are GL-comitants of these systems.

Notation 1.4. *Let $f,\ g\in\mathbb{R}[a,x,y]$ and*

$$(f,g)^{(k)}=\sum_{h=0}^{k}(-1)^h\binom{k}{h}\frac{\partial^k f}{\partial x^{k-h}\partial y^h}\ \frac{\partial^k g}{\partial x^h\partial y^{k-h}}.\tag{1.32}$$

$(f,g)^{(k)}\in\mathbb{R}[a,x,y]$ is called the transvectant of index k of (f,g).

Theorem 1.1. *Any GL-comitant of quadratic systems can be constructed from the elements of the set* (1.31) *by using the operations: $+,\ -,\ \times$, and by applying the differential operation $(*,*)^{(k)}$.*

Remark 1.5. *We point out that the elements of the set* (1.31) *generate the whole set of GL-comitants and hence also the set of affine comitants as well as of set of T-comitants.*