The period functions' higher order derivatives

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New Trends in Dynamical Systems - Salou 2012

-

for Jaume Llibre's first (integral) 60 years

Centers

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We consider a vector field $V(z) \in C^{\infty}(\Omega, \mathbb{R}^2)$, Ω open connected subset of the real plane, and the system.

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Figure : A central region of $x' = y, \quad y' = -\sin x$.

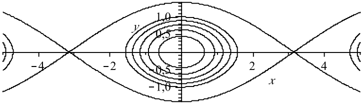

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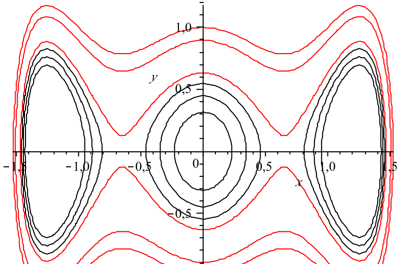
Figure : 3 central regions, 4 period annuli.



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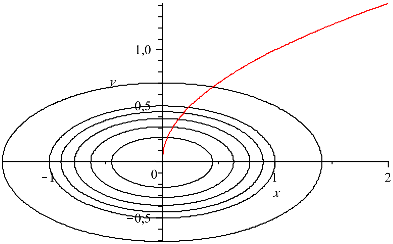
If $V(z) \in C^\infty(\Omega, \mathbb{R}^2)$, then $T \in C^\infty(\Omega, \mathbb{R})$.

Period function

Choose a transversal curve $\delta(s)$ of class C^∞, parametrize A's cycles using s, write T as a function of s: $T = T(s) = T(\delta(s))$. Then $T \in C^\infty$.



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3. Bifurcation: If $T'(z) \neq 0$, then at most a limit cycle bifurcates from γ_z.

Upper bounds to the number of critical periods

•) 2^{nd} order ODE's:

Chow, Sanders (1986)
Wang (1987)
Gavrilov (1993)
Bonorino, Brietzke, Lukaszczyk, Taschetto (2005)
Li, Lu (2008)
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•) Hamiltonian systems other than ODE's:

Chicone, Jacobs (1989)
Coppel, Gavrilov (1993)
Sabatini (2005)

Upper bounds to the number of critical periods

•) Other systems:

Chicone, Dumortier (1988)
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Francoise (1998)
Cima, Gasull, Mañosas (2000)
Garijo, Gasull, Jarque (2006)
Garijo, Gasull, Jarque (2010)
Gasull, Liu, Yang (2010)

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If W is a transversal normalizer of V, then the local flow $\phi_W(s, z)$ defined by W in Ω takes V-cycles into V-cycles:

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Theorem

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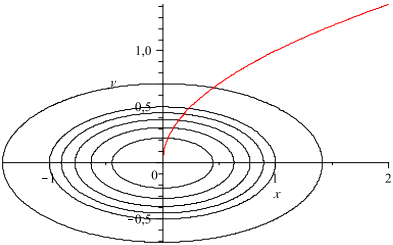
$$\mu > 0 \implies T' > 0.$$

Every section comes from a normalizer

For every transversal curve $\delta(s)$ there exists a normalizer W of V and a point $z \in A$ such that

$$\delta(s) = \phi_W(s, \gamma_{z^*})$$

Figure : Every $\delta(s)$ is an orbit of a normalizer .



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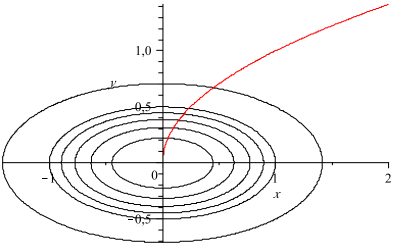

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$$x' = \alpha(H)\frac{H_x}{H_x^2 + H_y^2}, \qquad y' = \alpha(H)\frac{H_y}{H_x^2 + H_y^2},$$

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If V is the Hamiltonian system

$$x' = H_y, \qquad y' = -H_x,$$

and $\alpha(r) = 1$, then

$$\mu_H = \frac{(H_{yy} - H_{xx})H_x^2 - 4H_{xy}H_xH_y + (H_{xx} - H_{yy})H_y^2}{|\nabla H|^4},$$

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this produces an N-cofactor with separable variables,

$$\mu_{FGG} = \left(\frac{G(x)}{G'(x)}\right)' + \left(\frac{F(y)}{F'(y)}\right)' - 1.$$

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Villarini (1992) and S. (1997) proved the following

Theorem

A period annulus A of V is isochronous if and only V has a transversal commutator W.

C-factors

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Lemma

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Moreover, m is a C-factor if and only if

$$\mu = \frac{\partial_W m}{m}$$

is an N-cofactor.

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If m a C-factor, then

$$\partial_W^{(n)} T(z) = \int_0^{T(z)} \frac{\partial_W^{(n)} m(\phi_V(t,z))}{m(\phi_V(t,z))} dt.$$

(S., 2012)

C-factors

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For $n = 1$, one gets Freire, Gasull, Guillamon's result about T'.

Corollary

If m a C-factor, then

$$T'(z) = \partial_W T(z) = \int_0^{T(z)} \frac{\partial_W m(\phi_V(t,z))}{m(\phi_V(t,z))} dt$$

$$= \int_0^{T(z)} \mu(\phi_V(t,z)) dt.$$

Higher order derivatives via N-cofactors

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If one only knows an N-cofactor μ, define recursively:

$$\mu_1 = \mu,$$

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$$\partial_W^{(n)} T(z) = \int_0^T \mu_n(\phi_V(t,z))dt.$$

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$$\mu_5 = \mu^5 + 10\mu^3\mu' + 10\mu^2\mu'' + 15\mu\mu'^2 + 5\mu\mu''' + 10\mu'\mu'' + \mu^{(4)}.$$

C-factors vs N-cofactors

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$$\partial_W^{(n)} m > 0 \;\Longrightarrow\; \partial_W^{(n)} T > 0$$

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Remark: T may be W-convex, but not $\tilde{W}$-convex, for a different normalizer $\tilde{W}$.

Jacobian maps and annuli

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If $\Psi = (P, Q)$ is a jacobian map, then the Hamiltonian system with $H = \frac{P^2+Q^2}{2}$ as hamiltonian,

$$x' = PP_y + QQ_y, \qquad y' = -PP_x - QQ_x,$$

has a center at every extremum of H. It can also have non-central period annuli.

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A normalizer is the system W_δ

$$x' = \frac{PQ_y - QP_y}{\delta}, \qquad y' = \frac{-PQ_x + QP_x}{\delta}.$$

with C-factor and N-cofactor given by

$$m_s = \frac{1}{\delta}, \qquad \mu_s = -\frac{\partial_{W_\delta}\delta}{\delta}.$$

Systems with separable variables

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If $F(0) = 0$, $G(0) = 0$, $yF'(y) > 0$ for $x \neq 0$, $xG'(x) > 0$ for $y \neq 0$, $F''(0) > 0$, $G''(0) > 0$, the system

$$x' = F'(y), \qquad y' = -G'(x).$$

comes from the jacobian map $\Psi(x,y) = (P(x), Q(y))$ $= (s(x)\sqrt{2G(x)}, s(y)\sqrt{2F(y)})$; $s(t)$ sign function. It has a center at the origin O

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Its jacobian determinant is

$$\delta(x, y) = P'(x)Q'(y) = \frac{1}{2}\frac{s(x)G'(x)}{\sqrt{G(x)}}\frac{s(y)F'(y)}{\sqrt{F(y)}}$$

Systems with separable variables

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From the determinant we get the C-factor. It has separable variables.

$$m_s(x,y) = \frac{1}{P'(x)Q'(y)} = 2\frac{s(x)\sqrt{G(x)}}{G'(x)}\,\frac{s(y)\sqrt{F(y)}}{F'(y)}.$$

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From the C-factor we get the N-cofactor

$$\mu_s = -\partial_{W_\delta}\left(\ln P'(x)Q'(y)\right) = -\partial_{W_\delta}\ln P'(x) - \partial_{W_\delta}\ln Q'(y)$$

$$= -\partial_{W_\delta}\left(\ln\frac{2s(x)G'(x)}{\sqrt{2G(x)}}\right) - \partial_{W_\delta}\left(\ln\frac{2s(y)F'(y)}{\sqrt{2F(y)}}\right)$$

Systems with separable variables

Theorem

The period function of the system

$$x' = F'(y), \qquad y' = -G'(x)$$

satisfies

$$\partial^{(n)}_{W_\delta} T(z) = \int_0^{T(z)} \frac{\partial^{(n)}_{W_\delta} m_s(\phi_{V_\Psi}(t,z))}{m_s(\phi_{V_\Psi}(t,z))} dt,$$

where the integration is performed along the cycle $\phi_{V_\Psi}(t,z)$ *starting at* z*. (S. 2012)*

Systems with separable variables

Systems with separable variables

In order to study convexity, one can consider μ_{s2}, which is the sum of a mixed term (both x and y) plus two pure terms (only x or only y):

$$\mu_{s2} = 4\left[1 + 2\,\frac{GG''}{G'^2}\cdot\frac{FF''}{F'^2} + \right.$$
$$+\frac{3G^2G''^2 - 3GG'^2G'' - G^2G'G'''}{G'^4}$$
$$\left. + \frac{3F^2F''^2 - 3FF'^2F'' - F^2F'F'''}{F'^4}\right].$$

2^{nd} order ODE's

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In the case of 2^{nd} order ODE's, i. e. if $F(y) = \frac{y^2}{2}$, μ_{s2} reduces to the following form

$$\mu_{s2} = \frac{G'^4 - 8G'^2 G G'' + 12 G^2 G''^2 - 4 G^2 G''' G'}{G'^4}.$$

Systems with separable variables

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An alternative choice: use the normalizer

$$x' = \frac{H_x}{H_x^2 + H_y^2} = \frac{G'}{G'^2 + F'^2}, \qquad y' = \frac{H_y}{H_x^2 + H_y^2} = \frac{F'}{G'^2 + F'^2}.$$

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Its N-cofactor is

$$\mu_H = \frac{(F'' - G'')(G'^2 - F'^2)}{(G'^2 + F'^2)^2},$$

which does not separate variables, but does not require the non-degeneracy assumption .

End of Talk

END