Chaos in delay differential equations with applications in population dynamics

A. Ruiz-Herrera
University of Granada

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" A chaotic phenomenon occurs if it is possible to reproduce, within the system and varying the initial conditions, all the possible outcomes of a coin-fliping experiment"

S. Smale, *Finding horseshoe on the beaches of Rio*, Math. Intelligencer **20** (1998), 39–44

Definition of chaos

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Definition

Let (X, d) be a metric space. A continuous map $\psi : X \to X$ has **chaotic dynamics on two symbols** if $\exists$ disjoint compact sets $\mathcal{K}_0, \mathcal{K}_1 \subset X$ with this property:

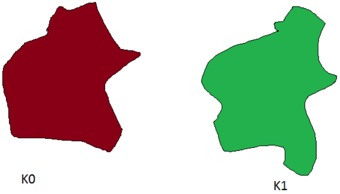
$\forall\ (s_i)_{i\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}},\ \exists\ (\omega_i)_{i\in\mathbb{Z}} \in (\mathcal{K}_0 \cup \mathcal{K}_1)^{\mathbb{Z}}$ s.t.

$$\omega_i \in \mathcal{K}_{s_i} \quad \text{and} \quad \omega_{i+1} = \psi(\omega_i) \ \text{ for all } i \in \mathbb{Z}. \tag{1}$$

Moreover, if the sequence $(s_i)_{i\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ is periodic then we can also find a periodic sequence $(\omega_i)_{i\in\mathbb{Z}} \in (\mathcal{K}_0 \cup \mathcal{K}_1)^{\mathbb{Z}}$ satisfying (1).

EXAMPLE

Take $\psi : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ a map with chaotic dynamics.



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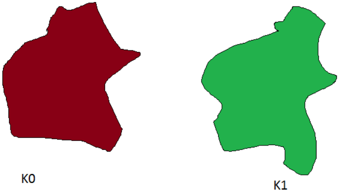

For the sequence $s = (\ldots, 1, 0, 1, 1, 0, 1, 1, 1, 0\ldots)$ we find a sequence $\omega = (\omega_i)_{i\in\mathbb{Z}} \in (\mathcal{K}_0 \cup \mathcal{K}_1)^{\mathbb{Z}}$: $\omega_0 \in \mathcal{K}_1$,

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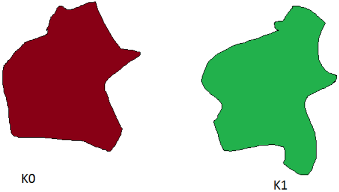

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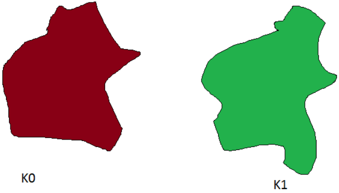

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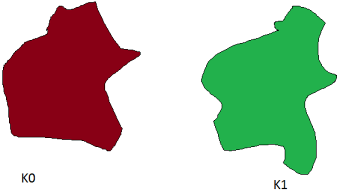

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For the sequence $\widetilde{s} = (\ldots, 1, 0, 1, 0, 1, 0, 1, \ldots)$ we find a sequence $\widetilde{\omega} = (\widetilde{\omega}_i)_{i\in\mathbb{Z}} \in (\mathcal{K}_0 \cup \mathcal{K}_1)^{\mathbb{Z}}$: $\widetilde{\omega}_0 \in \mathcal{K}_1$, $\widetilde{\omega}_1 = \psi(\widetilde{\omega}_0) \in \mathcal{K}_0$, $\widetilde{\omega}_2 = \psi^2(\widetilde{\omega}_0) \in \mathcal{K}_1$, $\widetilde{\omega}_3 = \psi^3(\widetilde{\omega}_0) \in \mathcal{K}_0 \ldots$

- This definition of chaos has been used by several authors, e.g Zgliczynski, Kirchgraber, Smale, Srzednicki, Mischaikow...Namely, our definition of chaos is the classical definition of chaos in the sense of coin tossing with the additional property of periodic points.

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- If our map is chaotic according to our definition is also chaotic in the sense of Li-Yorke and in the sense of Block-Coppel.
- Our definition guarantees natural properties of complex dynamics such as sensitive dependence on the initial conditions.

Definition

Given a set $\mathcal{A} \subset \mathbb{R}^2$ and a positive constant m,

$$X_{\mathcal{A},m} := \{(x, y) \in \mathbb{R} \times \mathcal{C}([-\tau, 0], \mathbb{R}) :$$

$$(x, y(0)) \in \mathcal{A}, |y(t) - y(0)| \leq m \textit{ for all } t \in [-\tau, 0]\}.$$

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$(\frac{1}{2}, \sin(t)) \in X_{\mathcal{A},1}$ but $(\frac{1}{2}, \sin(t)) \notin X_{\mathcal{A},1/3}$.

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$(\frac{3}{2}, \sin(t)) \notin X_{\mathcal{A},1}$.

Consider $P : \mathcal{D} \subset \mathbb{R} \times \mathcal{C}([-\tau, 0], \mathbb{R}) \longrightarrow \mathbb{R} \times \mathcal{C}([-\tau, 0], \mathbb{R})$ a compact operator, a rectangle $\mathcal{A} = [a, b] \times [c, d] \subset \mathbb{R}^2$, a constant $m > 0$ and a closed set $\mathcal{H} \subset X_{\mathcal{A},m}$.

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Definition

We say that $(\mathcal{H}, P)$ **stretches** $X_{\mathcal{A},m}$ along the paths and write

$$(\mathcal{H}, P) : X_{\mathcal{A},m} \Leftrightarrow\!\!\!\rightarrow X_{\mathcal{A},m},$$

if $\forall\ w : [0, 1] \longrightarrow X_{\mathcal{A},m}$ continuous with $w(0) \in \{(x, y) \in X_{\mathcal{A},m} : x = a\}$ and $w(1) \in \{(x, y) \in X_{\mathcal{A},m} : x = b\}$, there exists a subinterval $[t', t''] \subset [0, 1]$ satisfying that

- $w(t) \in \mathcal{H}, \;\; P(w(t)) \in X_{\mathcal{A},m}$ for all $t \in [t', t'']$,
- $P(w(t')) \in \{(x, y) \in X_{\mathcal{A},m} : x = a\}$ and $P(w(t'')) \in \{(x, y) \in X_{\mathcal{A},m} : x = b\}$ or viceversa.

Consider

$$Pro : \mathbb{R} \times \mathcal{C}([-\tau, 0], \mathbb{R}) \longrightarrow \mathbb{R}^2$$
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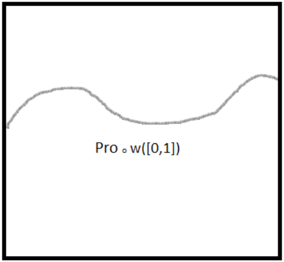
For all $w : [0, 1] \longrightarrow X_{\mathcal{A},m}$ continuous so that $Pro \circ w$ is a path in $\mathcal{A} = [a, b] \times [c, d]$ joining $a \times [c, d]$ and $b \times [c, d]$



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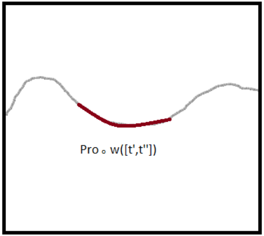
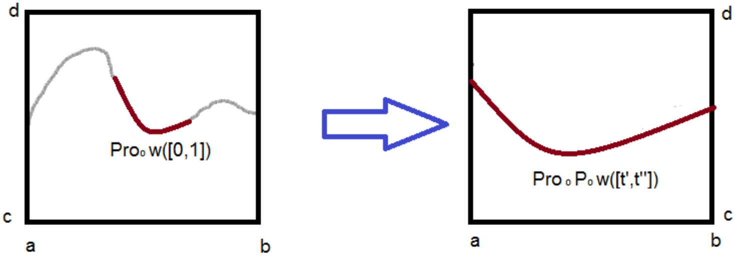


- $w([t', t'']) \subset \mathcal{H}$,
- $Pro \circ P \circ w([t', t''])$ is a path in $\mathcal{A}$ joining $\{a\} \times [c, d]$ and $\{b\} \times [c, d]$.



Theorem

Assume that there exist two disjoint compact sets $\mathcal{K}_0, \mathcal{K}_1 \subset \mathcal{A}$*, a constant* m*, and a compact operator* P *such that*

$$(X_{\mathcal{K}_0,m}, P) : X_{\mathcal{A},m} \Leftrightarrow\!\!\!\!\!\to X_{\mathcal{A},m},$$

$$(X_{\mathcal{K}_1,m}, P) : X_{\mathcal{A},m} \Leftrightarrow\!\!\!\!\!\to X_{\mathcal{A},m}.$$

Then P *induces chaotic dynamics on two symbols relative to* $P(X_{\mathcal{A},m}) \cap X_{\mathcal{K}_1,m}$ *and* $P(X_{\mathcal{A},m}) \cap X_{\mathcal{K}_0,m}$.

We use our method to prove the presence of chaotic dynamics in the classical model of predator prey with delay suggested by R. May in

- *Time-delay versus stability in population models with two and three trophic levels*, Ecology, 1973.

Consider the system

$$\begin{cases} x'(t) = x(t)(-a(t) - b(t)x(t) + c(t)y(t)) \\ y'(t) = y(t)(d(t) - e(t)x(t) - f(t)y(t-\tau)) \end{cases} \tag{2}$$

where $\tau > 0$ and all the coefficients are positive and T-periodic.

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$b(t)$ competition inside the species x.
$c(t)$ effect of the prey in the species x.
$f(t)$ competition inside the species y.
$e(t)$ effect of the predator in the species y.

Why is the delay introduced in the the previous model?

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The competition for the food inside the herbivore at time t must depend on $y(t-\tau)$ where τ is the growing time for the grass (food for the herbivore).

Forget the carnivorous population,

$$y'(t) = y(t)\underbrace{\left(d(t) - f(t)y(t-\tau)\right)}_{GROWTH\ RATE} \tag{3}$$

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SITUATION 1: BEFORE HUGE number of animals **at time** t 4 animals.
SITUATION 2: BEFORE SMALL number of animals **at time** t 10 animals.

If we don't introduce delay, **the situation 1** is much **better** for surviving than **the situation 2** because the **Growth rate** is larger.

We want to prove **analytically** the presence of chaos in

$$\begin{cases} x'(t) = x(t)(-a(t) - b(t)x(t) + c(t)y(t)) \\ y'(t) = y(t)(d(t) - e(t)x(t) - f(t)y(t-\tau)) \end{cases} \tag{4}$$

Two remarks:

- **For some parameters, our system is not chaotic**. See
  - A. R.-H, *Topological criteria of global attraction with applications in population dynamics*, Nonlinearity **25** (2012), 2823–2841.

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Two remarks:

- **For some parameters, our system is not chaotic**. See
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- Notice that the Poincaré operator for this system is

$$P : \mathcal{D} \subset \mathbb{R} \times \mathcal{C}([-\tau, 0], \mathbb{R}) \longrightarrow \mathbb{R} \times \mathcal{C}([-\tau, 0], \mathbb{R})$$

$$z = (\xi, \eta) \mapsto (x(T; z), y_T(z))$$

with $y_T(z)(t) := y(T + t; z)$,

Assume that the coefficients in the previous system are piece-wise constant in this way ($T_1 + T_2 = T$) (all the parameters are strictly positive)

$$\left\{ \begin{array}{l} x'(t) = x(t)(-a_1 + c_1 y(t)) \\ y'(t) = y(t)(d_1 - e_1 x(t)) \end{array} \right. \qquad \text{for } t \in [nT, nT + T_1[\tag{5}$$

$$\left\{ \begin{array}{l} x'(t) = x(t)(-a_2 - b_2 x(t) + c_2 y(t)) \\ y'(t) = y(t)(d_2 - f_2 y(t-\tau)) \end{array} \right. \qquad \text{for } t \in [nT + T_1, nT + T_1 + T_2[\tag{6}$$

Denote by (S) this T-periodic system.

Theorem

Consider system (S) with all parameters fixed except d_1, T_1, T_2, τ and suppose that

$$\frac{3d_2}{4f_2} < \frac{a_2}{c_2} < \frac{5d_2}{4f_2}.$$

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$$\frac{3d_2}{4f_2} < \frac{a_2}{c_2} < \frac{5d_2}{4f_2}. \tag{7}$$

Then there exist a constant T_2^* and three maps $d_1^*(\widetilde{T}_2)$, $T_1^*(\widetilde{d}_1, \widetilde{T}_2)$, and $\tau^*(\widetilde{d}_1, \widetilde{T}_1, \widetilde{T}_2)$ with the following property:

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In concrete examples we can give precise estimates of T_2^*, $d_1^*(T_2)$, $T_1^*(d_1, T_2)$ and $\tau^*(d_1, T_1, T_2)$ depending on the coefficients of the system.

Now we study the presence of chaos in this system when the coefficients are not necessarily piece-wise constant.

$$\begin{cases} x'(t) = x(t)(-a(t) - b(t)x(t) + c(t)y(t)) \\ y'(t) = y(t)(d(t) - e(t)x(t) - f(t)y(t-\tau)) \end{cases} \tag{8}$$

Theorem

Fix all parameters in (S) satisfying the conditions of the previous theorem, i.e. (7) and $0 < T_2 < T_2^*$, $d_1 > d_1^*(T_2)$, $T_1 > T_1^*(d_1, T_2)$ *and* $0 < \tau < \tau^*(d_1, T_1, T_2)$. *Then there exists* $\epsilon > 0$ *such that if the distance in* L^1_T *between the previous parameters in (S) and the coefficients of (8) is smaller than* ϵ, *the Poincaré map associated to (8) is chaotic.*

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Given two T-periodic integrable functions $f(t)$ and $g(t)$, their distance in L^1_T is given by $\int_0^T |f(t) - g(t)|\, dt$. In our setting, the assumptions mean that

$$\int_0^{T_1} |a(t) - a_1| dt + \int_{T_1}^{T} |a(t) - a_2| dt < \epsilon,$$

and so on (for the other coefficients).

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THANK YOU VERY MUCH!