Classification of Lattès maps on $\mathbf{P}^2$

Feng Rong

Department of Mathematics
Shanghai Jiao Tong University
Shanghai, P.R. China

New Trends in Dynamical Systems
Salou (Tarragona), Spain; Oct. 1, 2012

A Lattès map on $\mathbf{P}^k$ is, by definition, a holomorphic map f on $\mathbf{P}^k$ such that the following diagram commutes:

$$\begin{array}{ccc}
\mathbf{C}^k & \xrightarrow{l} & \mathbf{C}^k \\
\pi \downarrow & & \downarrow \pi \\
E^k & \xrightarrow{L} & E^k \\
\Psi \downarrow & & \downarrow \Psi \\
\mathbf{P}^k & \xrightarrow{f} & \mathbf{P}^k,
\end{array}$$

where E^k is a complex torus of dimension k, l is an affine map on $\mathbf{C}^k$ which induces an affine map L on E^k, and Ψ is a holomorphic map from E^k onto $\mathbf{P}^k$.

Main Theorem

Let g be a holomorphic map on $\mathbf{P}^1$ and f be a holomorphic map on $\mathbf{P}^2$. If there exists a holomorphic map $\pi : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^2$ such that $f \circ \pi = \pi \circ (g, g)$, then we call f the square map of g.

Main Theorem

Let g be a holomorphic map on $\mathbf{P}^1$ and f be a holomorphic map on $\mathbf{P}^2$. If there exists a holomorphic map $\pi : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^2$ such that $f \circ \pi = \pi \circ (g, g)$, then we call f the square map of g.

An algebraic web is given by a reduced curve $C \subset \check{\mathbf{P}}^2$, where $\check{\mathbf{P}}^2$ is the dual projective plane consisting of lines in $\mathbf{P}^2$. The web is invariant for a holomorphic map f on $\mathbf{P}^2$ if every line in $\mathbf{P}^2$ belonging to C is mapped to another such line.

Main Theorem

Let g be a holomorphic map on $\mathbf{P}^1$ and f be a holomorphic map on $\mathbf{P}^2$. If there exists a holomorphic map $\pi : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^2$ such that $f \circ \pi = \pi \circ (g, g)$, then we call f the square map of g.

An algebraic web is given by a reduced curve $C \subset \check{\mathbf{P}}^2$, where $\check{\mathbf{P}}^2$ is the dual projective plane consisting of lines in $\mathbf{P}^2$. The web is invariant for a holomorphic map f on $\mathbf{P}^2$ if every line in $\mathbf{P}^2$ belonging to C is mapped to another such line.

Theorem (Main Theorem)

If f is a Lattès map on $\mathbf{P}^2$, then either f or a suitable iteration of f is one of the following:
(i) a square map of a Lattès map on $\mathbf{P}^1$;
(ii) a holomorphic map preserving an algebraic web associated to a smooth cubic.

Complex crystallographic group

Let $E(n)$ be the complex motion group acting on $\mathbf{C}^n$. A complex crystallographic group is, by definition, a discrete subgroup of $E(n)$ with compact quotient. Let $U(n)$ be the unitary group of size n.

Complex crystallographic group

Let $E(n)$ be the complex motion group acting on $\mathbf{C}^n$. A complex crystallographic group is, by definition, a discrete subgroup of $E(n)$ with compact quotient. Let $U(n)$ be the unitary group of size n.

For $A \in U(2)$ and $r \in \mathbf{C}^2$, let $(A|r) \in E(2)$ denote the transformation: $z \to Az + r$. For a two dimensional complex crystallographic group Γ, $R := \{r;\ (1|r) \in \Gamma\}$ and $G := \{A;\ (A|r) \in \Gamma\}$ are called the lattice and the point group of Γ, respectively. If Γ has the representation $\{(A|r);\ A \in G,\ r \in R\}$, then Γ is called the semidirect product $G \ltimes R$ of the lattice and the point group.

Complex crystallographic group

Let $E(n)$ be the complex motion group acting on $\mathbf{C}^n$. A complex crystallographic group is, by definition, a discrete subgroup of $E(n)$ with compact quotient. Let $U(n)$ be the unitary group of size n.

For $A \in U(2)$ and $r \in \mathbf{C}^2$, let $(A|r) \in E(2)$ denote the transformation: $z \to Az + r$. For a two dimensional complex crystallographic group Γ, $R := \{r;\ (1|r) \in \Gamma\}$ and $G := \{A;\ (A|r) \in \Gamma\}$ are called the lattice and the point group of Γ, respectively. If Γ has the representation $\{(A|r);\ A \in G,\ r \in R\}$, then Γ is called the semidirect product $G \ltimes R$ of the lattice and the point group.

Let $G(m, p, 2) \subset U(2)$ denote the group generated by

$$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \begin{pmatrix} & \theta \\ \theta^{-1} & \end{pmatrix} \text{ and } \begin{pmatrix} \theta^p & \\ & 1 \end{pmatrix}, \quad \theta = e^{2\pi i/m}.$$

Theorem (Corollary of a theorem of Kaneko-Tokunaga-Yoshida)

If (Λ, G) is a pair such that $E^2 = \mathbf{C}^2/\Lambda$ and $E^2/G \cong \mathbf{P}^2$, then, up to conjugation, it is one of the following

$$\Lambda = L^2(\tau), \qquad G = G_1 := G(2,1,2),$$

$$\Lambda = L^2(\zeta), \qquad G = G_2 := G(3,1,2),$$

$$\Lambda = L^2(i), \qquad G = G_3 := G(4,1,2),$$

$$\Lambda = L^2(\zeta), \qquad G = G_4 := G(6,1,2),$$

$$\Lambda = L^2(i), \qquad G = G_5 := G(4,2,2) \ltimes \mathbf{Z}\frac{1+i}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$\Lambda = \Lambda_6 := L(\tau)\begin{pmatrix} -1 \\ 1 \end{pmatrix} + L(\tau)\begin{pmatrix} \zeta^2 \\ \zeta \end{pmatrix}, \quad G = G_6 := G(3,3,2).$$

Affine maps

Write $L = (A|r)$, where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad r = \begin{pmatrix} e \\ f \end{pmatrix},$$

with $a, b, c, d, e, f \in \mathbf{C}$. Set $\psi = \Psi \circ \pi$.

Affine maps

Write $L = (A|r)$, where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad r = \begin{pmatrix} e \\ f \end{pmatrix},$$

with $a, b, c, d, e, f \in \mathbf{C}$. Set $\psi = \Psi \circ \pi$.

For $z \in \mathbf{C}^2$ and $\lambda \in \Lambda$ we then have

$$\psi \circ (Az + r) = f \circ \psi(z) = f \circ \psi(z + \lambda) = \psi \circ (A(z + \lambda) + r).$$

This shows that

$$A\Lambda \subset \Lambda.$$

Affine maps

Write $L = (A|r)$, where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad r = \begin{pmatrix} e \\ f \end{pmatrix},$$

with $a, b, c, d, e, f \in \mathbf{C}$. Set $\psi = \Psi \circ \pi$.

For $z \in \mathbf{C}^2$ and $\lambda \in \Lambda$ we then have

$$\psi \circ (Az + r) = f \circ \psi(z) = f \circ \psi(z + \lambda) = \psi \circ (A(z + \lambda) + r).$$

This shows that

$$A\Lambda \subset \Lambda.$$

Since L carries a small domain of volume V to a domain of volume $|\det A|^2 V$, it follows that the map L has degree $|\det A|^2$. Let d_f be the algebraic degree of f and the (topological) degree of f is d_f^2. Since $f \circ \Psi = \Psi \circ L$, we get $d_f^2 = |\det A|^2$, and thus

$$|\det A| = d_f.$$

Admissible affine maps

By definition, an affine map L on $E^2 = \mathbf{C}^2/\Lambda$ induces a Lattès map f on $\mathbf{P}^2 \cong E^2/G$ if and only if

$$(\star) \quad \text{for any } g \in G, \text{ there exists } h \in G \text{ s.t. } L \circ g \equiv h \circ L \mod \Lambda.$$

Admissible affine maps

By definition, an affine map L on $E^2 = \mathbf{C}^2/\Lambda$ induces a Lattès map f on $\mathbf{P}^2 \cong E^2/G$ if and only if

$(\star)$ for any $g \in G$, there exists $h \in G$ s.t. $L \circ g \equiv h \circ L \mod \Lambda$.

For $L = (A|r)$, $g = (B|r')$ and $h = (C|r'')$, we have

$$L \circ g = (AB|Ar' + r) \quad \text{and} \quad h \circ L = (CA|Cr + r'').$$

Therefore $L \circ g \equiv h \circ L \mod \Lambda$ is equivalent to

$$AB = CA \quad \text{and} \quad Ar' + r \equiv Cr + r'' \mod \Lambda.$$

For the proof of the Main Theorem, we make use of the following two simple yet important observations:
1. condition $(\star)$ is satisfied if and only if it is satisfied for all generators of G;
2. if $g = (B|r')$ and $h = (C|r'')$ satisfy $L \circ g \equiv h \circ L \mod \Lambda$, then B and C are of the same order.

Square maps

Lemma

Let $L = (A|r)$ be an affine map on E^2 which induces a Lattès map on $\mathbf{P}^2$ in the case (Λ, G_i), $i = 1, 2, 3, 4, 5$. If A is of the form

$$\begin{pmatrix} \alpha & \\ & \beta \end{pmatrix} \text{ or } \begin{pmatrix} & \alpha \\ \beta & \end{pmatrix},$$

then L induces a map on $\mathbf{P}^1 \times \mathbf{P}^1$ of the form (g, g), where g is a Lattès map on $\mathbf{P}^1$.

Square maps

Lemma

Let $L = (A|r)$ be an affine map on E^2 which induces a Lattès map on $\mathbf{P}^2$ in the case (Λ, G_i), $i = 1, 2, 3, 4, 5$. If A is of the form

$$\begin{pmatrix} \alpha & \\ & \beta \end{pmatrix} \text{ or } \begin{pmatrix} & \alpha \\ \beta & \end{pmatrix},$$

then L induces a map on $\mathbf{P}^1 \times \mathbf{P}^1$ of the form (g, g), where g is a Lattès map on $\mathbf{P}^1$.

By the above lemma, we have the following commutative diagram

$$\begin{tikzcd} E^2 \arrow[r, "\pi_1"] \arrow[d, "L"'] & \mathbf{P}^1 \times \mathbf{P}^1 \arrow[r, "\pi_2"] \arrow[d, "(g,g)"'] & \mathbf{P}^2 \arrow[d, "f"] \\ E^2 \arrow[r, "\pi_1"] & \mathbf{P}^1 \times \mathbf{P}^1 \arrow[r, "\pi_2"] & \mathbf{P}^2. \end{tikzcd}$$

Maps preserving an algebraic web

Let g be a holomorphic map on $\mathbf{P}^1$. Let C be a smooth conic in $\mathbf{P}^2$, which we identify with $\mathbf{P}^1$. For a point $P \in \mathbf{P}^2$, let l_1 and l_2 be the tangent lines to C which pass through P and let Q_1 and Q_2 be the points of contact. Let l_1' and l_2' be the tangent lines to C at $g(Q_1)$ and $g(Q_2)$ and define $f(P)$ to be the intersection point of l_1' and l_2'. It is easy to see that f preserves the algebraic web associated to the dual curve of C.

Maps preserving an algebraic web

Let g be a holomorphic map on $\mathbf{P}^1$. Let C be a smooth conic in $\mathbf{P}^2$, which we identify with $\mathbf{P}^1$. For a point $P \in \mathbf{P}^2$, let l_1 and l_2 be the tangent lines to C which pass through P and let Q_1 and Q_2 be the points of contact. Let l_1' and l_2' be the tangent lines to C at $g(Q_1)$ and $g(Q_2)$ and define $f(P)$ to be the intersection point of l_1' and l_2'. It is easy to see that f preserves the algebraic web associated to the dual curve of C.

If $L = (A|r)$ induces a Lattès map in the case (Λ, G_6) then we have that either L, L^2, L^3 or L^6 is of the form (l, l), where $l : x \to ax + e$, with $3e \equiv 0 \mod L(\tau)$, is an affine map on E.

Let $\eta : E \to \check{\mathbf{P}}^2$ be an embedding of E into $\check{\mathbf{P}}^2$ and denote the image by C, which is a smooth cubic. Since $3e \equiv 0 \mod L(\tau)$, the map l preserves collinearity. Thus (l, l) induces a holomorphic map on $\mathbf{P}^2$, through $\phi : E \times E \to \mathbf{P}^2$, which preserves the algebraic web associated to C. Here ϕ maps $(x, y) \in E \times E$ to the line joining $\eta(x)$ and $\eta(y)$.

Associated Orbifold

Let X be a complex manifold and denote by $\mathcal{H}(X)$ the space of irreducible analytic subvarieties of codimension 1 in X. Let r be a function defined on $\mathcal{H}(X)$ with values in $\mathbf{N}^+$, which is equal to 1 outside a locally finite family of analytic subvarieties of X. We call the pair (X, r) an orbifold. We say that the orbifold (X, r) is parabolic, if there exists a ramified covering $f : X \to X$ such that

$$r(f(H)) = m_f(H) \cdot r(H)$$

for every $H \in \mathcal{H}(X)$, where $m_f(H)$ is the multiplicity of f along H (i.e. the multiplicity of f at a generic point of H). (Note that $m_f(H) = 1$ if $H \not\subset C_f$.)

Associated Orbifold

Let X be a complex manifold and denote by $\mathcal{H}(X)$ the space of irreducible analytic subvarieties of codimension 1 in X. Let r be a function defined on $\mathcal{H}(X)$ with values in $\mathbf{N}^+$, which is equal to 1 outside a locally finite family of analytic subvarieties of X. We call the pair (X, r) an orbifold. We say that the orbifold (X, r) is parabolic, if there exists a ramified covering $f : X \to X$ such that

$$r(f(H)) = m_f(H) \cdot r(H)$$

for every $H \in \mathcal{H}(X)$, where $m_f(H)$ is the multiplicity of f along H (i.e. the multiplicity of f at a generic point of H). (Note that $m_f(H) = 1$ if $H \not\subset C_f$.)

Lemma

If f is a Lattès map on $\mathbf{P}^k$, then there exists a parabolic orbifold $(\mathbf{P}^k, r)$ associated with f.

An example

The following is an example in the case $(L^2(i), G(2,1,2))$.

The affine map is given by $L = (A|r)$, where

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, r \equiv e \begin{pmatrix} 1 \\ 1 \end{pmatrix} \bmod L^2(i), \quad e \equiv -e \bmod L(i).$$

To obtain f, we need to express $\wp(u+v) + \wp(u-v)$ and $\wp(u+v)\wp(u-v)$ in terms of $\wp(u) + \wp(v)$ and $\wp(u)\wp(v)$. Here $\wp$ is the Weierstrass $\wp$-function associated to $L(i)$. From

$$\wp(u+v) + \wp(u-v) = \frac{2(\wp(u)\wp(v) - 1)(\wp(u) + \wp(v))}{(\wp(u) - \wp(v))^2},$$

and

$$\wp(u+v)\wp(u-v) = \frac{(\wp(u)\wp(v) + 1)^2}{(\wp(u) - \wp(v))^2},$$

we get

$$f : [x : y : z] \longrightarrow [2x(y-z) : (y+z)^2 : x^2 - 4yz].$$

Thank You!

&

Happy Birthday, Jaume!