New doubly-symmetric families of comet like periodic orbits in the spatial restricted $(N+1)$–Body Problem

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C A P E S

Introduction

The main objective of classical Celestial Mechanics is the study of the $N-$body problem, which consists of describing the motion of N point masses moving in the Euclidian $3-$dimensional space under the action of their mutual newtonian gravitational forces. The formulation of the $N-$body problem appears at first time in this treatise *Philosophiae Naturalis Principia Mathematica* of Newton (1687). It is in this treatise where the laws of mechanics and the universal gravitational attraction law allowed to formulate the $N-$body problem as a system of differential equations.

The main goal of this paper is to present new families of periodic solutions for spatial restricted $(N+1)$–body problems. We distinguish one of the masses that is far way from the others. This mass is called $comet$ and the others $primaries$. We pointed that the comet has infinitesimal mass and a small parameter ε is considered as a parameter scale related with the ratio between the distance of two primaries and the distance of one primary and the comet.

We assume that all the primaries have the same mass m. They are in a planar central configuration given by the vertices of an $N-$regular polygon inscribed on the unit circle rotating around the center of mass with angular velocity ω. We obtain the existence of a new family of doubly–symmetric periodic solutions for the motion of the infinitesimal body by using the Poincaré's continuation method. This periodic solution is a perturbation of the circular solution of the Kepler problem.

1. Modeling

First we consider that N primaries with equal mass m are in a central configuration at the vertices of an N–regular polygon. Then we deal with the problem of $N+1$ bodies moving in the space $\mathbb{R}^3$ such that the only forces acting on them are the ones coming from their mutual gravitational attractions.

Assume that the comet is very far from the other N primaries. Moreover the primaries are in a planar central configuration that satisfies: the N primaries have the same mass m, and the kth particle is located at the vertex $\rho_k = (\cos\theta_k, \sin\theta_k, 0)$, $\theta_k = 2\pi k/N$ and $k = 1, \ldots, N$, of a N–regular polygon inscribed on the unit circle (see Figure 1 and [4]).

Note that the origin of the system of inertial coordinates is at the center of mass. The primaries are rotating around the center of mass with angular velocity ω and their motion is given by

$$\begin{pmatrix} X_k(t) \\ Y_k(t) \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_k \\ \sin\theta_k \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\theta_k+\omega t) \\ \sin(\theta_k+\omega t) \\ 0 \end{pmatrix}.$$

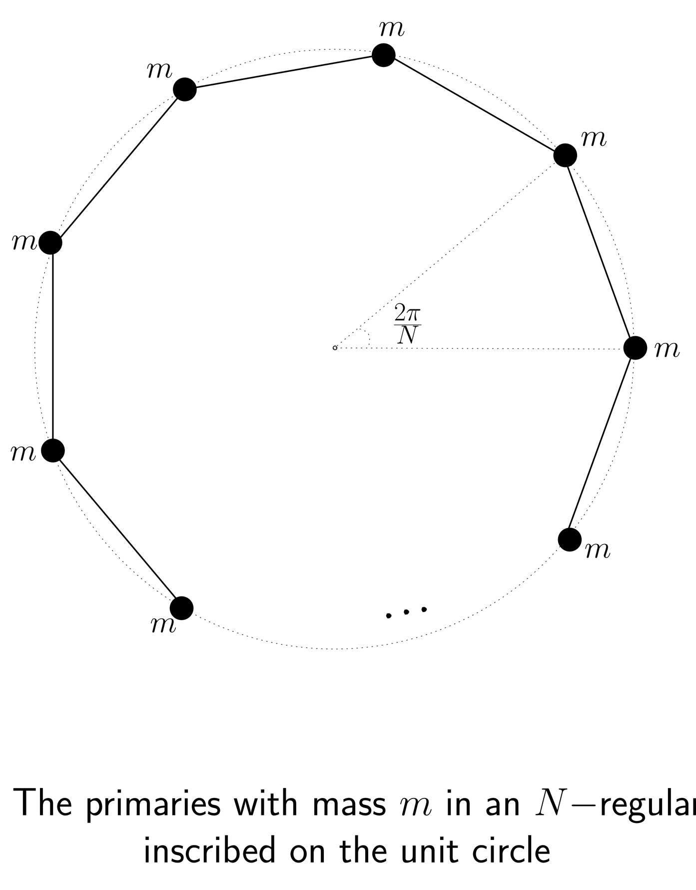

Figure 1: The primaries with mass m in an $N-$regular polygon inscribed on the unit circle

We consider (X,Y,Z) the coordinates of the infinitesimal mass and $(X_k, Y_k, 0)$ the coordinates of the kth primary in the inertial coordinates. We want to study the motion of the infinitesimal mass. Thus the equations of motion for the infinitesimal mass in inertial coordinates are

$$\begin{aligned} \ddot{X} &= -\sum_{k=1}^{N} \frac{m(X-\cos(\theta_k+\omega t))}{((X-\cos(\theta_k+\omega t))^2+(Y-\sin(\theta_k+\omega t))^2+Z^2)^{3/2}}, \\ \ddot{Y} &= -\sum_{k=1}^{N} \frac{m(Y-\sin(\theta_k+\omega t))}{((X-\cos(\theta_k+\omega t))^2+(Y-\sin(\theta_k+\omega t))^2+Z^2)^{3/2}} \qquad (1) \\ \ddot{Z} &= -\sum_{k=1}^{N} \frac{mZ}{((X-\cos(\theta_k+\omega t))^2+(Y-\sin(\theta_k+\omega t))^2+Z^2)^{3/2}}. \end{aligned}$$

Considering a rotating coordinate system and a special change of coordinates we have that the motion of the infinitesimal mass of this restricted $N+1$–body problem can be written as a Hamiltonian system

$$\begin{aligned} \dot{x_1} &= y_1 - x_2, \\ \dot{x_2} &= y_2 + x_1, \\ \dot{x_3} &= y_3, \\ \dot{y_1} &= -y_2 - \sum_{k=1}^{N} m \frac{x_1-\cos\theta_k}{((x_1-\cos\theta_k)^2+(x_2-\sin\theta_k)^2+x_3^2)^{3/2}}, \qquad (2) \\ \dot{y_2} &= y_1 - \sum_{k=1}^{N} m \frac{x_2-\sin\theta_k}{((x_1-\cos\theta_k)^2+(x_2-\sin\theta_k)^2+x_3^2)^{3/2}}, \\ \dot{y_3} &= -\sum_{k=1}^{N} m \frac{x_3}{((x_1-\cos\theta_k)^2+(x_2-\sin\theta_k)^2+x_3^2)^{3/2}}, \end{aligned}$$

with Hamiltonian function

$$H_1 = \frac{\|y\|^2}{2} - x^T K y - \sum_{k=1}^{N} \frac{m}{((x_1-\cos\theta_k)^2+(x_2-\sin\theta_k)^2+x_3^2)^{1/2}}, \qquad (3)$$

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ are the conjugate coordinates of x, and $K = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

As we consider the case that the infinitesimal mass is far from the primaries we take the scale of variables: $x \to \varepsilon^{-2}x$, $y \to \varepsilon y$, that means to replace x by $\varepsilon^{-2}x$ and y by εy in the Hamiltonian (3). This is a symplectic transformation with multiplier ε, see for more details [3]. Moreover, we consider the total mass Nm as the unit mass and work with Poincaré variables. Then

$$H_1 = \frac{-\varepsilon^3}{2(P_1+P_3)^2} - P_1 + \frac{1}{2}(P_2^2+Q_2^2) + \varepsilon^5 H_1^*(Q_1, Q_2, Q_3, P_1, P_2, P_3, \varepsilon, m),$$

and the equations of motion are

$$\begin{aligned} \dot{Q}_1 &= \frac{\partial H_1}{\partial P_1} = \frac{\varepsilon^3}{(P_1+P_3)^3} - 1 + \varepsilon^5 f_1, \\ \dot{Q}_2 &= \frac{\partial H_1}{\partial P_2} = P_2 + \varepsilon^5 f_2, \\ \dot{Q}_3 &= \frac{\partial H_1}{\partial P_1} = \frac{\varepsilon^3}{(P_1+P_3)^3} + \varepsilon^5 f_3, \qquad (4) \\ \dot{P}_1 &= -\frac{\partial H_1}{\partial Q_1} = 0 + \varepsilon^5 f_4, \\ \dot{P}_2 &= -\frac{\partial H_1}{\partial Q_2} = -Q_2 + \varepsilon^5 f_5, \\ \dot{P}_3 &= -\frac{\partial H_1}{\partial Q_3} = 0 + \varepsilon^5 f_6, \end{aligned}$$

where f_i is the appropriate partial derivative of H_1^* for $i = 1, \ldots, 6$.

2. Comet doubly-symmetric periodic orbits

By the symmetry of an N–regular polygon we have that the system (2) is invariant under the two anti–symplectic involutions:

$$\begin{aligned} \mathcal{R}_1 &: (t, x_1, x_2, x_3, y_1, y_2, y_3) \to (-t, x_1, -x_2, -x_3, -y_1, y_2, y_3), \\ \mathcal{R}_2 &: (t, x_1, x_2, x_3, y_1, y_2, y_3) \to (-t, x_1, -x_2, x_3, -y_1, y_2, -y_3). \end{aligned}$$

The fixed sets by the involutions $\mathcal{R}_1$ and $\mathcal{R}_2$ are $\mathcal{L}_1 = \{(x_1, 0, 0, 0, y_2, y_3)\}$ and $\mathcal{L}_2 = \{(x_1, 0, x_3,\ 0,\ y_2,\ 0)\}$, respectively. If a solution starts in one of these sets at time $t = 0$ and hits the other set at later time $t = T$, then the solution is $4T$–periodic and the orbit of this solution is invariant by both symmetries. We shall call such a periodic solution *doubly–symmetric*. Geometrically an orbit intersects $\mathcal{L}_1$ if it hits the x_1–axis perpendicularly and it intersects $\mathcal{L}_2$ if it hits the (x_1, x_3)–plane perpendicularly.

In Poincaré variables we obtain a circular orbit when $Q_2 = P_2 = 0$. The set $\mathcal{L}_1$ in Poincaré variables is defined by $Q_2 = 0$, $Q_1 \equiv Q_3 \equiv 0 \pmod{\pi}$, and $\mathcal{L}_2$ in Poincaré variables is defined by $Q_2 = 0$, $Q_1 \equiv 0 \pmod{\pi}$, $Q_3 \equiv \pi/2 \pmod{\pi}$.

Note that with the scaled variables when the parameter ε tends to zero the distance between the infinitesimal mass and the primaries goes to infinity. So the differential equations (4) degenerate. Then we need to obtain a solution in a neighborhood of $\varepsilon = 0$, and to do this we need the approximate solutions to this system of differential equations and good estimates. Moreover, since we are looking for periodic solutions far from the primaries and therefore of long period, we need these approximate solutions for large values of t and small values of ε.

In the first approximation we ignore the ε^5 terms of equations (4), and we get

$$\begin{aligned} \dot{Q}_1 &= \frac{\varepsilon^3}{(P_1+P_3)^3} - 1, & \dot{P}_1 &= 0, \\ \dot{Q}_2 &= P_2, & \dot{P}_2 &= -Q_2, \qquad (5) \\ \dot{Q}_3 &= \frac{\varepsilon^3}{(P_1+P_3)^3}, & \dot{P}_3 &= 0. \end{aligned}$$

These are the equations of motion for the Kepler problem in the scaled, rotating Poincaré variables around the center mass of the primaries.

After study in which conditions a solution of (5) in $t = 0$ hits $\mathcal{L}_1$ and in $t = T$ hits $\mathcal{L}_2$ we conclude that solution

$$\begin{aligned} Q_1(t) &= \left(\frac{\varepsilon^3}{(p_1+p_3)^3} - 1\right) t + i\pi, & P_1(t) &= p_1, \\ Q_2(t) &= 0, & P_2(t) &= 0, \\ Q_3(t) &= \left(\frac{\varepsilon^3}{(p_1+p_3)^3}\right) t + j\pi, & P_3(t) &= p_3, \end{aligned}$$

with p_3 arbitrário, $k = -\frac{(m+1/2)}{\varepsilon^3} + m + \frac{1}{2}$, p_1 given by $(p_1+p_3)^3 = 1$ is a dobly–symmetric $4T$–periodic solution and $T = (m-k+1/2)\pi$.

We shall show that the solution of the approximate equations (5) are actually approximations of doubly–symmetric periodic solutions of the true equations (4). Thus we obtain our main result is the following one.

3. Main Result

Theorem 1. *There exist doubly–symmetric periodic solutions of the spatial restricted $(N+1)$–body problem for the primaries with equal mass m at the vertex of an $N-$regular polygon inscribed on the unit circle.*

The next lemma is used to conclude Theorem 1.

Lemma 2. *Let $(q_1, q_2, q_3, p_1, p_2, p_3)$ be initial conditions such that for the equations of the first approximation (5) the solutions remain bounded and bounded away from singularities. Let $\phi(t) = (\phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t), \phi_5(t), \phi_6(t))$ be a solution of (4) satisfying $(\phi_1(0), \phi_2(0), \phi_3(0), \phi_4(0), \phi_5(0), \phi_6(0)) = (q_1, q_2, q_3, p_1, p_2, p_3)$. Then this solution is of the form*

$$\begin{aligned} \phi_1(t) &= \left(\varepsilon^3/(p_1+p_3)^3 - 1\right) t + q_1 + \varepsilon^2 g_1, \\ \phi_2(t) &= q_2\cos(t) + p_2\sin(t) + \varepsilon^2 g_2, \\ \phi_3(t) &= \left(\varepsilon^3/(p_1+p_3)^3\right) t + q_3 + \varepsilon^2 g_3, \qquad (6) \\ \phi_4(t) &= p_1 + \varepsilon^2 g_4, \\ \phi_5(t) &= -q_2\sin(t) + p_2\cos(t) + \varepsilon^2 g_5, \\ \phi_6(t) &= p_3 + \varepsilon^2 g_6, \end{aligned}$$

for $0 \le t \le \gamma\varepsilon^{-3}$, where γ is a constant independent of ε, and where the $g_i = g_i(t, q_1, q_2, q_3, p_1, p_2, p_3, \varepsilon)$ are uniformly bounded in $t \in [0, \gamma\varepsilon^{-3}]$ as ε approaches zero.

A solution $\phi(t)$ of Lemma 2 that in $t = 0$ hits $\mathcal{L}_1$ and in $t = T$ hits $\mathcal{L}_2$ (i.e. a doubly–periodic solution) must satisfy the conditions: in $t = 0$ given by $Q_1 = i\pi$, $Q_2 = 0$, $Q_3 = j\pi$; and in $t = T$ given by $Q_1 = (i+k)\pi$, $Q_2 = 0$, $Q_3 = (j+m+1/2)\pi$.

By Arenstorf´s Implicit Function Theorem [1] we conclude that for $\varepsilon \in \mathcal{W} = \{\varepsilon \in B \setminus \{0\} : \varepsilon^3 = 1/n$, for $n \in \mathbb{Z}^*\}$ and p_3 arbitrary we can find functions $T(p_1(\varepsilon), p_3, \varepsilon) = \tilde{T}(p_3, \varepsilon) \cong \left(m_0 + \frac{1}{2} + \frac{2m_0}{\varepsilon^3}\right)\pi$, $p_2(p_3, \varepsilon) \cong 0$, and $p_1(\varepsilon)$ such that $(p_1+p_3)^3 \cong 4m_0/(2m_0+1)$ and

$$\begin{aligned} \left(\frac{\varepsilon^3}{p_1(\varepsilon)+p_3} - 1\right)\tilde{T}(p_3, \varepsilon) - \frac{2m_0\pi}{\varepsilon^3} + \varepsilon^2 g_1 &= 0, \\ p_2(p_3, \varepsilon)\sin(\tilde{T}(p_3, \varepsilon)) + \varepsilon^2 g_2 &= 0, \\ \left(\frac{\varepsilon^3}{p_1(\varepsilon)+p_3}\right)\tilde{T}(p_3, \varepsilon) - \left(m_0 + \frac{1}{2}\right)\pi + \varepsilon^2 g_3 &= 0. \end{aligned}$$

Then for each $\varepsilon \in \mathcal{W}$, since for $\varepsilon^3 = 1/n$ we have $2m_0\pi/\varepsilon^3 = k\pi$ for k an integer, we obtain the necessary period and the initial conditions.

The results of this paper generalize the results of Howison and Meyer [2] from $N = 2$ to any $N \ge 2$.

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