Study of the equilibrium points of the restricted three-body problem with an oblate primary

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Introduction

We consider the restricted three–body problem with the more massive primary as an oblate spheroid. The primaries are moving in a keplerian circular orbit about their center of mass, the equatorial plane of the oblate primary coinciding with the plane of motion of the binaries.

We study the dynamics of a third particle of infinitesimal mass in space under the gravitational attraction of the binary, looking at the existence and stability of the equilibrium points.

The planar problem has been studied by several authors. They have confirmed the existence of five equilibrium points, three of them being collinear and two in a triangular configuration, as in the problem without oblateness. The collinear ones are unstable in the Liapunov sense; in the interval of linear stability of the triangular ones it has been shown stability except for two values of the mass parameter and the critical mass [4], [6], [7]. These results follow from Arnold Theorem [1].

In the spatial case, a three-degree of freedom system, Arnold Theorem does not apply but we can still try to establish stability results in a weaker formulation such as formal stability and stability of finite type. Our idea is to use normal forms techniques and the theory developed in [3]. We have expanded the potential in power series up to fourth order in the oblateness parameter, the eccentricity of the spheroid.

This spatial problem has been studied with an approximation of the potential to second order in the oblateness parameter [4], [5]. Recently, Markellos and Douskos, found two new equilibrium points outside the equatorial plane, nearly above and below the oblate primary [2]. We comment about this point here.

1. Equations of motion

Considering a rotating coordinate system (ξ, η, ζ), with origin at the center of mass of the primaries m_1 and m_2, angular velocity ω and $\mu = m_2/(m_1+m_2)$, the equations of motion for the third particle m_3 are,

$$\begin{aligned}\xi'' - 2\omega\eta' &= \frac{\partial \Omega_0}{\partial \xi} + \frac{\partial W_\epsilon}{\partial \xi},\\ \eta'' + 2\omega\xi' &= \frac{\partial \Omega_0}{\partial \eta} + \frac{\partial W_\epsilon}{\partial \eta},\\ \zeta'' &= \frac{\partial \Omega_0}{\partial \zeta} + \frac{\partial W_\epsilon}{\partial \zeta},\end{aligned} \tag{1}$$

where,

$$\Omega_0 = \frac{\omega^2}{2}\left(\xi^2+\eta^2\right) + W_0, \qquad W_0 = \frac{1-\mu}{r_1} + \frac{\mu}{r_2}, \tag{2}$$

$$r_1 = \sqrt{(\xi+\mu)^2+\eta^2+\zeta^2}, \quad r_2 = \sqrt{(\xi+\mu-1)^2+\eta^2+\zeta^2}, \tag{3}$$

and W_ϵ is the disturbed potential due to the action of the m_1 oblateness.

The angular velocity ω is affected by the oblateness, more precisely,

$$\omega = \sqrt{1+3\alpha\epsilon^2}, \tag{4}$$

where ϵ is the eccentricity of the oblate primary and α is defined so that $\alpha\epsilon^2 = (R_E^2 - R_P^2)/10R^2$, R_E is the equatorial radius, R_P is the polar radius, and R is the distance between the primaries.

Differentiating equations (2) and (3) equations (1) become,

$$\begin{aligned}\xi'' - 2\omega\eta' &= \xi\left(\omega^2 + \frac{1}{r_1}\frac{\partial W_0}{\partial r_1} + \frac{1}{r_2}\frac{\partial W_0}{\partial r_2}\right) + \\ &\quad \frac{\mu}{r_1}\frac{\partial W_0}{\partial r_1} + \frac{\mu-1}{r_2}\frac{\partial W_0}{\partial r_2} + \frac{\partial W_\epsilon}{\partial \xi},\\ \eta'' + 2\omega\xi' &= \eta\left(\omega^2 + \frac{1}{r_1}\frac{\partial W_0}{\partial r_1} + \frac{1}{r_2}\frac{\partial W_0}{\partial r_2}\right) + \frac{\partial W_\epsilon}{\partial \eta},\\ \zeta'' &= \zeta\left(\frac{1}{r_1}\frac{\partial W_0}{\partial r_1} + \frac{1}{r_2}\frac{\partial W_0}{\partial r_2}\right) + \frac{\partial W_\epsilon}{\partial \zeta}.\end{aligned} \tag{5}$$

The disturbed part of the potential W_ϵ is due to the oblateness effect of m_1 on m_3 and is obtained from the potential function,

$$V(u,\epsilon) = k^2 \int \frac{dm}{D}, \tag{6}$$

where k^2 is the gravitational constant, dm is the element of mass of the oblateness body S with total mass m_1, and D is the distance from dm to the body of mass m_3.

The development of V in powers of ϵ has only even terms, that is,

$$V(u,\epsilon) = V_0(u) + V_1(u)\epsilon^2 + V_2(u)\epsilon^4 + \ldots, \tag{7}$$

where,

$$V_0(u) = \frac{k^2 m_1}{D_1},$$

is the potential due to the action of m_1 as a spherical body with radius equal to the major semi-axis a of the ellipsoid, and D_1 is the distance from the center of mass of m_1 to m_3.

Here we study the problem up to order four in ϵ and so we get the representation of the binomial series of ω in (4),

$$\omega = 1 + \frac{3\alpha}{2}\epsilon^2 - \frac{9\alpha^2}{8}\epsilon^4 + O(\epsilon^6), \tag{8}$$

and we consider,

$$W_\epsilon(u,\mu) = V_1(u,\mu)\epsilon^2 + V_2(u,\mu)\epsilon^4 + O(\epsilon^6), \tag{9}$$

which is obtained from binomial expansion of $1/D$ in (6), up to order four in r followed by the expansion of $r = b/\sqrt{1-\epsilon^2\cos^2\phi}$ up to order six in ϵ, that is,

$$\begin{aligned}V_1(u,\mu) &= \frac{k^2 b^5 \pi\sigma}{r_1^3}\left(-\frac{4}{15} + \frac{2\left((\xi+\mu)^2+\eta^2\right)}{5r_1^2}\right)\\ V_2(u,\mu) &= \frac{k^2 b^5 \pi\sigma}{r_1^3}\left[-\frac{8}{15} + \frac{4\left((\xi+\mu)^2+\eta^2\right)}{5r_1^2} + b^2\left(\frac{4}{35r_1^2} - \right.\right.\\ &\quad \left.\left.\frac{4\left((\xi+\mu)^2+\eta^2\right)}{7r_1^4} + \frac{\left((\xi+\mu)^2+\eta^2\right)^2}{2r_1^6}\right)\right].\end{aligned} \tag{10}$$

We note that terms with r_1^{-n} come from terms $y^{(n-1)/2}$ in the binomial expansion of $1/D$. Moreover, each term of order ϵ^k contains all powers of r_1^{-n}, $n = 3,5,7,...,2k+1$, in such a way that terms of order y^l, with $l > k$ does not contribute to the term of order ϵ^k.

Thus, with these considerations, from (8) and (9), the equations of motion (5) become,

$$\begin{aligned}\xi'' - 2\eta' &= \xi\left(1 + \frac{1}{r_1}\frac{\partial W_0}{\partial r_1} + \frac{1}{r_2}\frac{\partial W_0}{\partial r_2}\right) + \frac{\mu}{r_1}\frac{\partial W_0}{\partial r_1} + \frac{\mu-1}{r_2}\frac{\partial W_0}{\partial r_2} + \\ &\quad \left(3\alpha\eta' + 3\alpha\xi + \frac{\partial V_1(u,\mu)}{\partial \xi}\right)\epsilon^2 + \\ &\quad \left(-\frac{9\alpha^2}{4}\eta' + \frac{\partial V_2(u,\mu)}{\partial \xi}\right)\epsilon^4 + O(\epsilon^6),\\ \eta'' + 2\xi' &= \eta\left(1 + \frac{1}{r_1}\frac{\partial W_0}{\partial r_1} + \frac{1}{r_2}\frac{\partial W_0}{\partial r_2}\right) + \\ &\quad \left(-3\alpha\xi' + 3\alpha\eta + \frac{\partial V_1(u,\mu)}{\partial \eta}\right)\epsilon^2 + \\ &\quad \left(\frac{9\alpha^2}{4}\xi' + \frac{\partial V_2(u,\mu)}{\partial \eta}\right)\epsilon^4 + O(\epsilon^6),\\ \zeta'' &= \zeta\left(\frac{1}{r_1}\frac{\partial W_0}{\partial r_1} + \frac{1}{r_2}\frac{\partial W_0}{\partial r_2}\right) + \\ &\quad \frac{\partial V_1(u,\mu)}{\partial \zeta}\epsilon^2 + \frac{\partial V_2(u,\mu)}{\partial \zeta}\epsilon^4 + O(\epsilon^6).\end{aligned} \tag{11}$$

2. Equilibrium points

The equilibrium points satisfy $\xi' = \eta' = \zeta' = 0$, and consequently, looking at system (11) where V_1 and V_2 are given by (10), the equilibrium points satisfy,

$$\begin{aligned}&\left(\xi - \frac{\mu(-1+\mu+\xi)}{r_2^3} - \frac{(1-\mu)(\mu+\xi)}{r_1^3}\right) + \\ &\left(3\alpha\xi - \frac{2\beta(\mu+\xi)\left(-4\zeta^2+\eta^2+(\mu+\xi)^2\right)}{5r_1^7}\right)e^2 - \\ &\beta(\mu+\xi)\left(\frac{4\left(-4\zeta^2+\eta^2+(\mu+\xi)^2\right)}{5r_1^7} + \right.\\ &\left.\frac{3b^2\left(8\zeta^4 - 12\zeta^2\left(\eta^2+(\mu+\xi)^2\right) + \left(\eta^2+(\mu+\xi)^2\right)^2\right)}{14r_1^{11}}\right)e^4 = 0\end{aligned}$$

$$\begin{aligned}&\eta\left[1 - \frac{\mu}{r_2^3} - \frac{(1-\mu)}{r_1^3} + \left(3\alpha - \frac{2\beta\left(-4\zeta^2+\eta^2+(\mu+\xi)^2\right)}{5r_1^7}\right)e^2 - \right.\\ &\beta\left(\frac{4\left(-4\zeta^2+\eta^2+(\mu+\xi)^2\right)}{5r_1^7} + \right.\\ &\left.\left.\frac{3b^2\left(8\zeta^4 - 12\zeta^2\left(\eta^2+(\mu+\xi)^2\right) + \left(\eta^2+(\mu+\xi)^2\right)^2\right)}{14r_1^{11}}\right)e^4\right] = 0\end{aligned}$$

$$\begin{aligned}&\zeta\left[\left(-\frac{\mu}{r_2^3} - \frac{(1-\mu)}{r_1^3}\right) + \frac{2\beta\left(2\zeta^2 - 3\left(\eta^2+(\mu+\xi)^2\right)\right)}{5r_1^7}e^2 + \right.\\ &\beta\left(\frac{4\left(2\zeta^2 - 3\left(\eta^2+(\mu+\xi)^2\right)\right)}{5r_1^7} - \right.\\ &\left.\left.\frac{b^2\left(8\zeta^4 - 40\zeta^2\left(\eta^2+(\mu+\xi)^2\right) + 15\left(\eta^2+(\mu+\xi)^2\right)^2\right)}{14r_1^{11}}\right)e^4\right] = 0\end{aligned}$$

where, $\beta = b^5k^2\pi\sigma$, $b = a\sqrt{1-e^2}$ and a is the major semi-axis of the ellipsoid.

3. Results

- We prove that the potential function (7) of the oblate body S satisfies the equality,

  $$V_z(u,\epsilon) = z\left(-\frac{k^2M}{r_1^2} + h(u,\epsilon)\epsilon^2\right),$$

  where h is an analytic function of (u,ϵ) and $u = (x,y,z)$ is the coordinate of a point relative to the center of the oblate body S.

  Thus, for ϵ sufficiently small in the region $\Omega : a < |u| \leq R$, $zV_z < 0$ and so the vertical components of the resulting forces on a particle inside Ω is downward in the upper half–space and upward in the lower one.

  Therefore, there is no equilibrium points outside the equatorial plane in the region Ω because a particle in such a position with a horizontal velocity equal to that of the rotating system would have a vertical displacement in the direction of the equatorial plane.

- Regarding the previous considerations we recall that [2] found the existence of out of plane equilibrium points nearly above and below the oblate primary using the development of the potential up to order two in ϵ. We are analyzing this question considering the development up to order four in ϵ.

- There exists three collinear equilibrium points $L_i = (\xi_i, 0, 0), i = 1,2,3$,

  $$L_3 \in (-\infty, -\mu), \quad L_1 \in (-\mu, 1-\mu), \quad L_2 \in (1-\mu, \infty)$$

  Each coordinate $\xi_i = \xi_i(\epsilon)$ satisfies a polynomial equation that have only one root (in the respective interval) just as in the collinear equilibrium of the undisturbed case.

- There exist two triangular equilibrium points in the plane of motion $L_{4,5}$ that satisfy

  $$r_1 \approx 1, \quad r_2 \approx 1, \quad \xi_0 \approx 1/2, \quad \eta_0 \approx \pm\sqrt{3}/2$$

  near the triangular equilibrium of the undisturbed case.

- By continuity, we can extend the following result in [3] about stability in the spatial circular restricted three body problem.

  The triangular libration points are stable for the majority of initial conditions (in the sense of Lebesgue measure), for all μ in the region of stability in the first approximation (excluding μ_1 and μ_2).

  In fact, the fourth order determinant D_4 in [3] is nonzero when $\epsilon = 0$ in the region of stability except for two values of the parameter μ.

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