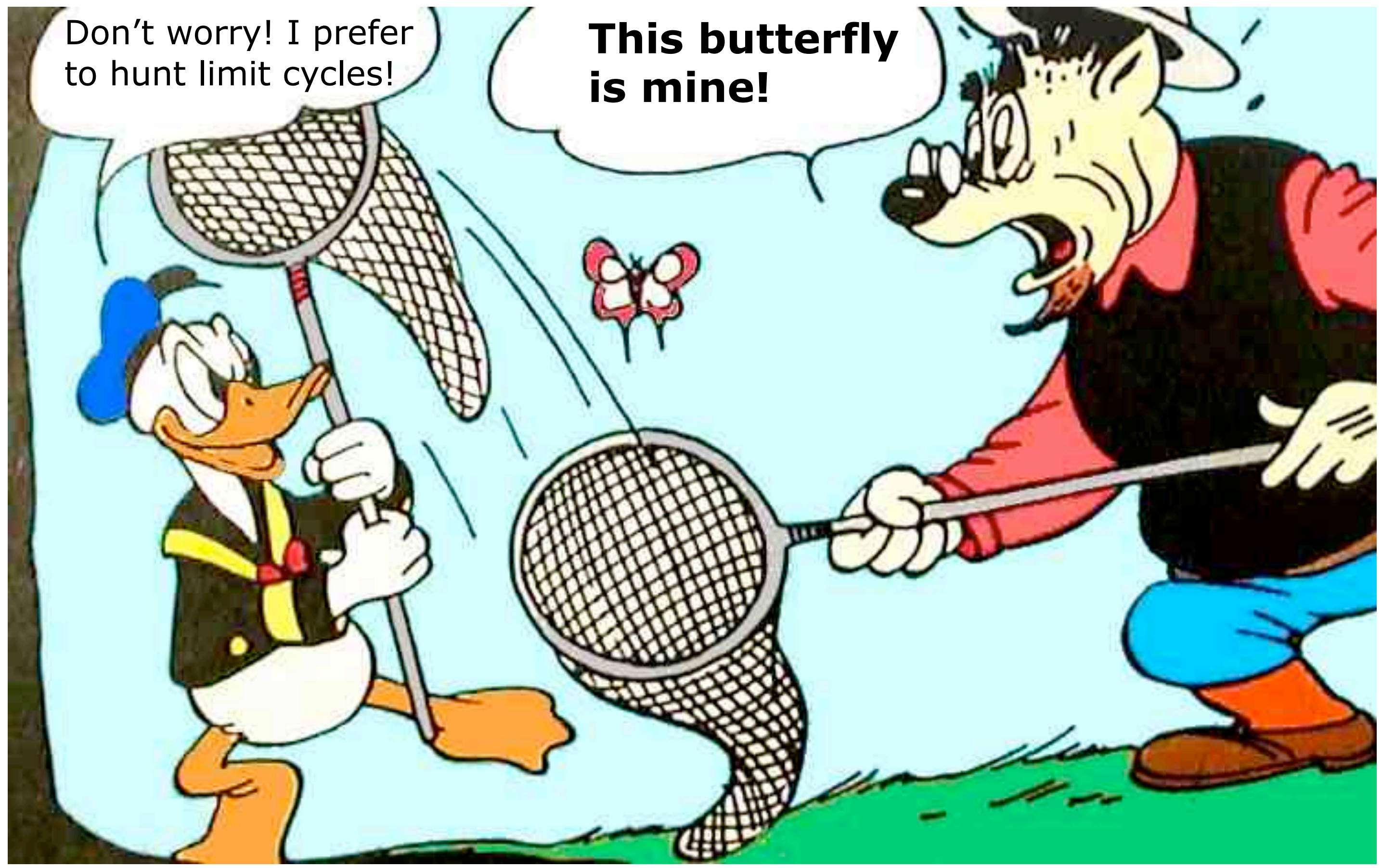
New Trends in Dynamical Systems

(On the occasion of Jaume Llibre's 60th birthday)



Hunting three limit cycles with only two linear foci

E. Freire, E. P. & F. Torres (U. Sevilla)

Salou, October 1st-5th, 2012

Summary

- We consider the family of planar discontinuous piecewise-linear systems with two linearity zones separated by a straight line.
- An example in this family has been recently reported to have three nested limit cycles, so breaking a natural conjecture on being two the maximum number of limit cycles.
- We exploit a Liénard-like canonical form that contains seven parameters, one of them characterizing the sliding set, which allows to pave the way for the systematic study of planar discontinuous piecewise-linear systems.
- The relevant case for the study is the focus-focus dynamics, where a reduced canonical form with only five parameters is useful.
- We look for the parameter regions where can be proved the existence of three limit cycles surrounding the whole sliding set.

The Huan-Yang example

The planar non-smooth piecewise linear differential system with two zones separated by a straight line corresponding to Example 5.1 of Huan and Yang is

$$\dot{\mathbf{x}} = \begin{cases} A^-\mathbf{x} & \text{if } x < 1, \\ A^+\mathbf{x} & \text{if } x \geq 1, \end{cases}$$

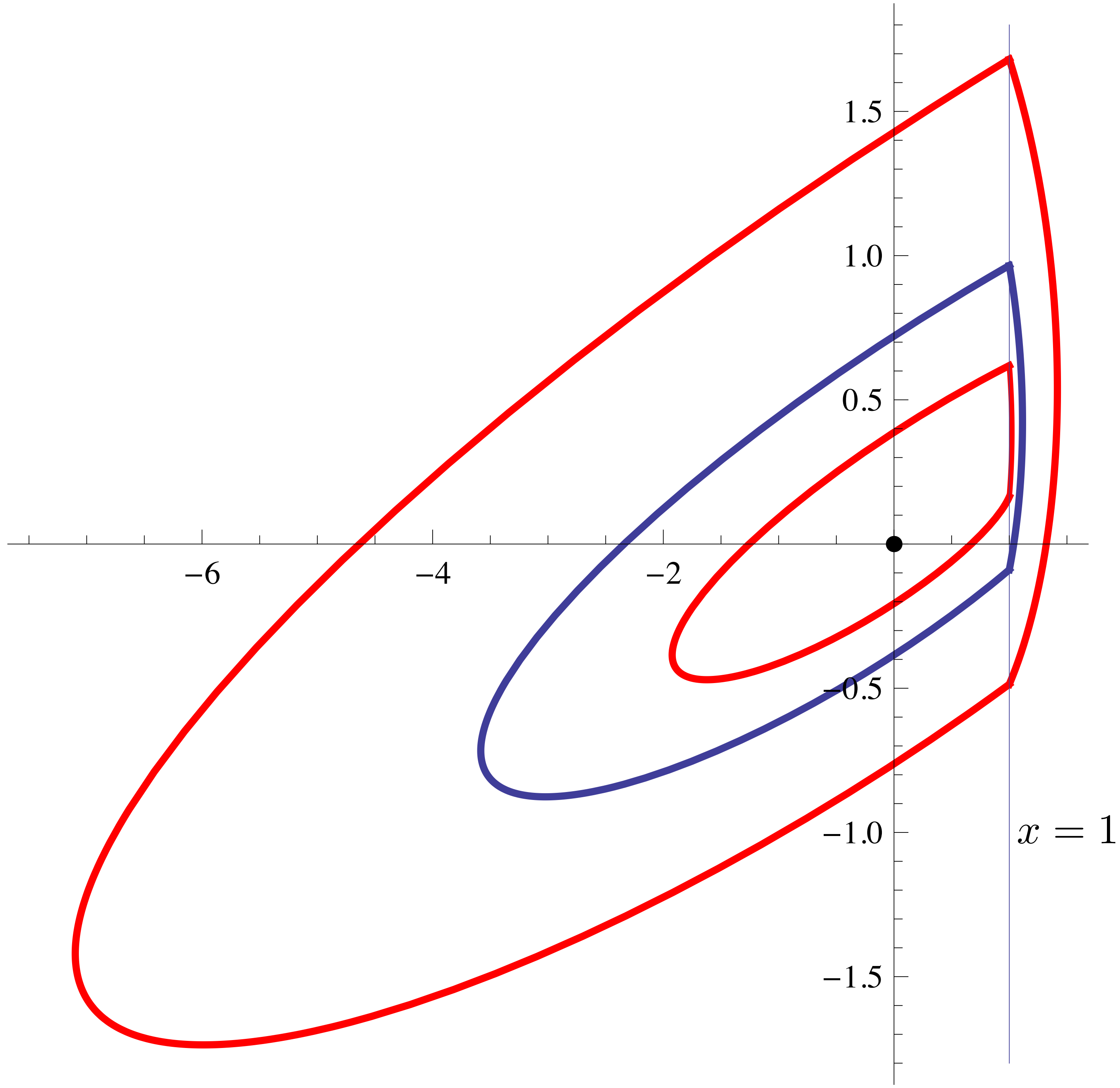
where $\mathbf{x} = (x, y)^T$ with

$$A^- = \begin{pmatrix} 1 & -5 \\ \dfrac{377}{1000} & -\dfrac{13}{10} \end{pmatrix}, \quad \text{and} \quad A^+ = \begin{pmatrix} \dfrac{19}{500} & -\dfrac{1}{10} \\ \dfrac{1}{10} & \dfrac{19}{500} \end{pmatrix}.$$

Theorem (J. Llibre & E.P.) The above planar non–smooth piecewise linear differential system with two zones has 3 limit cycles surrounding its unique equilibrium point located at the origin.

S.-M. Huan and X.-S. Yang, *On the number of limit cycles in general planar piecewise linear systems*, Discrete and Continuous Dynamical Systems-A **32** (2012) pp. 2147–2164.

J. Llibre and E. P., *Three Nested Limit Cycles In Discontinuous Piecewise Linear Differential Systems With Two Zones*, Dynamics of Continuous, Discrete and Impulsive Systems-B **19** (2012) pp. 325–335.



$$\boxed{\lambda = -\frac{1}{5} \pm i}$$

$$\boxed{\lambda = \frac{19}{50} \pm i}$$

$$A^- = \begin{pmatrix} \frac{4}{3} & -\frac{20}{3} \\ \frac{377}{750} & -\frac{26}{15} \end{pmatrix}, \quad \text{and} \quad A^+ = \begin{pmatrix} \frac{19}{50} & -1 \\ 1 & \frac{19}{50} \end{pmatrix}.$$

(after rescaling time, differently in each side)

The three limit cycles as a result of a boundary equilibrium bifurcation

It should be noticed that both linear vector fields, being homogeneous, share the origin as their common equilibrium point.

Thus, by considering the one-parameter family

$$\dot{\mathbf{x}} = \begin{cases} A^{-}\mathbf{x} & \text{if } x < \varepsilon, \\ A^{+}\mathbf{x} & \text{if } x \geq \varepsilon, \end{cases}$$

the following result can be stated.

Theorem (D.C. Braga & L.F. Mello) The above one-parameter family of piecewise linear systems with two zones has
(a) One unstable focus at the origin and no limit cycle when $\varepsilon < 0$.
(b) One unstable focus at the origin and no limit cycle when $\varepsilon = 0$.
(c) One stable focus at the origin and three limit cycles surrounding the origin for each $\varepsilon > 0$. One limit cycle is stable and the other two are unstable.

D. C. Braga and L. F. Mello, *Limit cycles in a family of discontinuous piecewise linear differential systems with two zones in the plane*, Preprint (2012).

The three limit cycles can be obtained also as in...

C.A. Buzzi, C. Pessoa and J. Torregrosa,
Piecewise Linear Perturbations of a linear center,
Preprint, 2012.

E. Freire, E.P., J. Torregrosa and F. Torres,
The Hopf bifurcation at infinity in discontinuous planar piecewise linear systems, In preparation.

Hunting the limit cycles one-by-one

The system can be written in the focus-focus canonical form proposed in Freire et al. (2012) for the case of one equilibrium in the left side.

Example 5.1 of Huan & Yang is equivalent to

$$\dot{\mathbf{x}} = \begin{cases} A^-\mathbf{x} + B^- & \text{if } x < 0, \\ A^+\mathbf{x} + B^+ & \text{if } x \geq 0, \end{cases}$$

with

$$A^- = \begin{pmatrix} -\frac{2}{5} & -1 \\ \frac{26}{25} & 0 \end{pmatrix}, B^- = \begin{pmatrix} 0 \\ \frac{26}{25} \end{pmatrix}, A^+ = \begin{pmatrix} \frac{19}{25} & -1 \\ \frac{2861}{2500} & 0 \end{pmatrix}, B^+ = \begin{pmatrix} \frac{6}{5} \\ \frac{2861}{375} \end{pmatrix}.$$

We will look for bifurcations leading to one limit cycle in the general family of piecewise linear systems with two zones.

E. Freire, E.P. and F. Torres, *Canonical Discontinuous Planar Piecewise Linear Systems*, SIAM J. Applied Dynamical Systems **11** (2012) 181–211.

Planar PWL Filippov Systems (Utkin 1992, Kuznetsov et al. 2003)

- We consider one discontinuity boundary defined by

$$\Sigma = \{(x, y) \in \mathbb{R}^2 : x = 0\}$$

- The boundary induces the partition of the phase plane into

$$S^- = \{(x, y) \in \mathbb{R}^2 : x < 0\},$$

$$S^+ = \{(x, y) \in \mathbb{R}^2 : x > 0\}.$$

The systems to be studied become

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \begin{cases} \mathbf{F}^+(\mathbf{x}) = \left(F_1^+(\mathbf{x}), F_2^+(\mathbf{x})\right)^T = A^+\mathbf{x} + \mathbf{b}^+, & \text{if } \mathbf{x} \in S^+, \\ \mathbf{F}^-(\mathbf{x}) = \left(F_1^-(\mathbf{x}), F_2^-(\mathbf{x})\right)^T = A^-\mathbf{x} + \mathbf{b}^-, & \text{if } \mathbf{x} \in S^-. \end{cases}$$

Planar PWL Filippov Systems (cont'd)

- As usual we define the Σ-subsets

$$\Sigma^c = \{(0,y) : F_1^+(0,y)F_1^-(0,y) > 0\} \quad \text{(crossing set)}$$

$$\Sigma^s = \{(0,y) : F_1^+(0,y)F_1^-(0,y) \leqslant 0\} \quad \text{(sliding set)}$$

- We also compute the Filippov vector field

$$\dot{x} = 0, \quad \dot{y} = g(y) = \frac{F_1^+(\mathbf{x})F_2^-(\mathbf{x}) - F_1^-(\mathbf{x})F_2^+(\mathbf{x})}{F_1^+(\mathbf{x}) - F_1^-(\mathbf{x})}, \quad \mathbf{x} \in \Sigma^s.$$

- Some other standard definitions follow:

Points $(0,\bar{y}) \in \Sigma^s$ with $g(\bar{y}) = 0$ act in some sense as equilibria of our system and they are called *pseudo-equilibria*.

A double invisible tangency point with close orbits spiraling around it, is called a *pseudo-focus* or *fused focus*.

A pseudo-equilibrium in the attractive part of the sliding set with $g'(y) < 0$ is a stable *pseudo-node*, being a *pseudo-saddle* if $g'(y) > 0$.

Similarly, a pseudo-equilibrium in the repulsive part with $g'(y) > 0)$ is an unstable *pseudo-node*, being again a *pseudo-saddle* if $g'(y) < 0$.

Note that at pseudo-equilibria $(0,\bar{y})$ which are neither boundary equilibrium nor tangency points we have

$$\frac{F_2^-(0,\bar{y})}{F_1^-(0,\bar{y})} = \frac{F_2^+(0,\bar{y})}{F_1^+(0,\bar{y})},$$

and so the two vector fields $\mathbf{F}^+$ and $\mathbf{F}^-$ are anticollinear.

So many parameters only allow particular studies...

$$\Sigma = \{(x,y) \in \mathbb{R}^2 : x = 0\}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_{11}^- x + a_{12}^- y + b_1^- \\ a_{21}^- x + a_{22}^- y + b_2^- \end{pmatrix} \quad \Bigg| \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_{11}^+ x + a_{12}^+ y + b_1^+ \\ a_{21}^+ x + a_{22}^+ y + b_2^+ \end{pmatrix}$$

F. Giannakopoulos and K. Pliete, *Closed trajectories in planar relay feedback systems*, Dynamical Systems, **17** (2002), 343–358

J. Llibre, E. P. and F. Torres, *On the existence and uniqueness of limit cycles in Liénard differential equations allowing discontinuities*, Nonlinearity **21** (2008), 2121–2142.

M. Han, W. Zhang, *On Hopf Bifurcation in non-smooth Planar Systems*, J. Differential Equations **248** (2010), 2399–2416.

Tangencies and sliding set

$$\Sigma = \{(x,y) \in \mathbb{R}^2 : x = 0\}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_{11}^- x + a_{12}^- y + b_1^- \\ a_{21}^- x + a_{22}^- y + b_2^- \end{pmatrix} \quad \Bigg| \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_{11}^+ x + a_{12}^+ y + b_1^+ \\ a_{21}^+ x + a_{22}^+ y + b_2^+ \end{pmatrix}$$

We will assume $a_{12}^-, a_{12}^+ \neq 0$ to avoid 'wall' cases.

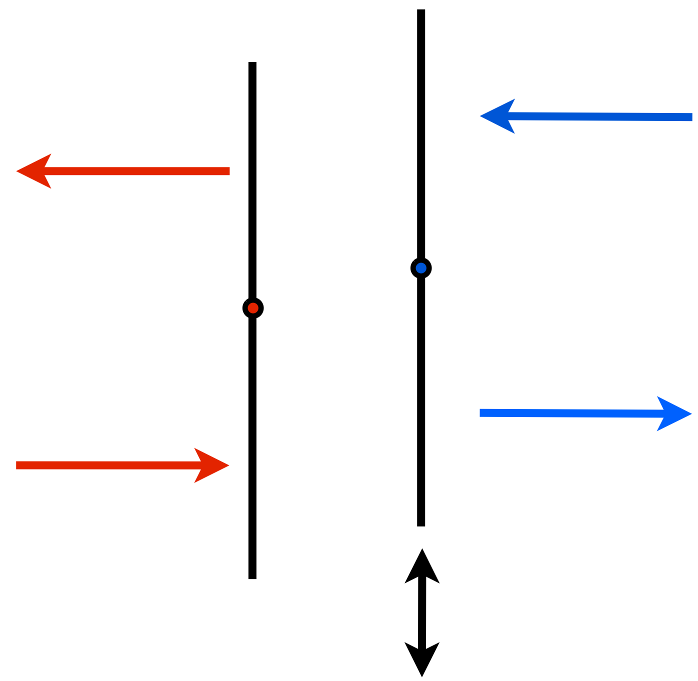
We have a tangency point in Σ when $\dot{x}|_{x=0} = a_{12} y + b_1$ vanishes.

At tangency points, we speak of visible (invisible) tangency depending on the sign of $\ddot{x}$. Since $\ddot{x}|_{x=0} = a_{11}(a_{12} y + b_1) + a_{12}(a_{21} y + b_2)$, we obtain

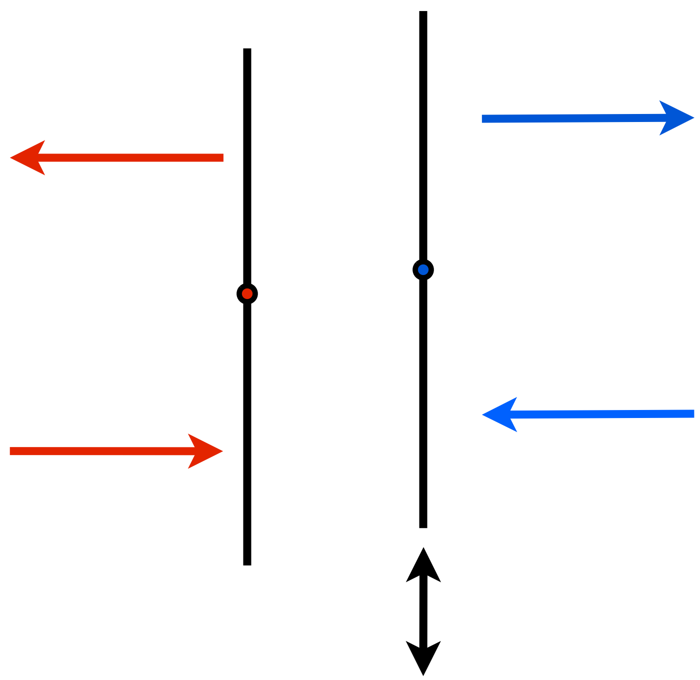
$$\ddot{x}|_{\dot{x}=0} = a_{12} b_2 - a_{21} b_1$$

Tangencies and sliding set (cont'd)

Assuming $a_{12}^{-} < 0$, there are two possibilities for a_{12}^{+}:



$(a_{12}^{+} < 0$: bounded sliding$)$



$(a_{12}^{+} > 0$: bounded crossing$)$

For a non-smooth system... a non-smooth change!

We do a continuous piecewise linear change of variables $\mathbf{u} = f(\mathbf{x})$, where

$$\mathbf{u} = -a_{12}^{+} \begin{pmatrix} x \\ a_{22}^{-}x - a_{12}^{-}y \end{pmatrix} + a_{12}^{+} \begin{pmatrix} 0 \\ b_1^{-} \end{pmatrix}, \quad x < 0,$$

and

$$\mathbf{u} = -a_{12}^{-} \begin{pmatrix} x \\ a_{22}^{+}x - a_{12}^{+}y \end{pmatrix} + a_{12}^{+} \begin{pmatrix} 0 \\ b_1^{-} \end{pmatrix}, \quad x > 0,$$

and afterwards rename the variable $\mathbf{u}$ to $\mathbf{x}$.

This change is a global homeomorfism that conjugates the vector field in each halfplane, separately. Such a conjugacy cannot be extended to the sliding vector field (but it works for our purpose)

M. Guardia, T.M. Seara, and M. A. Teixeira, *Generic bifurcations of low codimension of planar Filippov Systems*, Journal of Differential Equations **250** (2011) 1967–2023.

The discontinuous canonical form

Liénard canonical form for DPWL systems. Assume that $a_{12}^{+}a_{12}^{-} > 0$ (bounded sliding set). Then the system can be written in the form,

$$\dot{\mathbf{x}} = \begin{pmatrix} T^{-} & -1 \\ D^{-} & 0 \end{pmatrix}\mathbf{x} - \begin{pmatrix} 0 \\ a^{-} \end{pmatrix} \text{ if } \mathbf{x} \in S^{-},$$

$$\dot{\mathbf{x}} = \begin{pmatrix} T^{+} & -1 \\ D^{+} & 0 \end{pmatrix}\mathbf{x} - \begin{pmatrix} -b \\ a^{+} \end{pmatrix} \text{ if } \mathbf{x} \in S^{+},$$

where T,D stand for trace and determinant, and

$$a^{-} = a_{12}^{+}(a_{12}^{-}b_{2}^{-} - a_{22}^{-}b_{1}^{-}), \quad a^{+} = a_{12}^{-}(a_{22}^{+}b_{1}^{+} - a_{12}^{+}b_{2}^{+}), \quad b = a_{12}^{+}b_{1}^{-} - a_{12}^{-}b_{1}^{+}.$$

This system has as its tangency points (0,0) and (0,b).

Apart from the linear invariants, the other three parameters are associated to the x-coordinates of the equilibrium points (a^{+} and a^{-}) and the size and stability of the sliding set (b).

False Filippov systems

- When $a^{+}=a^{-}$ and $b=0$, we get a continuous piecewise linear system even if the original system was discontinuous.

- In particular, homogeneous systems with bounded sliding set and $\mathbf{b}^{+} = \mathbf{b}^{-} = 0$, can always be transformed in a continuous system. Thus the class of bimodal systems considered in

Y. Zou, T. Kupper and W. J. Beyn, *Generalized Hopf Bifurcations for Planar Filippov Systems Continuous at the origin,* J. Nonlinear Science **16** (2006), 159–177.

Y. Iwatani and S. Hara, *Stability Analysis and Stabilization for Bimodal Piecewise Linear Systems Based on Eigenvalue Loci* Mathematical Engineering Technical Reports Web page http://www.i.u.-tokyo.ac.jp/mi-e.htm.

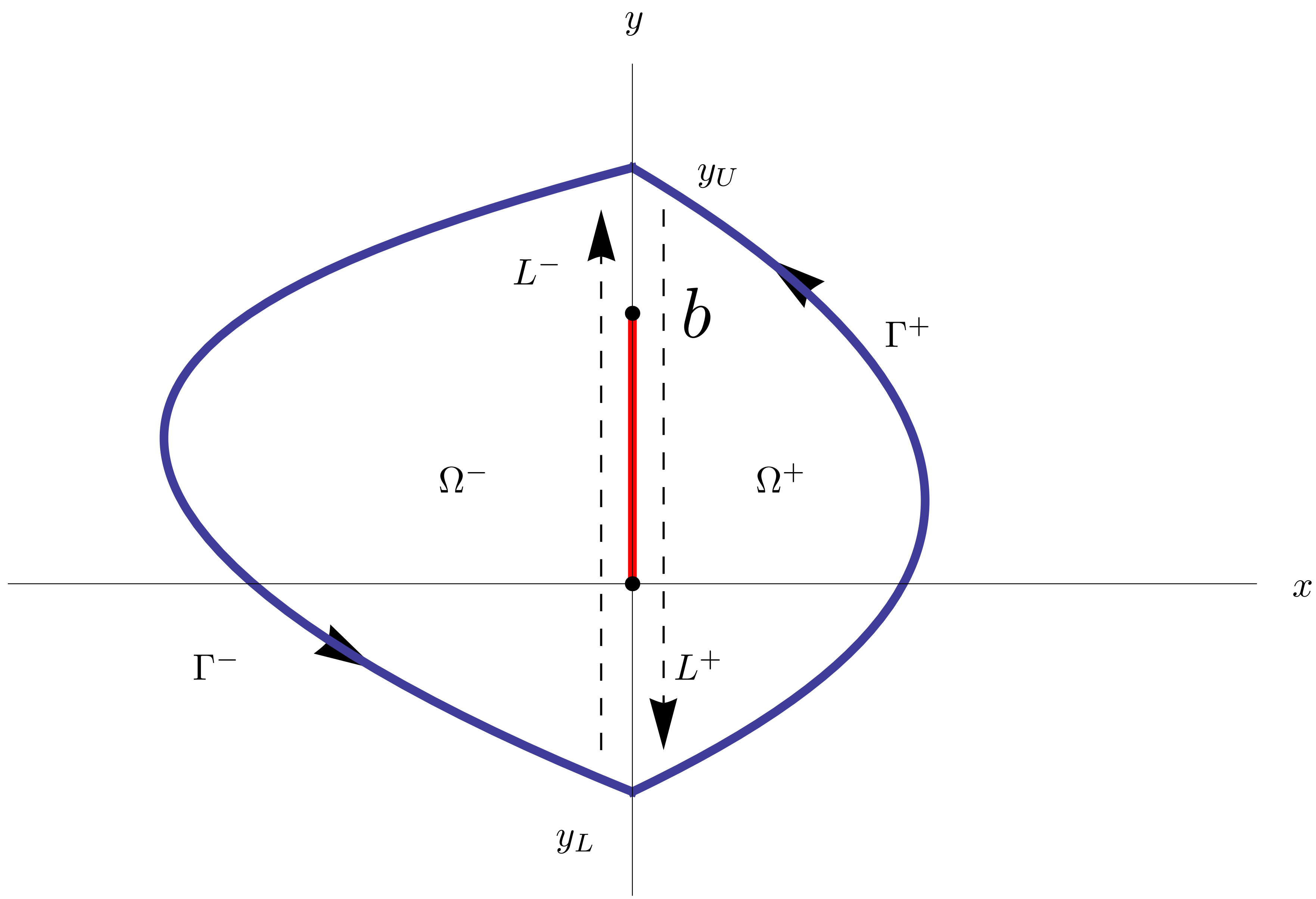
could be analyzed just by using the results in

E. Freire, E. P., F. Rodrigo and F.Torres, *Bifurcation Sets of Continuous Piecewise Linear Systems with Two Zones,* Int. J. Bifurcation and Chaos **8** (1998), 2073–2097.

A necessary condition for crossing periodic orbits

Proposition Defining the values $\sigma^- = \text{area }(\Omega^-)$, $\sigma^+ = \text{area }(\Omega^+)$ and $h = y_U - y_L$, then we have

$$T^-\sigma^- + T^+\sigma^+ + bh = 0.$$



The canonical form in the focus-focus case

Assume $T^{\pm} = 2\alpha^{\pm}$, $D^{\pm} = (\alpha^{\pm})^2 + (\omega^{\pm})^2$ with $\omega^{\pm} > 0$ in the canonical form, so that the corresponding eigenvalues are $\lambda^{\pm} = \alpha^{\pm} \pm i\omega^{\pm}$, and introduce the parameters

$$\gamma_R = \frac{\alpha^+}{\omega^+}, \quad \gamma_L = \frac{\alpha^-}{\omega^-}, \quad a_R = \frac{a^+}{\omega^+}, \quad a_L = \frac{a^-}{\omega^-}.$$

Then the previous canonical form can be written in the form

$$\dot{\mathbf{x}} = \begin{pmatrix} 2\gamma_L & -1 \\ 1+\gamma_L^2 & 0 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 0 \\ a_L \end{pmatrix} \text{ if } \mathbf{x} \in S^-,$$

$$\dot{\mathbf{x}} = \begin{pmatrix} 2\gamma_R & -1 \\ 1+\gamma_R^2 & 0 \end{pmatrix} \mathbf{x} - \begin{pmatrix} -b \\ a_R \end{pmatrix} \text{ if } \mathbf{x} \in S^+.$$

It suffices to do a new non-smooth change of variables

$$(x, y, t) \to \left(\frac{x}{\omega(x)}, y, \frac{t}{\omega(x)}, \right), \text{ where } \omega(x) = \begin{cases} \omega^- \text{ if } x < 0, \\ \omega^+ \text{ if } x > 0. \end{cases}$$

A first step: results for the case without inner equilibria ($a_R \leq 0 \leq a_L$)

Stability of the origin in systems without sliding set Assuming $b = 0$ and $a_R \leqslant 0 \leqslant a_L$, the following statements hold.

(a) The origin is asymptotically stable for $a_L\gamma_R < a_R\gamma_L$ and unstable for $a_L\gamma_R > a_R\gamma_L$.

(b) If $a_L\gamma_R = a_R\gamma_L$, then the origin is unstable for $\gamma_R+\gamma_L > 0$, asymptotically stable for $\gamma_R + \gamma_L < 0$, and a global center for $\gamma_R + \gamma_L = 0$.

Results without inner equilibria ($a_R \leq 0 \leq a_L$)

(Crossing periodic orbits without sliding set) Assuming the conditions $b = 0$ and $a_R \leqslant 0 \leqslant a_L$, the following statements hold.

(a) If $\gamma_R + \gamma_L = 0$, then there is a global nonlinear center around the origin for $a_L\gamma_R = a_R\gamma_L$, and no crossing periodic orbits when $a_L\gamma_R \neq a_R\gamma_L$.

(b) If $\gamma_R + \gamma_L \neq 0$ and $\gamma_R\gamma_L \geqslant 0$, then there are no crossing periodic orbits.

(c) If $\gamma_R + \gamma_L \neq 0$ and $\gamma_R\gamma_L < 0$, then for $(\gamma_R + \gamma_L)(a_L\gamma_R - a_R\gamma_L) < 0$, there is only one crossing periodic orbit which is stable for $\gamma_R + \gamma_L < 0$ and unstable for $\gamma_R + \gamma_L > 0$. When $(\gamma_R + \gamma_L)(a_L\gamma_R - a_R\gamma_L) \geqslant 0$ there are no crossing periodic orbits.

(Global asymptotic stability of the origin) Under the conditions $b = 0$ and $a_R \leqslant 0 \leqslant a_L$, the origin is globally asymptotically stable in the two following cases: (i) when $a_L\gamma_R < a_R\gamma_L$ and $\gamma_R + \gamma_L \leqslant 0$; (ii) when $a_L\gamma_R = a_R\gamma_L$ and $\gamma_R + \gamma_L < 0$.

Results without inner equilibria ($a_R \leq 0 \leq a_L$)

Theorem (Systems with escaping sliding set, $b > 0$) Assuming the conditions $a_R \leqslant 0 \leqslant a_L, b > 0$, the following statements hold.

(a) If $\gamma_R\gamma_L \geqslant 0$ then there are no crossing periodic orbits for $\gamma_R + \gamma_L \geqslant 0$, while for $\gamma_R + \gamma_L < 0$ there is only one crossing periodic orbit which is stable.

(b) If $\gamma_R\gamma_L < 0$, then the following cases arise.

(b1) If $\gamma_R + \gamma_L \geqslant 0$ and $a_L\gamma_R \geqslant a_R\gamma_L$, then there are no crossing periodic orbits.

(b2) If $\gamma_R + \gamma_L = 0$, $a_L\gamma_R < a_R\gamma_L$ and we define the value

$$b_\infty = 2(a_L + a_R)\frac{\gamma_L}{1+\gamma_L^2} = -2(a_L + a_R)\frac{\gamma_R}{1+\gamma_R^2},$$

then $b_\infty > 0$ and there is only one crossing periodic which is stable for $0 < b < b_\infty$ and no periodic orbits for $b \geqslant b_\infty$.

(b3) If $\gamma_R + \gamma_L > 0$ and $a_L\gamma_R < a_R\gamma_L$ then, there are two hyperbolic crossing periodic orbits for b sufficiently small, while there are no crossing periodic orbits for b sufficiently big.

If additionally $a_R a_L = 0$ with $a_R + a_L \neq 0$ there exists a value b_{SN} such that the system has exactly two hyperbolic crossing periodic orbits for $0 < b < b_{SN}$, only one crossing periodic orbit which is semistable for $b = b_{SN}$ and no crossing periodic orbits for $b > b_{SN}$.

(b4) If $\gamma_R + \gamma_L < 0$, then there is always a stable crossing periodic orbit. If in addition $a_R a_L = 0$, then the above crossing periodic orbit is unique.

Results without inner equilibria ($a_R \leq 0 \leq a_L$)

Theorem (Systems with attractive sliding set, $b < 0$) Assuming the conditions $a_R \leqslant 0 \leqslant a_L$ and $b < 0$, the following statements hold.

(a) If $\gamma_R \gamma_L \geqslant 0$ then for $\gamma_R + \gamma_L \leqslant 0$ there are no crossing periodic orbits, while for $\gamma_R + \gamma_L > 0$ there is only one crossing periodic orbit which is unstable.

(b) If $\gamma_R \gamma_L < 0$, then the following cases arise.

(b1) If $\gamma_R + \gamma_L \leqslant 0$ and $a_L \gamma_R \leqslant a_R \gamma_L$, then there are no crossing periodic orbits.

(b2) If $\gamma_R + \gamma_L = 0, a_L \gamma_R > a_R \gamma_L$ and we define the value

$$b_\infty = 2(a_L + a_R)\frac{\gamma_L}{1+\gamma_L^2} = -2(a_L + a_R)\frac{\gamma_R}{1+\gamma_R^2},$$

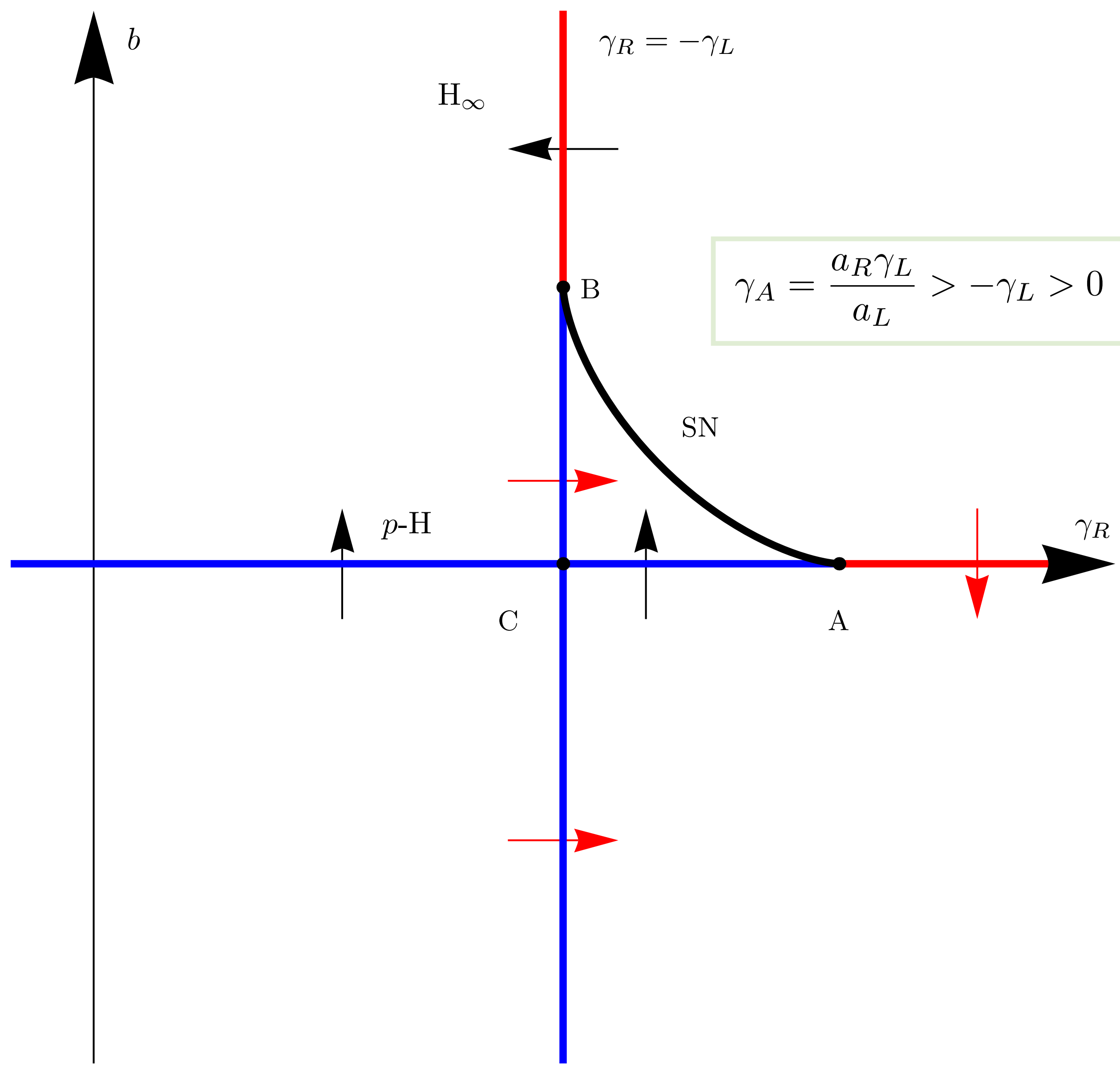
then $b_\infty < 0$ and there is only one crossing periodic which is unstable for $b_\infty < b < 0$, and no periodic orbits for $b \leqslant b_\infty$.

(b3) If $\gamma_R + \gamma_L < 0$ and $a_L \gamma_R > a_R \gamma_L$ then, there are two hyperbolic crossing periodic orbits for $|b|$ sufficiently small, while there are no crossing periodic orbits for $|b|$ sufficiently big.

If additionally $a_R a_L = 0$ with $a_R + a_L \neq 0$ there exists a value b_{SN} such that the system has exactly two hyperbolic crossing periodic orbits for $b_{SN} < b < 0$, only one crossing periodic orbit which is semistable for $b = b_{SN}$ and no crossing periodic orbits for $b < b_{SN}$.

(b4) If $\gamma_R + \gamma_L > 0$, then there is always a unstable crossing periodic orbit. If in addition $a_R a_L = 0$, then that crossing periodic orbit is unique.

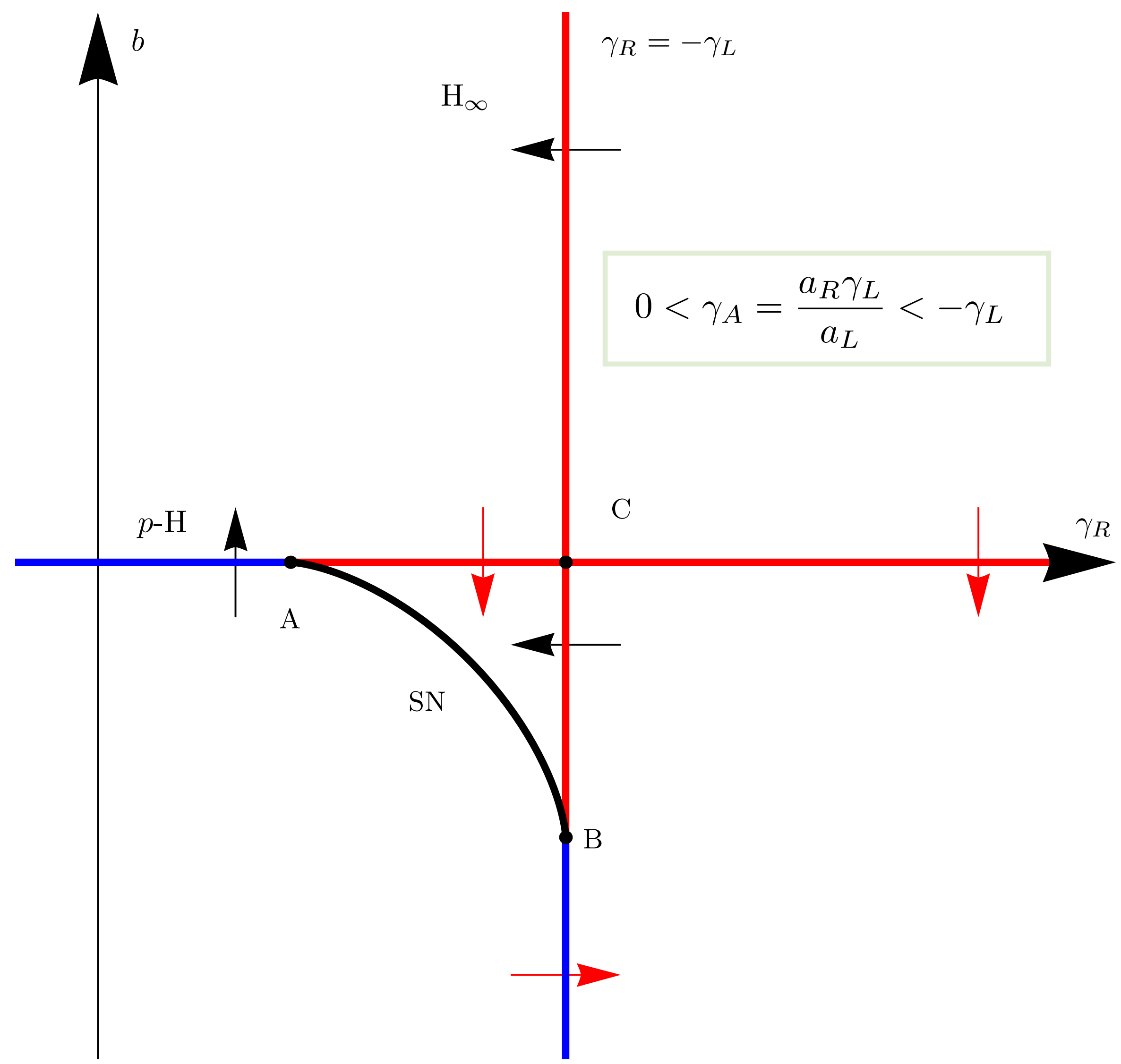
Results without inner equilibria ($a_R \leq 0 \leq a_L$)



$$\gamma_A = \frac{a_R \gamma_L}{a_L} > -\gamma_L > 0$$

Bifurcation set in the plane (γ_R, b) **for** $\gamma_L < 0$ **and** $a_R < 0 < a_L < -a_R$**, that is** $\gamma_A > -\gamma_L$. Different bifurcation curves appear: H_∞ stands for Hopf at infinity, SN indicates *saddle-node* of periodic orbits, and p-H means pseudo-Hopf. There are also two co-dimension two bifurcation points $A = (\gamma_A, 0), B = (-\gamma_L, b_\infty)$ and another $C = (-\gamma_L, 0)$ where two different bifurcations (one local and and another global) simultaneously appear.

Results without inner equilibria ($a_R \leq 0 \leq a_L$)

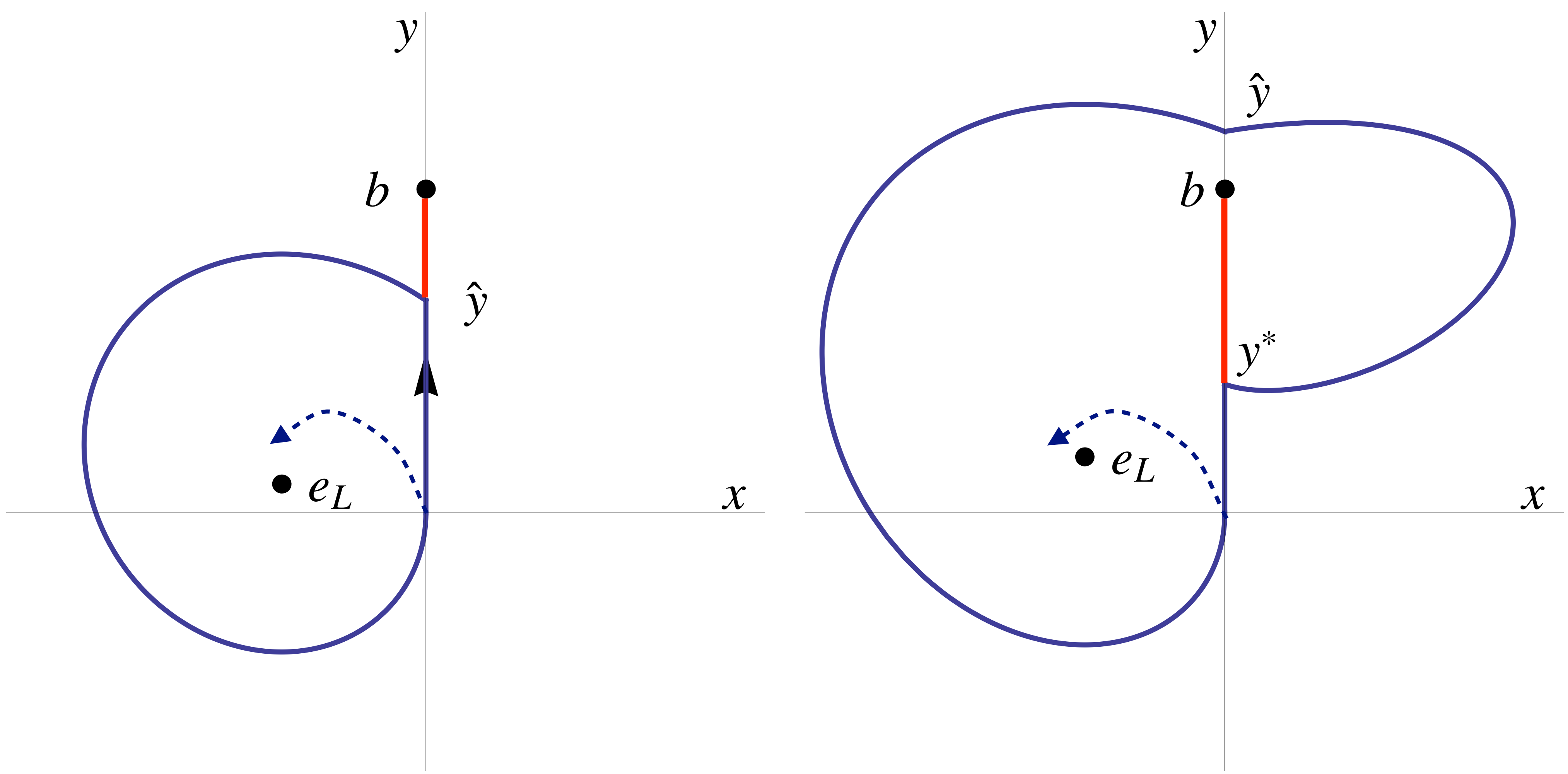


Bifurcation set in the plane (γ_R, b) **for the case** $\gamma_L < 0$ **and** $-a_L < a_R < 0 < a_L$**, that is** $\gamma_A < -\gamma_L$**.** Labels have the same meaning as in previous figure.

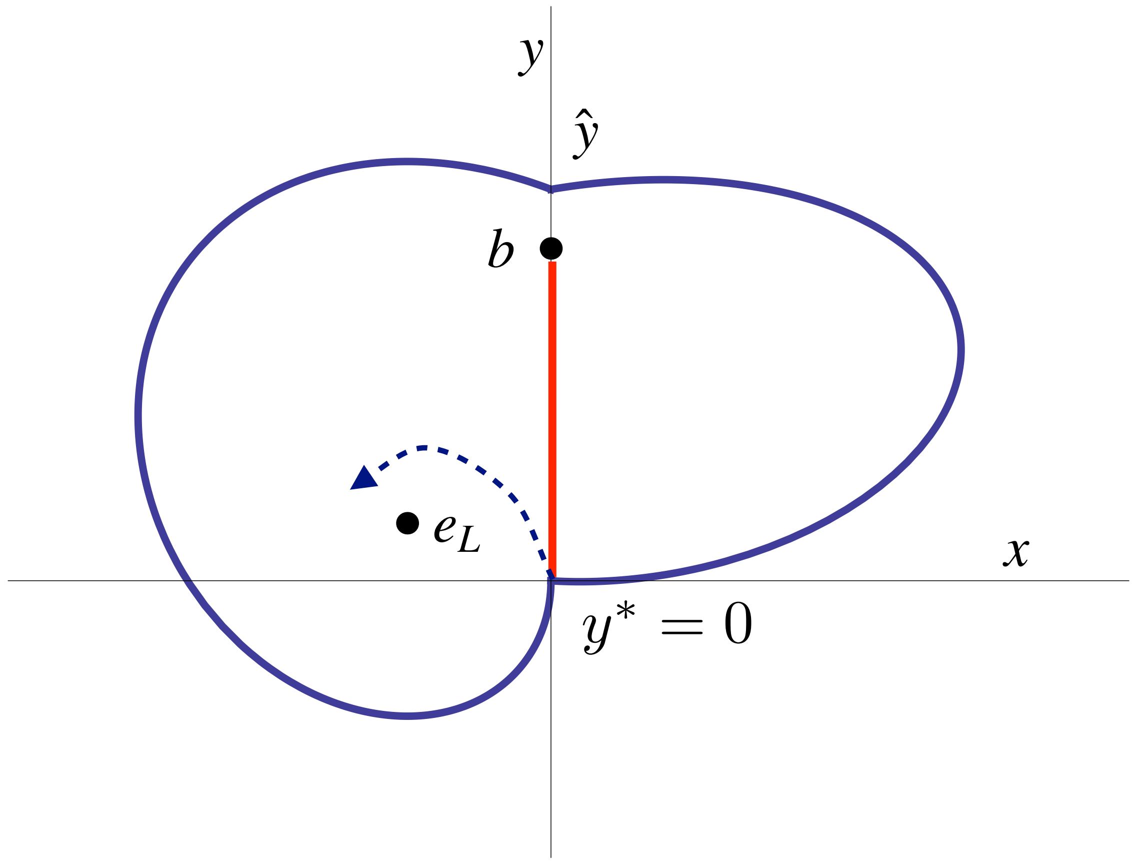
E. Freire, E.P. and F. Torres, *Canonical Discontinuous Planar Piecewise Linear Systems*, SIAM J. Applied Dynamical Systems **11** (2012) 181–211.

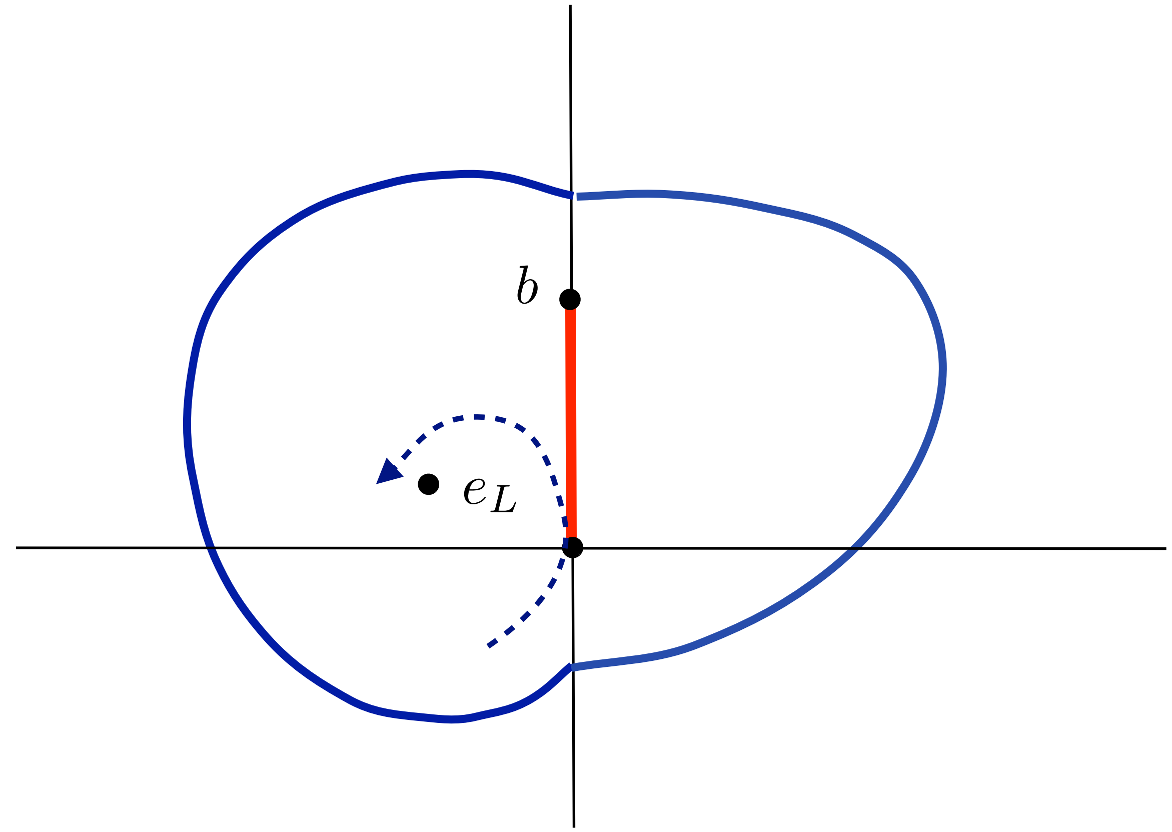
The case with one inner equilibrium ($a_L \cdot a_R > 0$)

We assume $a_L < 0$ and $a_R < 0$, so that there is one focus in the left side at $\mathbf{e}_L = (x_L, y_L)$ with $a_L = (1+\gamma_L^2)x_L$, $y_L = 2\gamma_L x_L$, and a (virtual) focus governing the right dynamics at $\mathbf{e}_R = (x_R, y_R)$ with $a_R = (1+\gamma_R^2)x_R$, $y_R = 2\gamma_R x_R + b$.



Sliding limit cycles of one and two zones

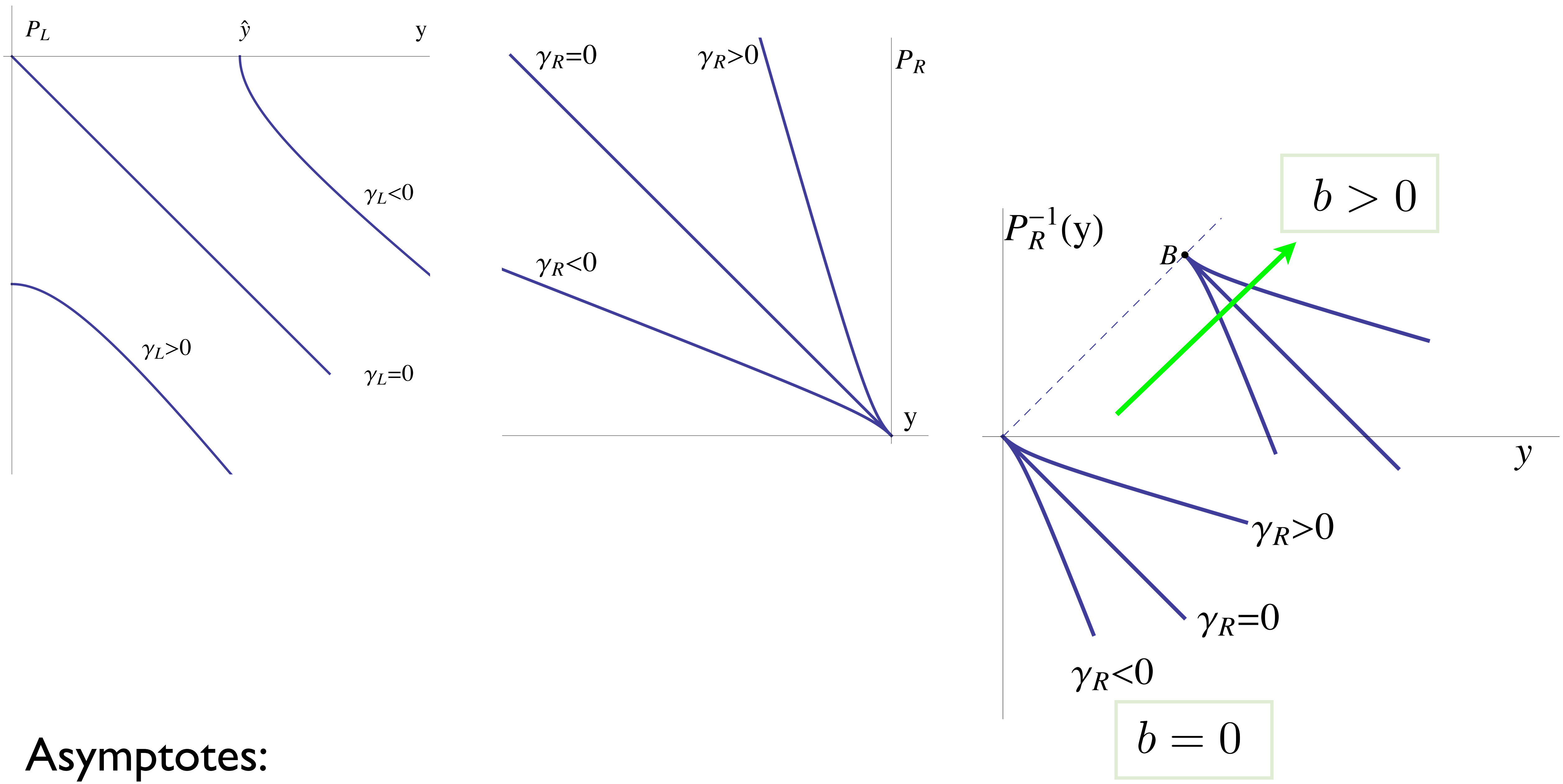




Crossing critical cycle (homoclinic to visible tangency)
and standard crossing limit cycle

M. Guardia, T.M. Seara, and M. A. Teixeira,
Generic bifurcations of low codimension of planar Filippov Systems,
Journal of Differential Equations **250** (2011) 1967–2023.

The half-return maps and their dependence on parameters



Asymptotes:

$$A_L(y) = -e^{\gamma_L \pi} y + 2x_L \gamma_L (1 + e^{\gamma_L \pi})$$

$$A_{R^{-1}}(y) = -e^{-\gamma_R \pi} y + (b + 2x_R \gamma_R)(1 + e^{-\gamma_R \pi})$$

Hunting limit cycles through elementary bifurcation analysis

Proposition (sliding periodic orbits) Assume that $x_L < 0$, $x_R < 0$, $b > 0$. When $\gamma_L \geqslant 0$ the system has no sliding periodic orbits; if $\gamma_L < 0$ and regarding the values $\hat{y} = P_L^{-1}(0) > 0$ and $y^* = P_R^{-1}(\hat{y}) < b$, the following statements hold.

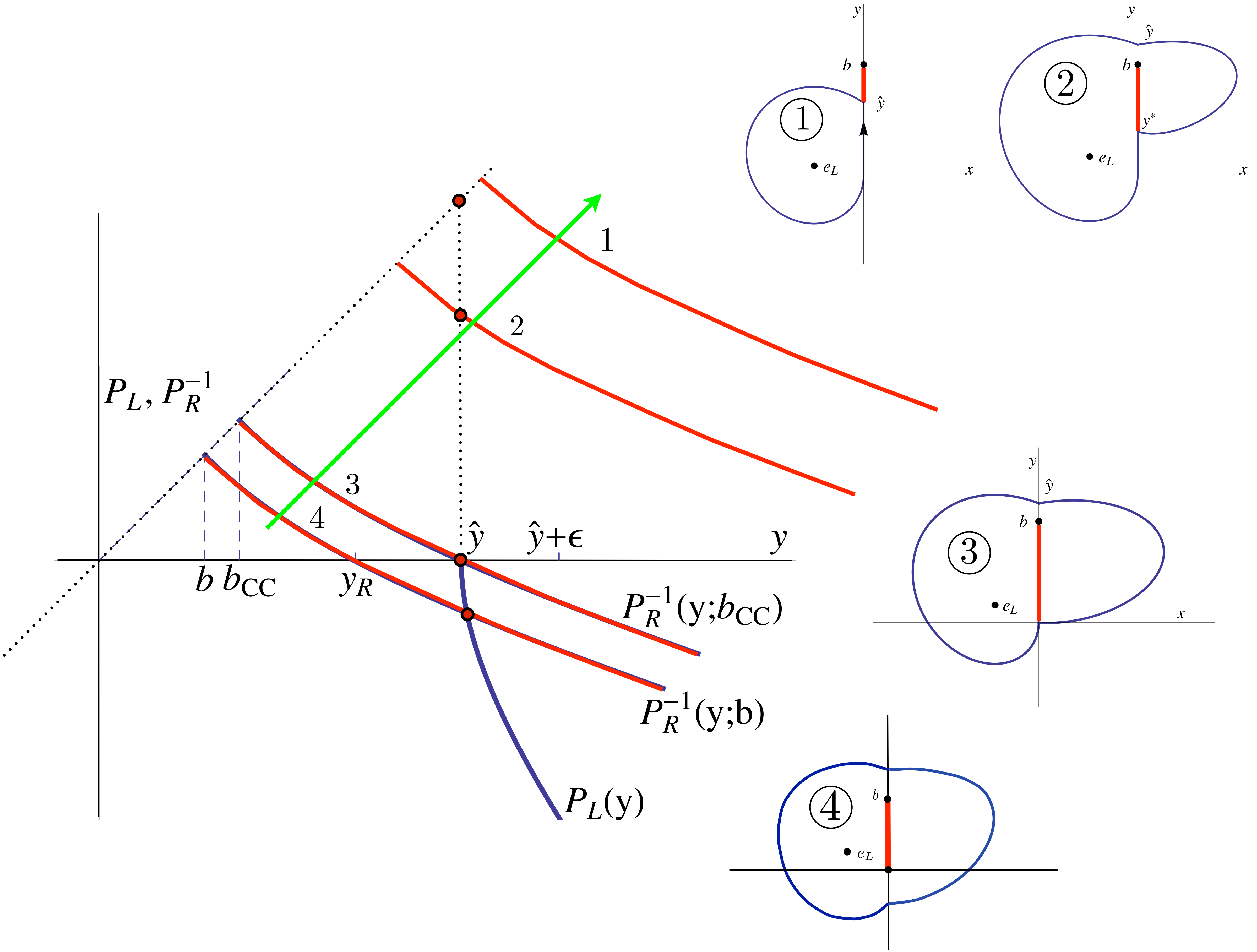
(a) If $\hat{y} < b$ then there is one sliding orbit backward in time which lives in the left zone and it is unstable.

(b) If $\hat{y} = b$ then there is one sliding orbit both backward and forward in time, which lives in the left zone and it is unstable.

(c) If $\hat{y} > b$ then the following cases arise.

- **(i)** If $0 < y^* < b$ then there is one sliding periodic orbit backward in time which lives in the two zones and it is unstable.
- **(ii)** If $y^* = 0$ then there is one crossing critical cycle which is unstable.
- **(iii)** If $y^* < 0$ then there are no sliding periodic orbits.

The b-bifurcation through the crossing critical cycle



The crossing critical cycle curve in the parameter plane (γ_R, b)

We assume $x_L < 0$, $\gamma_L < 0$ and $x_R < 0$ fixed, and look for possible bifurcations leading to one crossing limit cycle by moving parameters b and/or γ_R.

Proposition Assume that $x_L < 0, x_R < 0$ and $\gamma_L < 0$. Then there exists one smooth function $b = b_{CC}(\gamma_R)$ with $0 < b_{CC}(\gamma_R) < \hat{y}$ and $b_{CC}(0) = \hat{y}/2$, defined for every value of γ_R such that for $b = b_{CC}(\gamma_R)$ the system has one unstable crossing critical cycle.
In addition there exists $\varepsilon > 0$ such that for $b_{CC}(\gamma_R) - \varepsilon < b < b_{CC}(\gamma_R)$ there exists one unstable crossing periodic orbit which bifurcates from the crossing critical cycle.

Results that can be deduced from stability of the point at infinity

Theorem (stable equilibrium and extremal values of b) Assuming $x_L < 0$, $x_R < 0$, $\gamma_L < 0$ and $\gamma_R > 0$, and defining $b_\infty = 2(x_L + x_R)\gamma_L$, the following statements hold.

(a) If $\gamma_L + \gamma_R < 0$ and $b \geqslant b_{CC}$, then there is at least one stable crossing periodic orbit.

(b) If $\gamma_L + \gamma_R \leqslant 0$ and $b < 2x_L\gamma_L$, then there are no crossing periodic orbits.

(c) If $\gamma_L + \gamma_R > 0$ and $b < b_{CC}$, then there is at least one unstable crossing periodic orbit.

(d) If $\gamma_L + \gamma_R \geqslant 0$, then there exists a constant $M > 0$ such that for all $b > M$ there are no crossing periodic orbits.

(e) If $b < b_\infty$, then there exist $\varepsilon_1 > 0$ such that for $-\gamma_L < \gamma_R < -\gamma_L + \varepsilon_1$, there is at least one unstable crossing periodic orbit and when $b > b_\infty$, then there exist $\varepsilon_2 > 0$ such that for $-\varepsilon_2 - \gamma_L < \gamma_R < -\gamma_L$, there is at least one stable crossing periodic orbit.

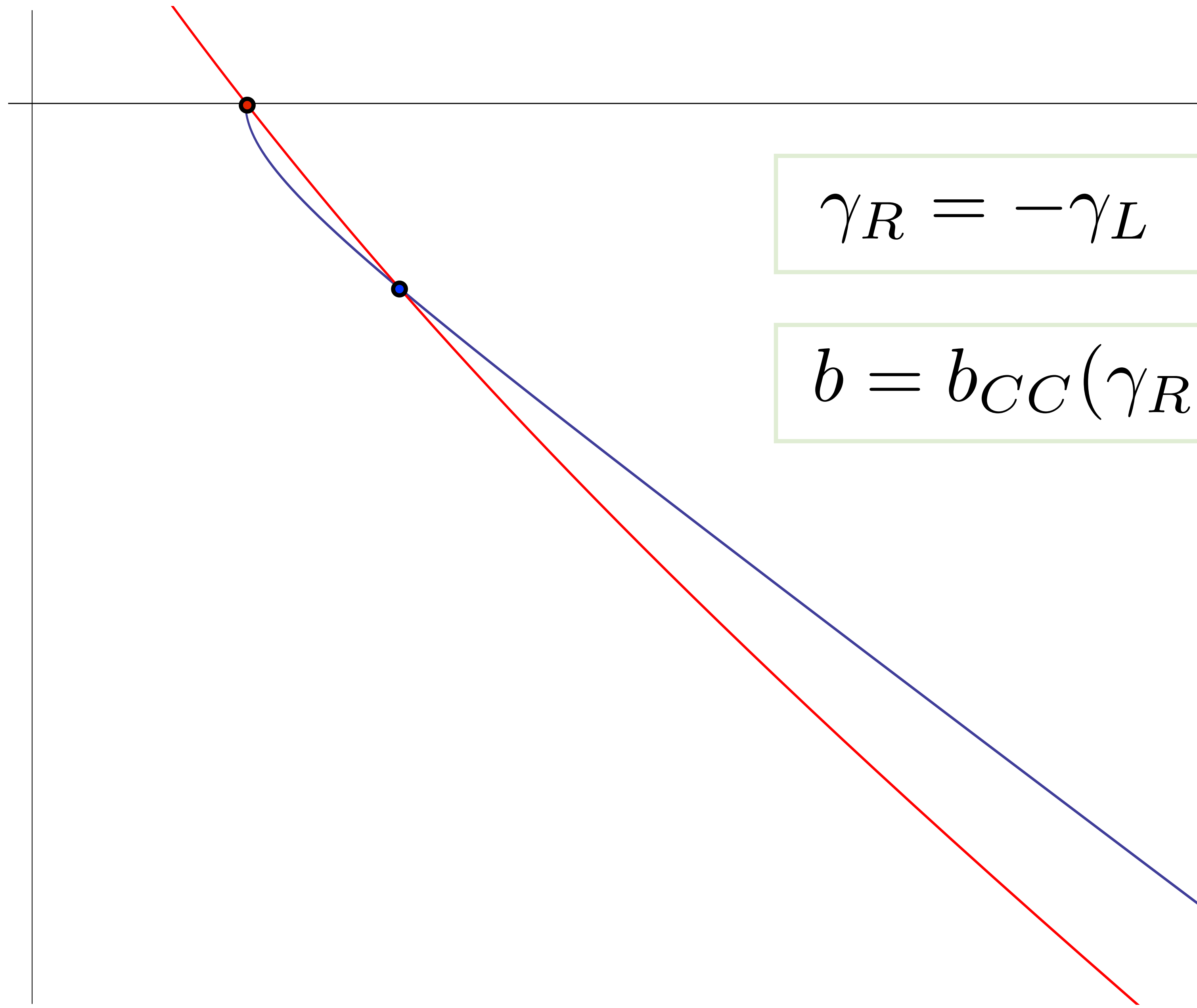
Getting our aim by combining local and global results...

Theorem (stable equilibrium, b near b_{CC}) Assuming that $x_L < 0$, $x_R < 0$, $\gamma_L < 0$ and $\gamma_R > 0$, the following statements hold.

(a) If $\gamma_L + \gamma_R < 0$ there exists $\varepsilon > 0$ such that for $b_{CC} - \varepsilon < b < b_{CC}$ the system has at least two crossing periodic orbits with opposite stabilities.

(b) Provided that $\hat{y} < b_\infty$, the following statements also hold.

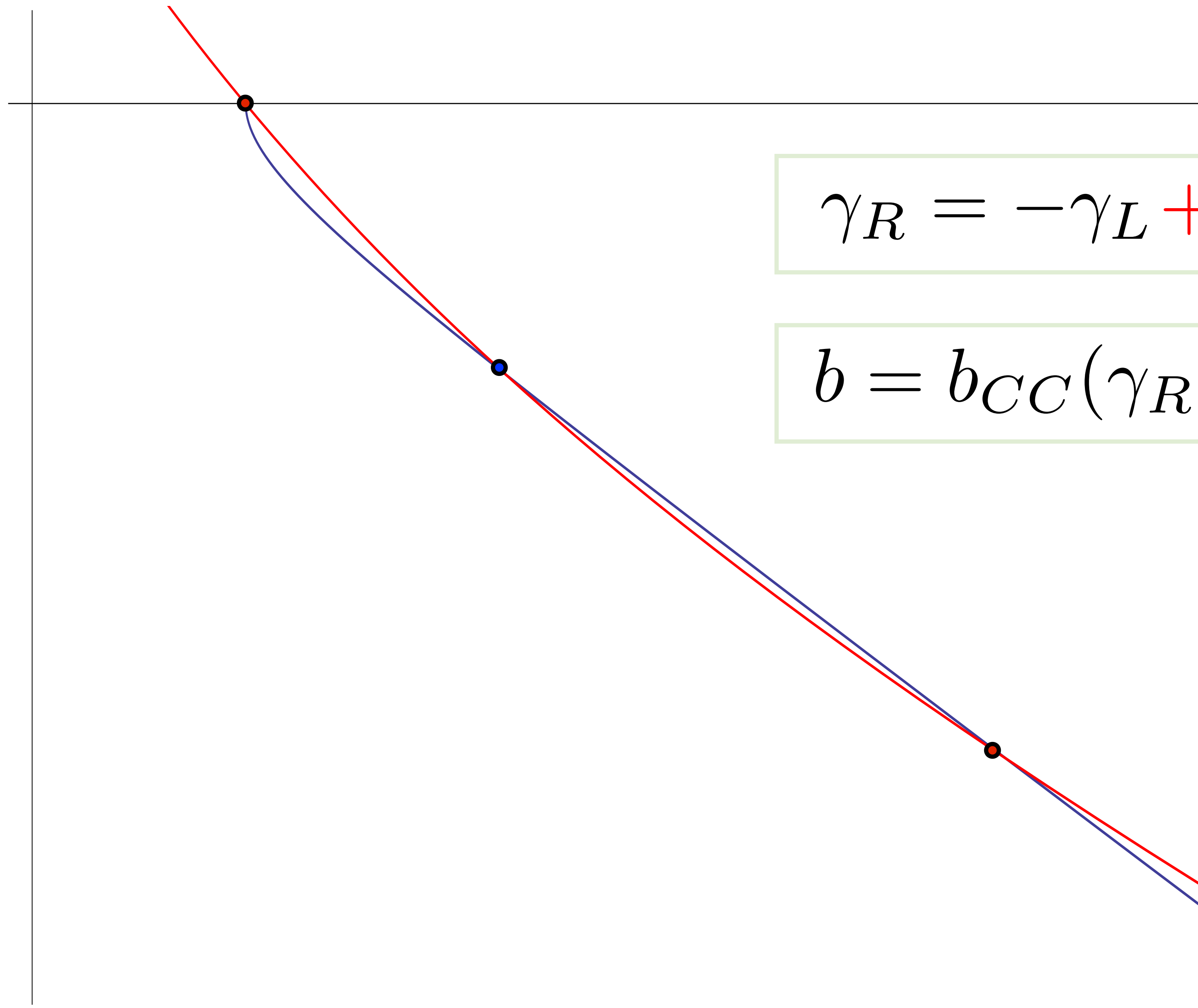
- **(i)** Assume $\gamma_R = -\gamma_L$. Then, there exists $\varepsilon_0 > 0$ such that for $b_{CC} \leqslant b < b_{CC} + \varepsilon_0$ the system has at least a stable crossing periodic orbit. In addition, there exists $\varepsilon_1 > 0$ such that for $b_{CC} - \varepsilon_1 < b < b_{CC}$ the system has at least two crossing periodic orbits with opposite stabilities.
- **(ii)** There exists $\varepsilon_2 > 0$ such that for $-\gamma_L < \gamma_R < -\gamma_L + \varepsilon_2$ and $b = b_{CC}(\gamma_R)$ the system has at least two crossing periodic orbits with opposite stabilities. Furthermore, for $-\gamma_L < \gamma_R < -\gamma_L + \varepsilon_2$ there exists $\varepsilon_3(\gamma_R) > 0$ such that for $b = b_{CC}(\gamma_R) - \varepsilon_3(\gamma_R)$ the system has at least **three nested crossing periodic orbits** being stable the intermediate one and unstable the two other.



$$\gamma_R = -\gamma_L$$

$$b = b_{CC}(\gamma_R)$$

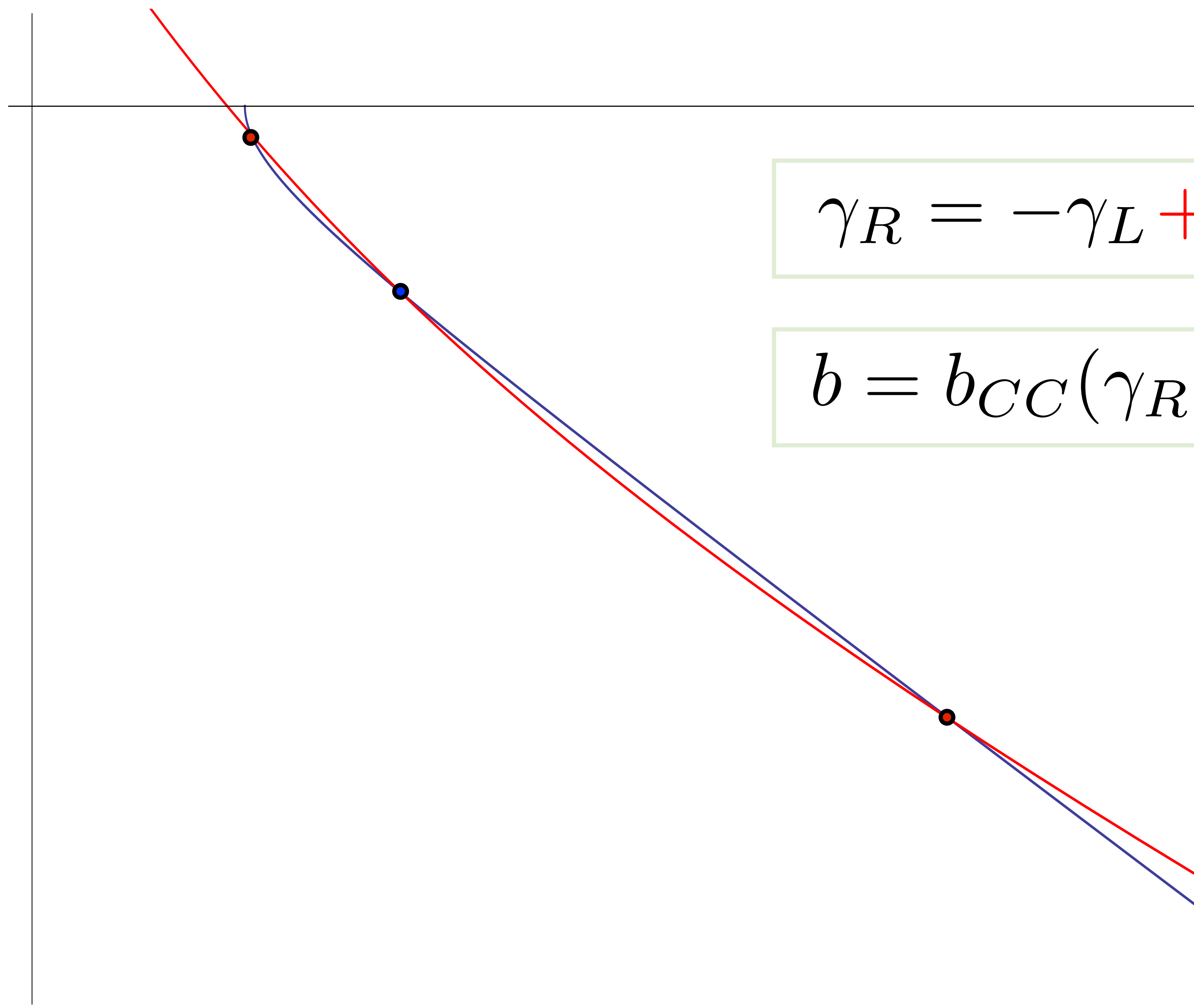
Hunting the three crossing limit cycles



$$\gamma_R = -\gamma_L + \varepsilon$$

$$b = b_{CC}(\gamma_R)$$

Hunting the three crossing limit cycles

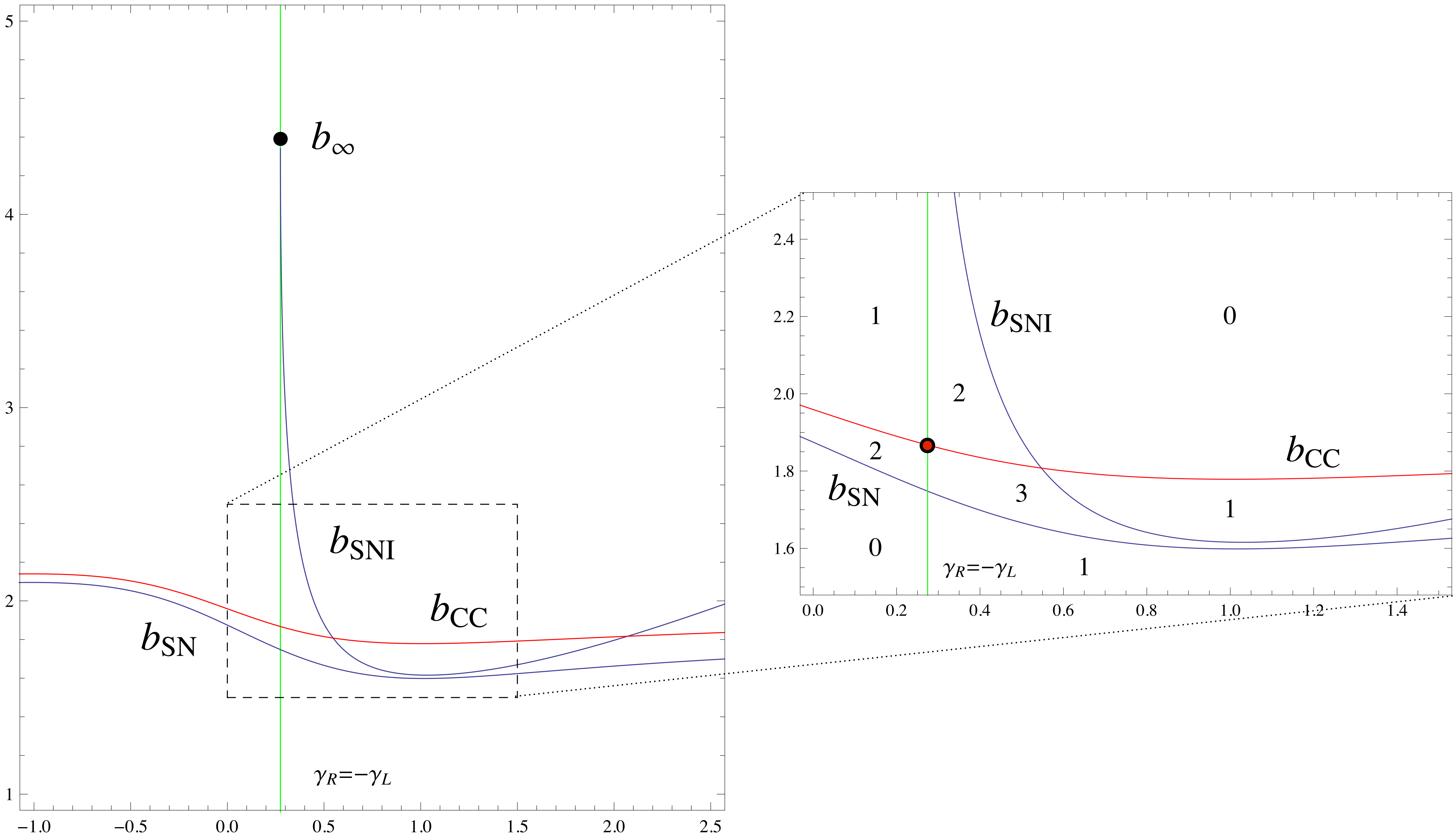


$$\gamma_R = -\gamma_L + \varepsilon$$

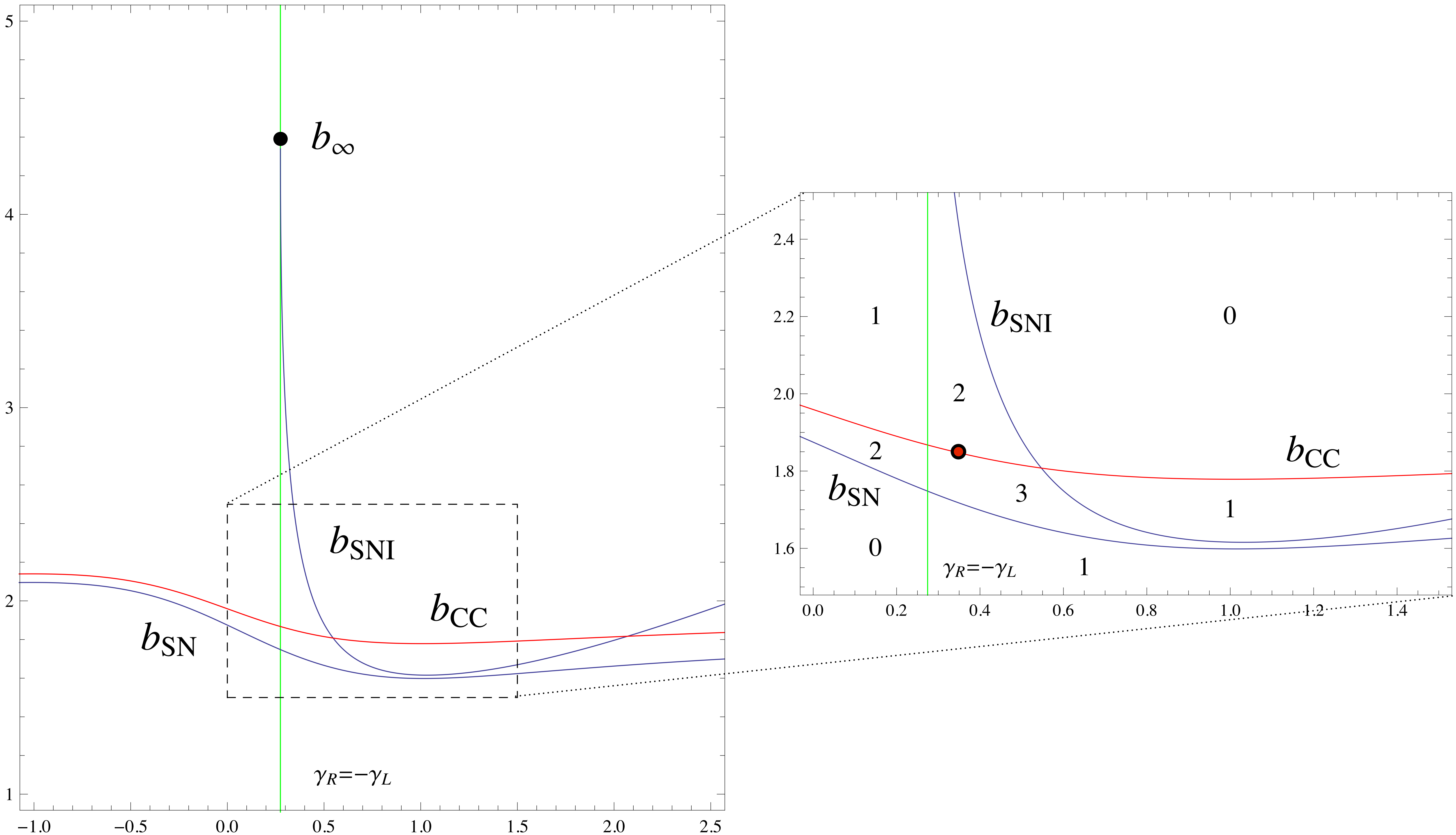
$$b = b_{CC}(\gamma_R) - \bar{\varepsilon}$$

Hunting the three crossing limit cycles

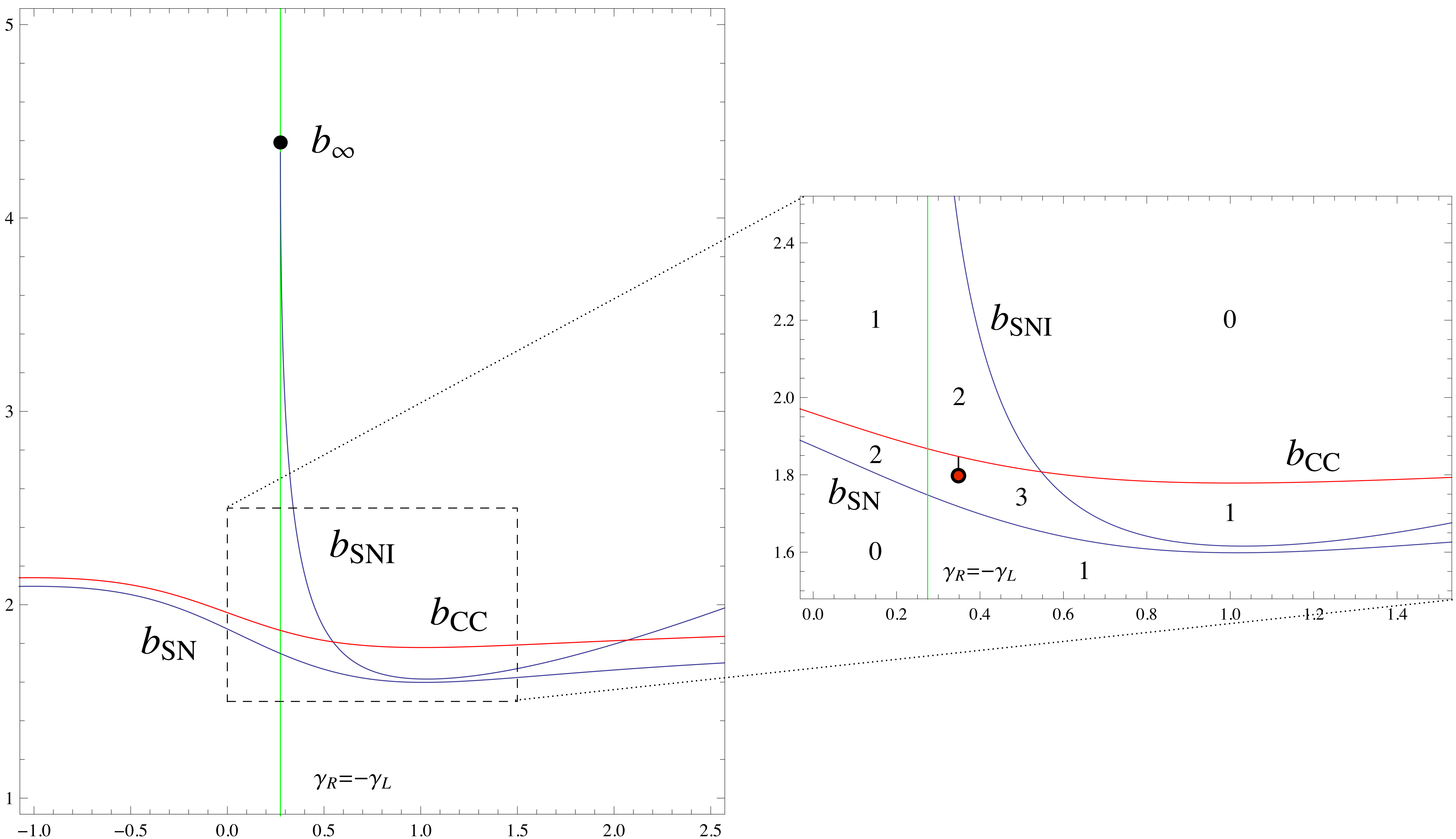
Hunting the three crossing limit cycles in a real case



Hunting the three crossing limit cycles in a real case



Hunting the three crossing limit cycles in a real case



Conclusions

- The study of all possible dynamics in discontinuous PWL systems with two zones is by no means a trivial task, involving a high number of parameters.
- Some headway is made in this problem thanks to a canonical form with fewer parameters. Nevertheless, the full rigorous analysis of this simpler family remains a formidable challenge.
- We have limited here our study to discontinuous piecewise linear systems whose dynamics rotates around the sliding set, resembling the one of a smooth focus. Certain bifurcation sets show different codimension-two Hopf bifurcation points that deserve a further study.
- Some location of the parameter region where three nested crossing limit cycles are possible, has been gained. It is a remaining problem to determine whether three is the maximum number of limit cycles to be found in the family.

On behalf of the west-andalousian dynamical systems group:

Happy birthday, Jaume, thanks for so many things, and...