The Dynamics of Quasi-periodically Forced Circle Maps in Unexplored Regions of Parameter Space.

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Introduction

Quasi-periodically forced circle maps are maps of the form:

$$x_n = f(x_{n-1}, \theta_{n-1}) \quad (1)$$
$$\theta_n = g(\theta_{n-1}) \quad (2)$$

Where

$$f(x_n, \theta_n) = a + \frac{b}{2\pi}\ \sin(2\pi x_{n-1}) + c\ \sin(2\pi\theta_{n-1}\ mod\ 1) \quad (3)$$

And

$$g(\theta_n) = \theta_{n-1} + \omega\ mod\ 1 \quad (4)$$

And $(x_n, \theta_n) \in S^1$ x S^1, parameters a, b and c are real numbers and ω is an irrational number (we chose $\omega = \frac{\sqrt{5}+1}{2}$ to avoid synchronization).

This maps have shown a variety of interesting features such as strange non-chaotic attractors, mode locking, intermittency, etc. And are a good model to study the transition to chaotic dynamics.

They are also interesting from the applied point of view, as they have been used to model a broad range of phenomena, for example, the periodic stimulation of cardiac oscillations and lung inflation by a mechanical ventilator [1].

1. Rotation Numbers

A common approach for the analysis of the dynamics of quasi-periodically forced circle maps consist of observing rotation numbers. Rotation numbers are often thought of as a measure of how much a point is translated in each direction, on average, with each iteration of the map.

We look at the rotation number in the x direction, as the other one is simple a rigid irrational rotation and therefore its rotation number is always ω.

$$\rho = \lim_{n\to\infty} \frac{F^n(x_0) - x_0}{n} \quad (5)$$

where F^n refers to nth iteration of the lift of f.

Clearly, ρ depends on parameters a, b and c, and also on the initial x_0 and θ_0.

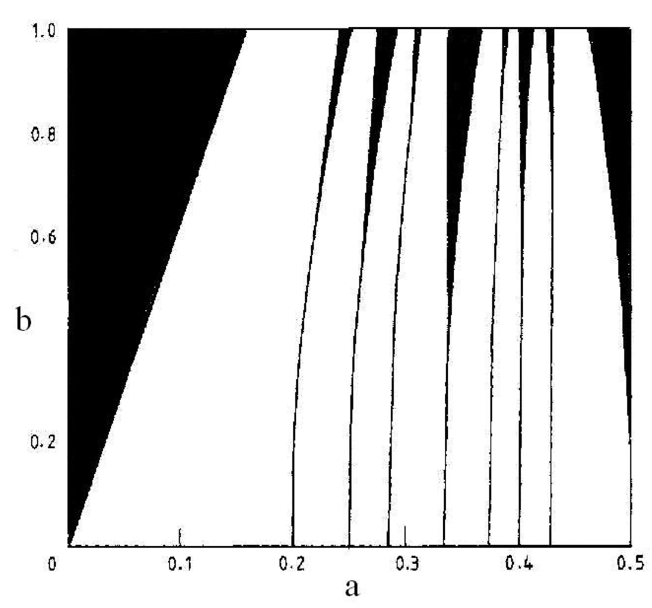
It possible to ignore the dependence on the initial x_0, θ_0, and only think of ρ as a function of parameters because either:

1. The rotation number is unique (for invertible maps, i.e. $b < 1$)
2. The rotation number may not be unique but they form a closed interval, in which case we look only at the borders of rotation intervals ($b > 1$).

The appearance of rotation intervals is important because it opens the possibility of chaotic dynamics, and so, we are interested on how this transition occurs.

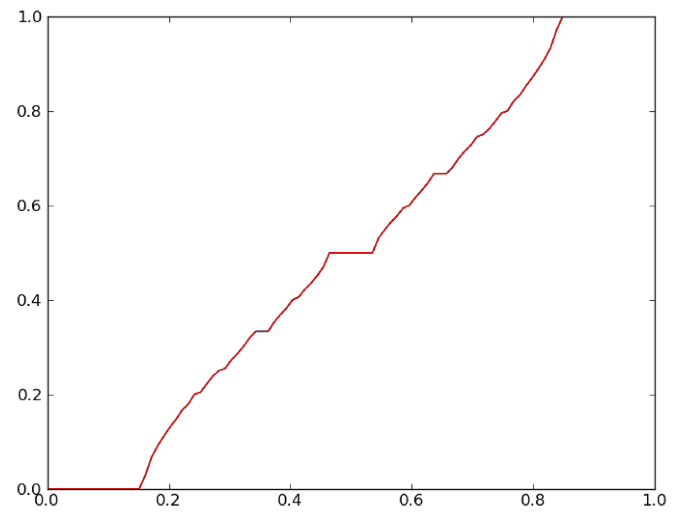
A question that arises with this approach is: **how are rotation numbers organized in parameter space?**

The structure for small values of parameter 'c' remains very close to that on the unforced case, this means that set of parameters with a same rational rotation number form cone shaped closed regions, with non-trivial interior called 'tongues' while the set of parameters with shared irrational rotation number are lines connecting $b = 0$ to $b = 1$.



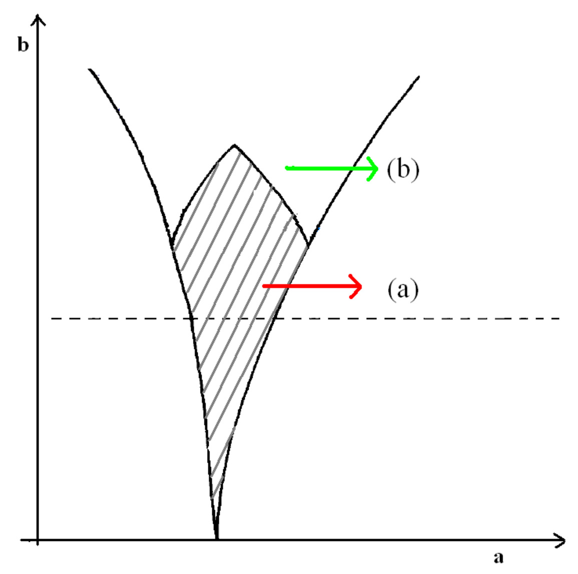
Working out the rotation number for large regions of parameter space is very time consuming, so, for their study is quite common to fix one of the parameters (usually b) while moving along a path in a.

The graph bellow shows the rotation number as a function of parameter a for a fixed $b = 0.9$

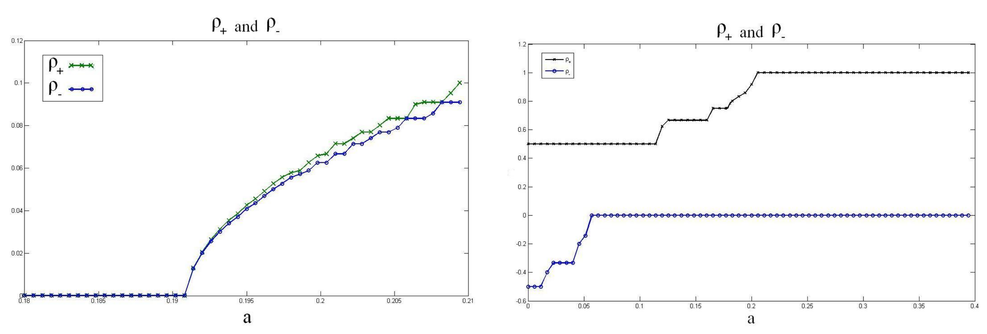


It is known that tongues continue to extend into the non-invertible region, overlaping at some point.
In consequence, there is a small region with a unique rotation number on top of the tongue.

We may see two qualitatively different pictures of the evolution of the borders of rotation intervals for a fixed $b > 1$ and changing a corresponding to the paths (a) and (b) in the following figure.



Whose rotation borders evolve as follows (to the left path (a), to right path (b)):

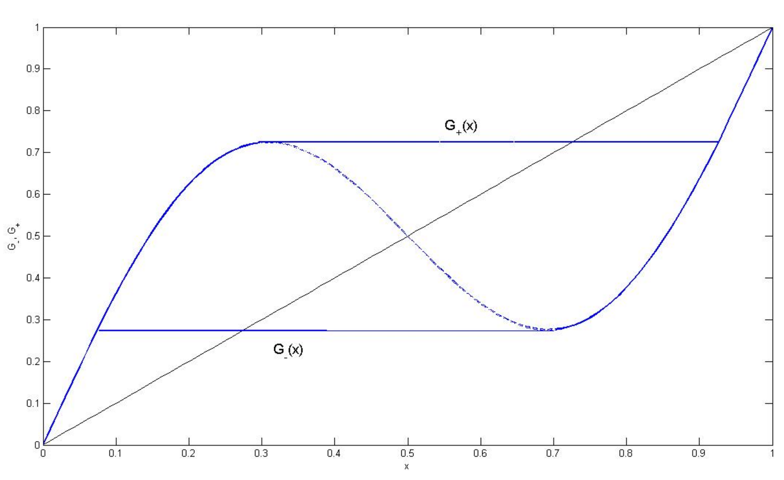


How are rotation numbers approximated?

Unlike the unforced circle map, there is no measure of good an approximation is by just iterating the map. However Stark et al [2] developed an algorithm, for monotone maps, which makes it possible to obtain a approximation a good as desired.

This algorithm consist on averaging the iteratives over several values of θ_0.

To calculate the borders of rotation numbers in the non-invertible case, we used this algorithm on a pair of invertible maps G_+, and G_- whose rotation number corresponds to the borders of the rotation interval of th original map. These maps are defined in [3]:



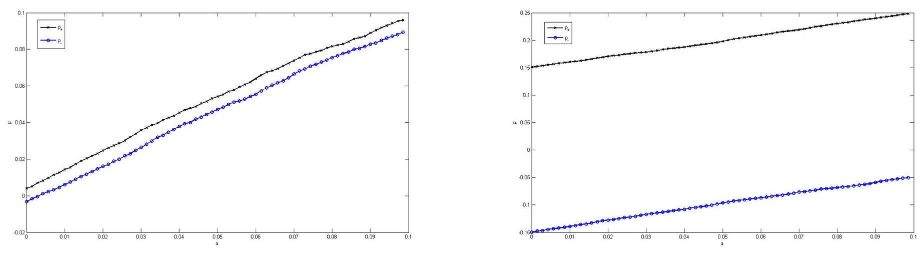
2. Results

As the value of the parameter c gets lager, the structure of rotation numbers changes (for an explanation of small c values see [4]).

Ding et al [5] tried to gain a insight into the dynamics for larger values of c by completely neglecting the term $\frac{b}{2\pi}\sin(2\pi x)$, which lead them to conjecture that:

$$\rho = a \quad (6)$$

We calculated the rotation number of the invertible maps corresponding to the borders of rotation intervals for non-invertible maps using large values of c, along the same paths presented before. This is how the borders of rotation intervals look:



Note that for each value of b a different length can be observed. However, this length remains constant for every value of a.

Further analysis (see [6]) led us to the conjecture that

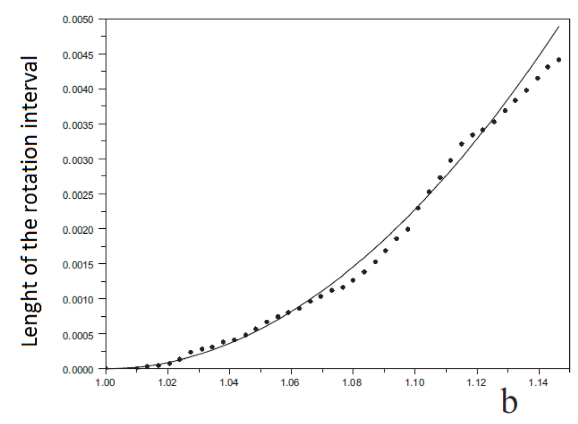
$$\rho_\pm \sim a = \int_0^1 g(x)_\pm - x \quad (7)$$

with g as in the maps defined in [3].

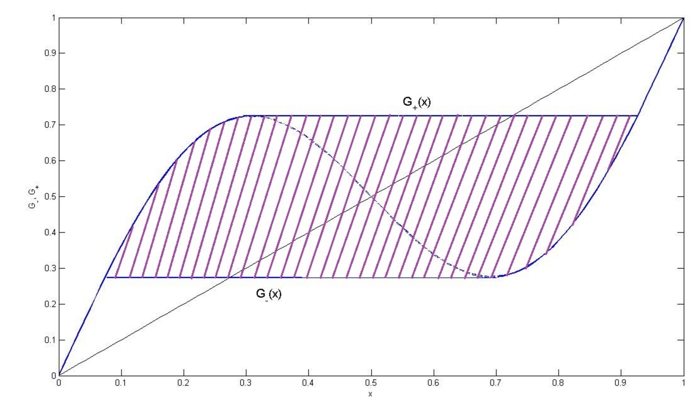
This formulation of $\rho_\pm$, in turn, gives a scaling function for the growth of rotation intervals as:

$$length[\rho_-, \rho_+] = \frac{9}{4\pi^2}(b-1)^2 \quad (8)$$

Which we can see that agrees with the numerical computation of the integrals:



And, of course, corresponds to the area between the graphs of $G_+(x)$ and $G_-(x)$.



References

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