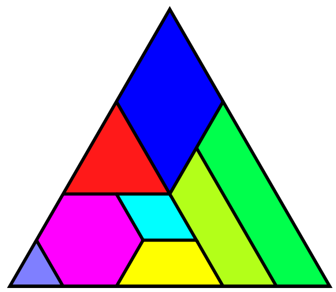

NEWTONIAN FEW-BODY PROBLEM CENTRAL CONFIGURATIONS WITH GRAVITATIONAL CHARGES OF BOTH SIGNS

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ABSTRACT

The Newtonian n-Body Problem is modified assuming positive inertial masses but different sign for the interacting force which is assumed with the possibility of two different signs for the gravitational masses, according to the prescription two masses with same sign attract one to the other, two masses of different sign repel one to the other. As in electrostatics the signed mass is called charge. The inertial mass is always positive. The two body problem behaves as the similar Coulomb problem of charged particles with two equal charges. The solution is a central configuration with almost same behavior that the Newton two-body problem for hyperbolic orbits. The 3-Body problem was found with collinear solutions. The four body case of charged central configurations [1] has only the planar and collinear solutions.

1 Introduction

In this paper we present the study of central configurations of few particles (2, 3, 4) that obey Newton's three laws of motion and Newton's gravitational force law [2], [3], assuming the possibility of gravitational charges of both signs, according to the prescription (which is opposite to that in electrostatics for electric charges where G would have a negative value) that the force between two masses with charge of the same sign is attractive, while the force between two masses with charge of opposite sign is repulsive. More explicitly, we assume the equations of motion

$$m_j \frac{d^2\mathbf{r}_j}{dt^2} = \sum_{l \neq j} \frac{G e_l e_j}{r_{lj}^3} (\mathbf{r}_l - \mathbf{r}_j) \, , \quad \forall j \tag{1}$$

where $\mathbf{r}_j$ denotes the position vector of particle j in 3D-space, m_j is its positive mass, G is the positive constant of universal gravitation, $r_{lj} = |\mathbf{r}_l - \mathbf{r}_j|$ is the distance between particles j and l, and e_j is the charge of particle j such that $m_j = |e_j|$, with two possible choices of sign for the charge e_j.

In the following we quote some fundamental equations of classical mechanics that are essential in Physics [2], [3].

Equation of motion (1) expresses Newton's second law equating the positive inertial mass times the acceleration to Newton's gravitational force. This force obeys the action-reaction law or Newton's third law: the force vector that particle j exerts on particle l is of equal magnitude and opposite sign to the force that particle l exerts on particle j. As a consequence, the sum over all j of the various equations of motion is the null vector

$$\sum_{\forall j} m_j \frac{d^2\mathbf{r}_j}{dt^2} = \mathbf{0} \, . \tag{2}$$

The few bodies' masses m_1, m_2,... are positive and generally different in value, but some may be equal.

The total mass is

$$m = \sum_{\forall j} m_j \, , \tag{3}$$

The center of mass position is defined as

$$\mathbf{c} = \frac{1}{m} \sum_{\forall j} m_j \mathbf{r}_j \, . \tag{4}$$

Equation (2) implies

$$\frac{d^2\mathbf{c}}{dt^2} = \mathbf{0}\,. \tag{5}$$

which asserts that the center of mass moves with constant velocity.

With no loss of generality we assume in this paper that

$$\frac{d\mathbf{c}}{dt} = \mathbf{0}\,, \quad \mathbf{c} = \mathbf{0}\,, \quad \sum_{\forall j} m_j \mathbf{r}_j = \mathbf{0}\,. \tag{6}$$

The center of mass is thus at the origin of the system of coordinates $\mathbf{r}_j$.

From equation (1) the conservation of total energy E follows, namely

$$\frac{1}{2}\sum_{\forall j} m_j \frac{d\mathbf{r}_j}{dt}\cdot\frac{d\mathbf{r}_j}{dt} - \sum_{l<j}\frac{Ge_je_l}{r_{lj}} = E\,. \tag{7}$$

In this expression the first term on the left hand side is the kinetic energy, involving the positive inertial masses, while the second term is the potential energy, which depends on the gravitational charges.

Equation (1) also implies conservation of angular momentum

$$\frac{d}{dt}\sum_{\forall j} m_j \mathbf{r}_j \times \frac{d\mathbf{r}_j}{dt} = \mathbf{0}\,, \tag{8}$$

which again contains the inertial masses.

2 The integrable two body problem

In this section the positions of the two particles are written in terms of the relative position $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ as

$$\mathbf{r}_1 = -\frac{m_2}{m}\mathbf{r}\,, \quad \mathbf{r}_2 = \frac{m_1}{m}\mathbf{r}\,. \tag{9}$$

The differential equations of motion are

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}\,, \quad \frac{d\mathbf{v}}{dt} = Gm\frac{1}{|\mathbf{r}|^3}\mathbf{r}\,. \tag{10}$$

With the constants of motion of energy and angular momentum written in terms of, the specific energy

$$E = \frac{1}{2}|\mathbf{v}|^2 + Gm\frac{1}{|\mathbf{r}|} \tag{11}$$

and the areal velocity

$$\mathbf{g} = \mathbf{r} \times \mathbf{v} = g(0, 0, 1)\,, \tag{12}$$

where g is twice the magnitude of the areal velocity. This last implies the orbit ($\mathbf{r}$ and $\mathbf{v}$) is in a plane orthogonal to the constant vector along $\mathbf{g}$. Polar coordinates: r, ψ, in this plane give us

$$\mathbf{r} = r(\cos\psi, \sin\psi, 0)\,, \quad \mathbf{v} = \dot{r}(\cos\psi, \sin\psi, 0) + r\dot{\psi}(-\sin\psi, \cos\psi, 0)\,, \tag{13}$$

where the dot on a letter denotes the time derivative, and

$$g = r^2\dot{\psi}\,, \quad E = \frac{1}{2}(\dot{r}^2 + r^2\dot{\psi}^2 + Gm\frac{1}{r}) = \frac{1}{2}(\dot{r}^2 + \frac{g^2}{r^2} + Gm\frac{1}{r})\,. \tag{14}$$

Dividing the second equation in (10) by the first equation in (14) one has

$$\frac{d\mathbf{v}}{d\psi} = \frac{1}{\dot{\psi}}\frac{d\mathbf{v}}{dt} = \frac{Gm}{g}(\cos\psi, \sin\psi, 0)\,, \tag{15}$$

which is integrated into

$$\mathbf{v} = \frac{Gm}{g}(\sin\psi, -\cos\psi, 0) + \mathbf{h}\,, \tag{16}$$

where $\mathbf{h}$ is a constant vector of integration, the Hamilton vector [4], in the plane of the orbit. Vector $\mathbf{v} - \mathbf{h}$ traces a circle in velocity space with center at $\mathbf{h}$ and radius $\frac{Gm}{g}$.

It is useful define the constant of motion called the Laplace-Runge-Lenz vector defined here in terms of other constants of motion as

$$\boldsymbol{\epsilon} = \frac{1}{Gm}\mathbf{h} \times \mathbf{g} = \frac{1}{Gm}\mathbf{v} \times \mathbf{g} + \frac{\mathbf{r}}{r} = \epsilon(1, 0, 0)\,, \tag{17}$$

which defines the direction of one coordinate axis in the plane of the orbit. Projecting vector $\mathbf{r}$ in this direction (in polar coordinates) lead to the orbit equation

$$\epsilon r\cos\psi = \frac{g^2}{Gm} + r\,, \tag{18}$$

which is a hyperbola with $\epsilon > 1$, the eccentricity and with $p = \frac{g^2}{Gm}$, the latus rectum. The specific energy becomes $E = \frac{Gm}{2a}$.

3 Central configurations of few charged masses. Are there equilateral few body central configurations?

We define a central configuration as one in which the particle positions, defined up to rotation around the center of mass and uniform dilatations, obey the equation

$$Bm_j\mathbf{r}_j = \sum_{l\neq j} \frac{Ge_l e_j}{r_{lj}^3}(\mathbf{r}_l - \mathbf{r}_j)\,, \quad \forall j\,, \tag{19}$$

where B is the same quantity for all the particles. In the right side we have the gravitational force acting on particle j due to the other charges. Using the fact that the sum of forces is the zero vector, we recover the third element of hypothesis (6): the origin of coordinates $\mathbf{r}_j$ is at the center of mass.

For the particular case of the previous section with just two particles the allowed hyperbolic motion is at any moment in a central configuration with $B = \frac{Gm}{r^3}$. In the following, one considers the cases with three and four particles.

The force of the right hand side of (19) is the gradient of the potential energy

$$\sum_{l\neq j} \frac{Ge_l e_j}{r_{lj}^3}(\mathbf{r}_l - \mathbf{r}_j) = \frac{\partial}{\partial \mathbf{r}_j} \sum_{l<k} \frac{Ge_k e_l}{r_{lk}}\,. \tag{20}$$

The left hand side of (19) is related with the gradient of the total moment of inertia, I,

$$m_j\mathbf{r}_j = \frac{\partial}{\partial \mathbf{r}_j} \frac{1}{2} \sum_{\forall k} m_k \mathbf{r}_k \cdot \mathbf{r}_k\,, \tag{21}$$

which may be written in terms of the relative distances as

$$I = \frac{1}{2} \sum_{\forall k} m_k \mathbf{r}_k \cdot \mathbf{r}_k = \frac{1}{2m} \sum_{k\neq l} m_k m_l r_{kl}^2\,. \tag{22}$$

It follows that the condition for central configurations of few particles in three dimensions may be expressed in terms of derivatives with respect to the relative distances r_{ij}^2 in the form

$$\frac{e_l e_j}{r_{lj}^3} = \sigma m_l m_j\,, \tag{23}$$

where σ is a quantity containing B and G. The left hand side of this equation is the derivative of the potential energy with respect to the square of the distance r_{lj}^2, while the right hand side is proportional to the derivative of the total moment of inertia with respect to the same variable r_{lj}^2.

Theorem 1

Equilateral central configurations do not exist for three or four particles with gravitational charges of different sign.

Proof: Equation (23) can not possibly be satisfied for all combinations of indexes because the left hand side takes different signs and the right hand side has the sign of σ because the m's are positive. One concludes that there is no equilateral central configuration for particles with different charges. Thus, the Lagrange equilateral triangle solution [5], and the Lehman-Filhés equilateral tetrahedron solution [6] valid for positive charges are not a central configuration for the case of positive and negative charges. □

4 Collinear Three-Body and planar Four-Body central configuratios for masses of different charge

Consider now the question of possible collinear/planar configurations for Three/Four-Bodies. The modified Dziobek equations for collinear/planar central configurations of positive and negative charges are

$$\frac{e_l e_j}{r_{lj}^3} = \sigma m_l m_j + \lambda S_l S_j \,, \tag{24}$$

where λ is a new parameter and S_k are the directed distances/areas of the segments/triangles having at their vertexes the two/three particles different from k. With respect to equation (23), the additional term takes into account the zero area/volume restriction necessary for collinear/planar configurations: $S_l S_k$ is a constant times the derivative with respect to r_{lk}^2 of the square of the area/volume of the triangle/tetrahedron formed by the three/four particles, which is expressed as the square of the Heron's equation/as the so-called Cayley-Menger determinant. A similar equation was obtained by Dziobek [7], but although the two terms on the right hand side of this equation are the same as in Dziobek's paper, in the left hand side the charges are replaced by masses.

Note the invariance of the modified Dziobek's equations with respect to a sign change of all the charges, as well as their invariance with respect to a sign change of the directed distances/areas

$$e_j \quad \longrightarrow \quad -e_j\,, \quad S_j \quad \longrightarrow \quad -S_j\,.$$

In order to prove that these equations are equivalent to the equations defining the central configurations, we need the collinear/planar conditions [7], [8]

$$\sum_{\forall j} S_j = 0\,, \quad \sum_{\forall j} S_j \mathbf{r}_j = \mathbf{0}\,. \tag{25}$$

Substituting the left hand side of (24) by the right hand side in (19), one obtains

$$Bm_j\mathbf{r}_j = G\sum_{l\neq j}^{4}(\mathbf{r}_l - \mathbf{r}_j)[\sigma m_l m_j + \lambda S_l S_j] = -G\sigma m m_j \mathbf{r}_j\,, \tag{26}$$

where we used the conditions (6) that the center of mass is at the origin, and the planar configuration properties (25). Thus from (26) we obtain that

$$B = -G\sigma m\,. \tag{27}$$

In addition, one has the property

$$\sum_{l=1}^{4}\sum_{j=1}^{4} S_l S_k r_{lj}^2 = \sum_{l=1}^{4}\sum_{j=1}^{4} S_l S_j(\mathbf{r}_l^2 - 2\mathbf{r}_l\cdot\mathbf{r}_j + \mathbf{r}_j^2) = 0\,, \tag{28}$$

where we use the collinear/planar configuration conditions (25). Note that the terms with $j = l$ in these summations are zero; therefore if we multiply both sides of equation (24) by r_{lj}^2 and we sum over all l and j with $l \neq j$ one obtains

$$\sum_{l,j,l\neq j} \frac{e_l e_j}{r_{lj}} = \sigma \sum_{l,j,l\neq j} m_l m_j r_{lj}^2\,. \tag{29}$$

In this case σ has not a definite sign because of the presence of charges with both signs.

5 Three-Body collinear central configurations

We ordered the coordinates of three particles as $x_1 < x_2 < x_3$.The Dziobek equations (24) for the collinear Three-Body Problem central configurations are

$$\frac{e_1e_2}{m_1m_2}\frac{1}{(x_2-x_1)^3} = \sigma + \beta\frac{m_3}{x_2-x_1} \tag{30}$$

$$\frac{e_2e_3}{m_2m_3}\frac{1}{(x_3-x_2)^3} = \sigma + \beta\frac{m_1}{x_3-x_2} \tag{31}$$

$$\frac{e_3e_1}{m_3m_1}\frac{1}{(x_3-x_1)^3} = \sigma + \beta\frac{m_2}{x_1-x_3}\,, \tag{32}$$

where β is

$$\beta = \lambda\frac{(x_3-x_2)(x_2-x_1)(x_1-x_3)}{m_1m_2m_3} \tag{33}$$

Theorem 2

It is impossible to have a collinear Three-Body central configuration of different charges with the middle charge equal to one of the other charges.

Proof: Assume the different charge is $e_3 = -m_3$. The previous equations (30-32) are in such a case

$$\frac{1}{(x_2-x_1)^3} = \sigma + \beta\frac{m_3}{x_2-x_1} \tag{34}$$

$$\frac{-1}{(x_3-x_2)^3} = \sigma + \beta\frac{m_1}{x_3-x_2} \tag{35}$$

$$\frac{-1}{(x_3-x_1)^3} = \sigma + \beta\frac{m_2}{x_1-x_3}\,. \tag{36}$$

Canceling σ between (34) and (36) we have

$$\frac{1}{(x_2-x_1)^3} + \frac{1}{(x_3-x_1)^3} = \beta\left(\frac{m_2}{x_3-x_1} + \frac{m_3}{x_2-x_1}\right)\,, \tag{37}$$

that implies $\beta > 0$.

Canceling σ between (36) and (35) we have

$$\frac{1}{(x_3-x_2)^3} - \frac{1}{(x_3-x_1)^3} = -\beta\left(\frac{m_2}{x_3-x_1} + \frac{m_1}{x_3-x_2}\right) \tag{38}$$

that implies $\beta < 0$, which is a contradiction. $\square$

On the contrary, when the middle charge is the one that has the opposite sign, the previous equations (30-32) are

$$-\frac{1}{(x_2-x_1)^3} = \sigma + \beta\frac{m_3}{x_2-x_1} \tag{39}$$

$$-\frac{1}{(x_3-x_2)^3} = \sigma + \beta\frac{m_1}{x_3-x_2} \tag{40}$$

$$\frac{1}{(x_3-x_1)^3} = \sigma + \beta\frac{m_2}{x_1-x_3}\,. \tag{41}$$

Canceling σ and β among these three equations we have the equation for collinear central configurations

$$m_2(x_3-x_1)^2(x_2-x_1)^3 + m_3(x_2-x_1)^2(x_3-x_2)^3 + m_3(x_2-x_1)^2(x_3-x_1)^3 =$$

$$m_2(x_3-x_1)^2(x_3-x_2)^3 + m_1(x_3-x_2)^2(x_3-x_1)^3 + m_1(x_3-x_2)^2(x_2-x_1)^3\,. \tag{42}$$

Particular collinear central configurations are

$$\frac{x_2-x_1}{x_3-x_2} = 2\,,\quad m_1 = 34 \quad m_2 = 10\,,\quad m_3 = 5\,.$$

$$\frac{x_2-x_1}{x_3-x_2} = \frac{3}{2}\,,\quad m_1 = 105 \quad m_2 = 84\,,\quad m_3 = 20\,.$$

6 Four-Body planar solutions

Write equation (24) divided by the product of masses $m_l m_j$ as

$$\frac{e_l e_j}{m_l m_j}\frac{1}{r_{lj}^3} = \sigma + \lambda A_l A_j\,, \tag{43}$$

where $A_j = S_j/m_j$ denotes, as in [8], weighted area. Noting that the square of $\frac{e_l e_j}{m_l m_j}$ equals positive one, this equation may also be written as

$$\left(\frac{e_l e_j}{m_l m_j}\frac{1}{r_{lj}}\right)^3 = \sigma + \lambda A_l A_j\,, \tag{44}$$

A similar equation is the basic tool to compute central configurations in reference [8] from the given weighted areas A_j. The difference is the replacement of distance r_{lj} by a sort of charged distance

$$r_{lj} \quad \longrightarrow \quad \frac{m_l m_j}{e_l e_j} r_{lj} \,. \tag{45}$$

The algorithm presented in [8] is also useful for the present case if allowance is made for these positively or negatively charged distances in the numerical computation. In that paper the algorithm to compute planar central configurations starting from the weighted areas was applied to several examples with positive charges. Since the areas of the triangles having its vertices at the positions of the particles were computed by Heron's formula in terms of the square of this distances, they are not modified by the extra factor $(m_l m_j)/(e_l e_j)$. At the end of the present paper the modified algorithm is used to obtain some numerical examples of central configurations combining it with the new algorithm recently proposed to compute central configurations from given masses [9] based on a new system of coordinates [10] which will be described in the next section. We refer the reader to those references for a more detailed account.

Substracting term by term equation (43) with subscripts l, j and l, k, and with subscripts n, j and n, k, with all subscripts different, yields

$$\frac{e_l e_j}{m_l m_j} \frac{1}{r_{lj}^3} - \frac{e_l e_k}{m_l m_k} \frac{1}{r_{lk}^3} = \lambda A_l (A_j - A_k) \,, \tag{46}$$

$$\frac{e_n e_j}{m_n m_j} \frac{1}{r_{nj}^3} - \frac{e_n e_k}{m_n m_k} \frac{1}{r_{nk}^3} = \lambda A_n (A_j - A_k) \,. \tag{47}$$

Elimination of λ between these two equations gives the fundamental relation

$$S_n e_l \left(\frac{e_j m_k}{r_{lj}^3} - \frac{e_k m_j}{r_{lk}^3} \right) = S_l e_n \left(\frac{e_j m_k}{r_{nj}^3} - \frac{e_k m_j}{r_{nk}^3} \right) \,. \tag{48}$$

The same equation was obtained directly from two of the defining equations for planar central configurations (19). A similar calculation may be found in reference [8].

Theorem 3

A planar central configuration of four particles with one charge opposite in sign to the other three leads to a concave configuration with the different sign charge in the convex hull of the other three.

Proof: Assume that the charge of particle 1 is of opposite sign to the charge of the other three particles. Using equation (48) with $j = 1$, we note that the quantity in the two parenthesis have the same sign and therefore

$$\frac{S_2}{S_3} > 0\,, \quad \frac{S_3}{S_4} > 0\,, \quad \frac{S_4}{S_2} > 0\,. \tag{49}$$

From the above equation we prove with no loss of generality that the sign of the charge of any particle may be made to coincide with the sign of the corresponding area.

$$S_2 > 0\,, \quad S_3 > 0\,, \quad S_4 > 0\,, \quad S_1 < 0\,.\square \tag{50}$$

Theorem 4

A planar central configuration of four particles with two pairs of particles of different sign leads to a convex configuration with charges of the same sign located at the ends of the two diagonals.

Proof: Consider now the case in which particles 1 and 2 have charges of one sign and particles 3 and 4 of the opposite sign. Again using equation (48) with charge j of one sign and charge k of the opposite sign, the quantity in the parenthesis have the same sign, so that

$$\frac{S_1}{S_3} < 0\,, \quad \frac{S_1}{S_4} < 0\,, \quad \frac{S_2}{S_3} < 0\,, \quad \frac{S_2}{S_4} < 0\,. \tag{51}$$

With no loss of generality we take

$$S_1 > 0\,, \quad S_2 > 0\,, \quad S_3 < 0\,, \quad S_4 < 0\,.\square \tag{52}$$

As a consequence, the quotient S_k/e_k may be considered to be always positive.

Theorem 5

To have a Four-Body central configuration determined by Dziobek-like equation (24), parameter λ is positive.

Proof: Divide both members of equation (24) by the product of charges $e_l e_j$ to yield

$$\frac{1}{r_{lj}^3} = \sigma \frac{m_l m_j}{e_l e_j} + \lambda \frac{S_l S_j}{e_l e_j}\,. \tag{53}$$

Choosing two of these equations, with subscripts say l, j and l, k, such that the term containing σ in each is of opposite sign, adding member by member we obtain

$$\left(\frac{1}{r_{lj}^3} + \frac{1}{r_{lk}^3}\right) = \lambda \left(\frac{S_l S_j}{e_l e_j} + \frac{S_l S_k}{e_l e_k}\right)\,. \tag{54}$$

Since the quantities in the parentheses are both positive, we have proved that

$$\lambda > 0\,. \quad \square \tag{55}$$

7 New coordinates in the Four-Body Problem

This section reviews the main ideas and results of the new four-body coordinates of [10], slightly expanded at a few spots but condensed to the minimum necessary.

We transform from the inertial referential system to the frame of principal axes of inertia by means of a three dimensional rotation $\mathbf{G}$ parameterized by three independent coordinates. In addition to this rotation, three more coordinates are introduced, as scale factors R_1, R_2, R_3, which are three directed distances closely related to the three principal moments of inertia through

$$I_1 = \mu(R_2^2 + R_3^2)\,, \quad I_2 = \mu(R_3^2 + R_1^2)\,, \quad \text{and} \quad I_3 = \mu(R_1^2 + R_2^2)\,, \tag{56}$$

where μ is the mass

$$\mu = \sqrt[3]{\frac{m_1\, m_2\, m_3\, m_4}{m_1 + m_2 + m_3 + m_4}}\,. \tag{57}$$

The first rotation changes nothing and after the scale change the resulting four-body configuration has a moment of inertia tensor with the three principal moments of inertia equal. The second rotation $\mathbf{G}'$ does not change this property.

The cartesian coordinates of the four particles, with the origin at the center of gravity, written in terms of the new coordinates are

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix} = \mathbf{G} \begin{pmatrix} R_1 & 0 & 0 \\ 0 & R_2 & 0 \\ 0 & 0 & R_3 \end{pmatrix} \mathbf{G}'^{\mathrm{T}} \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix}, \tag{58}$$

where $\mathbf{G}$ and $\mathbf{G}'$ are two rotation matrices, each a function of three independent coordinates such as the Euler angles, and where the column elements of the constant matrix

$$\mathbf{E} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix}, \tag{59}$$

are the coordinates of the four vertexes of a rigid orthocentric tetrahedron [11], having its center of mass at the origin of coordinates, namely:

$$\begin{array}{c} a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 = 0\,, \\ b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 = 0\,, \\ c_1 m_1 + c_2 m_2 + c_3 m_3 + c_4 m_4 = 0\,, \end{array} . \tag{60}$$

We introduce the mass matrix

$$\mathbf{M} = \begin{pmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{pmatrix}. \tag{61}$$

An equivalent condition for having three equal moments of inertia for the rigid tetrahedron is expressed as

$$\mathbf{E}\,\mathbf{M}\,\mathbf{E}^{\mathrm{T}} = \mu \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{62}$$

The system of coordinates for measuring the $\mathbf{G}'$ rotation can be chosen in various ways, from which we prefer to use the same coordinates as in reference

[9], namely, particle with mass m_1 along coordinate axis 3, the other three in a plane parallel to the coordinate plane containing axes 1 and 2 but that does not include the particle of mass m_1; the particle with mass m_2 on an orthogonal coordinate plane that contains the first particle and the center of mass, and the other two particles on a line that is parallel to coordinate axis 1 and perpendicular to the coordinate plane containing the first two particles. Particle 1 thus has coordinates

$$(a_1, b_1, c_1) = \left(0, 0, \sqrt{\frac{\mu(m - m_1)}{m_1 m}}\right) . \tag{63}$$

Particle 2 has coordinates

$$(a_2, b_2, c_2) = \left(0, \sqrt{\frac{\mu(m_3 + m_4)}{m_2(m - m_1)}}, -\sqrt{\frac{\mu m_1}{(m - m_1)m}}\right) . \tag{64}$$

Particle 3 has coordinates

$$(a_3, b_3, c_3) =$$

$$\left(\sqrt{\frac{\mu m_4}{m_3(m_3 + m_4)}}, -\sqrt{\frac{\mu m_2}{(m_3 + m_4)(m - m_1)}}, -\sqrt{\frac{\mu m_1}{(m - m_1)m}}\right) . \tag{65}$$

Particle 4 has coordinates

$$(a_4, b_4, c_4) =$$

$$\left(-\sqrt{\frac{\mu m_3}{m_4(m_3 + m_4)}}, -\sqrt{\frac{\mu m_2}{(m_3 + m_4)(m - m_1)}}, -\sqrt{\frac{\mu m_1}{(m - m_1)m}}\right) . \tag{66}$$

Note that $b_3 = b_4$ and $c_2 = c_3 = c_4$, as they should.

This rigid tetrahedron is the generalization of the rigid triangle of the Three-Body problem with the center of mass at the orthocenter discussed previously in [12]. The same triangle was used with different purposes by C. Simo [13].

Our coordinates are now adapted to the important and old subject [7] of planar configurations, with the four particles in a constant plane. Since the z-component of the four particles equals zero, the first rotation is just by one angle in the plane of motion and the scale factor associated with the third coordinate is zero, namely

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ 0 & 0 & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} R_1 & 0 & 0 \\ 0 & R_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{G}'^{\mathrm{T}} \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix} . \tag{67}$$

This equation simplifies to

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} =$$

$$\begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} R_1 & 0 & 0 \\ 0 & R_2 & 0 \end{pmatrix} \mathbf{G}'^{\mathrm{T}} \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix} , \tag{68}$$

in terms of six degrees of freedom.

The corresponding expression for the four directed areas in terms of these coordinates is

$$\begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{pmatrix} = C\mathbf{M}\mathbf{E}^{\mathrm{T}}\mathbf{G}' \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} , \tag{69}$$

where C is a constant with units of area divided by mass. Note that Equations (25) are satisfied from this expression of the directed areas since substitution of equations (68) and (69) in equations (25), and application of equation (62), yields an identity, independent of coordinates.

For $\mathbf{G}'$ two rotation angles are needed, for which we chose those required to express the unit vector in spherical coordinates

$$\mathbf{G}' \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} , \tag{70}$$

where θ and ϕ are the spherical coordinates determining this vector. Given the four masses, the four directed areas are functions, up to a multiplicative constant C depending on the choice of physical units, of this unit vector direction only.

The non-collinear planar central configurations are characterized in our coordinates by constant values of the $\mathbf{G}'$ matrix and of the constant value of the ratio R_1/R_2, which are not arbitrary, but they are determined by three independent quantities as discussed in the following.

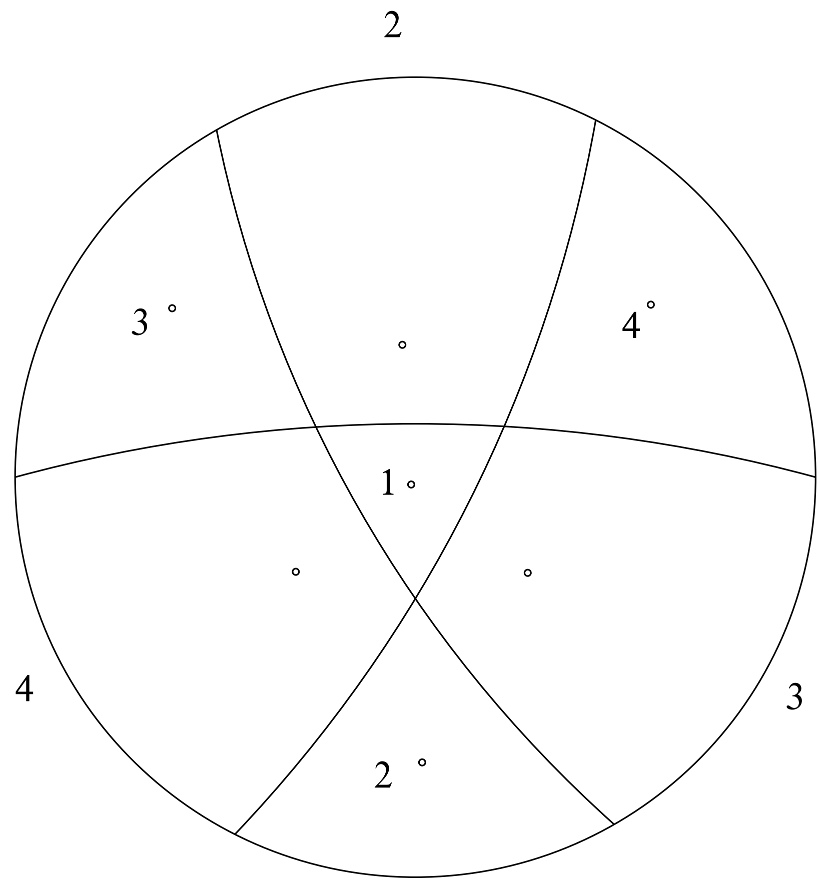

Figure 1: Stereographic projection of the hemisphere of the two angles motion of the orthocentric tetrahedron. The great circles represent the positions where three particles are collinear. The four spherical triangles are concave open sets labeled by the particle at the interior of the triangle. The spherical quadrilateral open sets correspond to convex configurations with the same order that the neighboring triangles. The isolated points are at the angles where a charged central configuration has been computed. The values of the masses are $m_1 = 10, m_2 = 13, m_3 = 15, m_4 = 17$. The sign of the charges is opposite for the position inside a concave spherical triangle set and apposite for the two particles along each diagonal when the particle is inside a rectangular convex region.

From (69) follows that in a planar solution the weighted directed areas are expressed as

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = C\mathbf{E}^{\mathrm{T}}\mathbf{G}' \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = C\mathbf{E}^{\mathrm{T}} \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} . \tag{71}$$

Therefore, the weighted directed areas are up to a normalization factor equal to the third rotated coordinate of the rigid tetrahedron. In terms of the vectors (63-66) and the angles θ and ϕ this equation is expressed simply as

$$Aj = C(a_j \sin\theta\cos\phi + b_j \sin\theta\sin\phi + c_j \cos\theta) . \tag{72}$$

Choosing $C = \sqrt{(m - m_1)/\mu}$ we have explicitly

$$A_1 = \frac{m - m1}{\sqrt{m_1 m}} \cos\theta ,$$

$$A_2 = -\sqrt{\frac{m_1}{m}} \cos\theta + \sqrt{\frac{m_3 + m_4}{m_2}} \sin\theta\sin\phi ,$$

$$A_3 = -\sqrt{\frac{m_1}{m}} \cos\theta - \sqrt{\frac{m_2}{m_3 + m_4}} \sin\theta\sin\phi + \sqrt{\frac{m_4(m - m_1)}{m_3(m_3 + m_4)}} \sin\theta\cos\phi$$

$$A_4 = -\sqrt{\frac{m_1}{m}} \cos\theta - \sqrt{\frac{m_2}{m_3 + m_4}} \sin\theta\sin\phi - \sqrt{\frac{m_3(m - m_1)}{m_4(m_3 + m_4)}} \sin\theta\cos\phi . \tag{73}$$

Note that a sign change of the unit vector (47) produces a simultaneous sign change in these four A_j's which does not to give a different central configuration. Therefore, it suffices to consider only the hemisphere $0 \leq \theta \leq \pi/2$.

8 Computing central configurations for positive and negative charges

In the case of positive charges, since the lengths and masses are defined up to arbitrary units, with no loss of generality we assumed in [8] that the parameter σ equals plus one. However, central configurations with charges of different sign obtained numerically always yield a negative σ, so that for those cases we used $\sigma = -1$ as follows

$$\frac{e_j\, e_k}{m_j\, m_k}\, r_{jk}^{-3} = -1 + \lambda A_j A_k \quad (j \neq k). \tag{74}$$

We recall that in the paper by Piña and Lonngi [8] the assumption that the directed weighted areas are known as four given constants was made. The equation which corresponds to (51) then gives the distances as functions of the unknown parameter λ. Through them, the areas of the four triangles become functions of λ, that should satisfy restrictions (25) for a planar solution. These restrictions allow in many cases to determine the value of λ and hence the values of the six distances and the four masses. This is an implicit way to deduce planar central configurations with four masses.

In contrast, in this paper, as well as in reference [9], we assume that the four masses are known from the beginning. The four weighted areas are then determined by expressions (73) in terms of the two tuning variables θ and ϕ. Particular values of these two angles determine the four constants A_j, up to a multiplicative factor, which in turn produce a central configuration with computed distances and masses. The computed masses are in general not equal (or proportional) to the starting values used to build the orthocentric tetrahedron. The two angles are then tuned until a numerical match is produced between the given and the computed masses. The distances between particles, computed for this central configuration, correspond to the given masses.

For the arbitrarily chosen values of the masses $m_1 = 10, m_2 = 13, m_3 = 15, m_4 = 17$ seven central configurations were found in [1].

We remark that, with the same values of the masses, the seven central configurations found in the case of positive values of the charges [9], reduces to one central configuration of different distribution of charges for each corresponding case. In all the cases the angles are different from the values previously computed in [9].

9 Collinear Four-Body central configurations

The collinear case is very similar to the planar case. Assume the particles are ordered in the line with coordinates: $r_1 < r_2 < r_3 < r_4$.In this case the

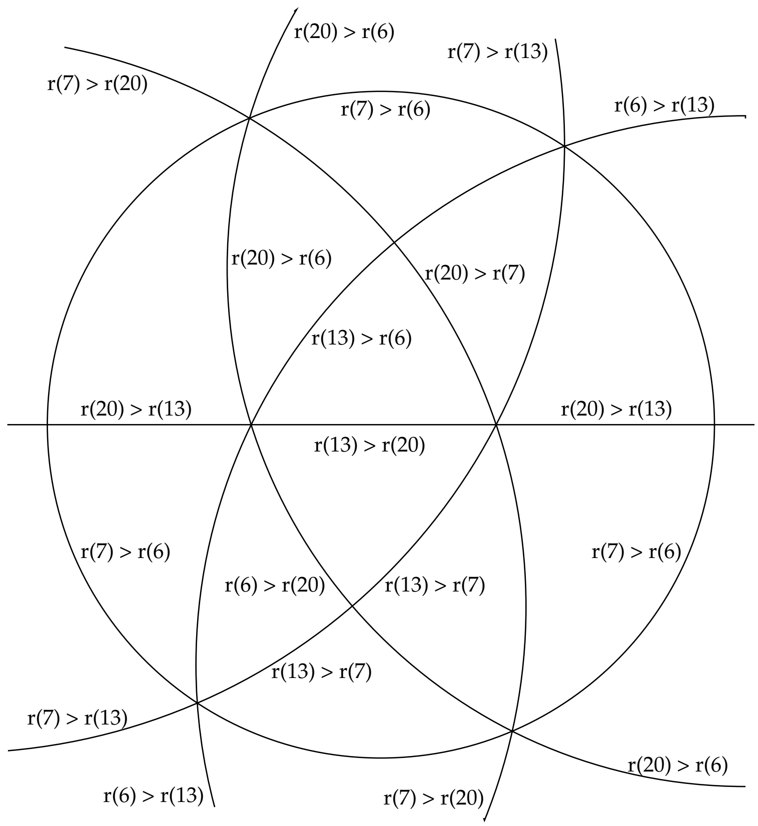

Figure 2: The set of the four-body collinear configurations in the central circle by the stereographic projection of the hemisphere of the two angles position of the orthocentric tetrahedron. The great circles represent the r_i positions where two particles have the same coordinate. Each point on the hemisphere represent a different collinear configuration. The particular permutation is determined by the inequalities of the coordinates associated to two masses on both sides of the great circles. Each of the interior of the twelve spherical triangles represent the set of different collinear configurations with the same permutation order. The values of the masses are $m_1 = 20, m_2 = 13, m_3 = 7, m_4 = 6$. The inequalities between coordinates in figure are labeled with the numerical value of the corresponding masses. To simplify compilation the r's in figure are not italic fonts.

configuration is determined by three distances between particles $a = r_2 - r_1$, $b = r_3 - r_2$, $c = r_4 - r_3$. Using the fact that the coordinates are defined with respect to the center of mass: $m_1 r_1 + m_2 r_2 + m_3 r_3 + m_4 r_4 = 0$, it is possible to write the positions r_j in terms of the masses and the distances a, b, c

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = \begin{pmatrix} 0 & -a & -(a+b) & -(a+b+c) \\ a & 0 & -b & -(b+c) \\ a+b & b & 0 & -c \\ a+b+c & b+c & c & 0 \end{pmatrix} \begin{pmatrix} \frac{m_1}{m} \\ \frac{m_2}{m} \\ \frac{m_3}{m} \\ \frac{m_4}{m} \end{pmatrix} . \tag{75}$$

The 4×4 matrix

$$\mathcal{A} = \begin{pmatrix} 0 & -a & -(a+b) & -(a+b+c) \\ a & 0 & -b & -(b+c) \\ a+b & b & 0 & -c \\ a+b+c & b+c & c & 0 \end{pmatrix} \tag{76}$$

is skew-symmetric with zero determinant. It is expressed in terms of the orthogonal vectors

$$\mathcal{A} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} \begin{pmatrix} -3a/2 - b - c/2 \\ a/2 - b - c/2 \\ a/2 + b - c/2 \\ a/2 + b + 3c/2 \end{pmatrix}^{\mathrm{T}} - \begin{pmatrix} -3a/2 - b - c/2 \\ a/2 - b - c/2 \\ a/2 + b - c/2 \\ a/2 + b + 3c/2 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}^{\mathrm{T}} . \tag{77}$$

Vector with entries r_j, or the parallel $\mathbf{F}$ with entries $\frac{F_j}{m_j G}$, will be perpendicular to two linearly independent vectors orthogonal to the vectors in this equation. Two particular, linearly independent vectors are

$$\mathbf{k}_1 = \begin{pmatrix} c \\ -c \\ -a \\ a \end{pmatrix} , \quad \mathbf{k}_2 = \begin{pmatrix} 0 \\ c \\ -(b+c) \\ b \end{pmatrix} \tag{78}$$

Theorem 6

It is impossible to have a collinear Four-Body central configuration with a different charge outside the other three charges.

Proof: Assume the different charge is at r_4, then the forces in terms of masses and distances are

$$\frac{F_1}{m_1 G} = \frac{m_2}{a^2} + \frac{m_3}{(a+b)^2} - \frac{m_4}{(a+b+c)^2} \tag{79}$$

$$\frac{F_2}{m_2 G} = -\frac{m_1}{a^2} + \frac{m_3}{b^2} - \frac{m_4}{(b+c)^2} \tag{80}$$

$$\frac{F_3}{m_3 G} = -\frac{m_1}{(a+b)^2} - \frac{m_2}{b^2} - \frac{m_4}{c^2} \tag{81}$$

$$\frac{F_4}{m_4 G} = \frac{m_1}{(a+b+c)^2} + \frac{m_2}{(b+c)^2} + \frac{m_3}{c^2}\,. \tag{82}$$

Interior product with the vector $\mathbf{k}_1 + \mathbf{k}_2$ is always positive, all the negative terms are exactly canceled by positive terms. It can never been zero. □

Theorem 7

It is impossible to have a collinear Four-Body central configuration with two charges of one sign on one side of the line and two charges of opposite sign on the other side.

Proof: Assume charges 1 and 2 are positive and charges 3 and 4 are negative. The forces for this distribution of charges are

$$\frac{F_1}{m_1 G} = \frac{m_2}{a^2} - \frac{m_3}{(a+b)^2} - \frac{m_4}{(a+b+c)^2} \tag{83}$$

$$\frac{F_2}{m_2 G} = -\frac{m_1}{a^2} - \frac{m_3}{b^2} - \frac{m_4}{(b+c)^2} \tag{84}$$

$$\frac{F_3}{m_3 G} = \frac{m_1}{(a+b)^2} + \frac{m_2}{b^2} + \frac{m_4}{c^2} \tag{85}$$

$$\frac{F_4}{m_4 G} = \frac{m_1}{(a+b+c)^2} + \frac{m_2}{(b+c)^2} - \frac{m_3}{c^2}\,. \tag{86}$$

The interior product with the vector $\mathbf{k}_2$ is always positive definite. It can never been zero. □

Theorem 8

It is impossible to have a collinear Four-Body central configuration with two charges of one sign inside the two charges of opposite sign.

Proof: Assume charges 1 and 4 are with different sign than charges 2 and 3. The forces for this distribution of charges are

$$\frac{F_1}{m_1 G} = -\frac{m_2}{a^2} - \frac{m_3}{(a+b)^2} + \frac{m_4}{(a+b+c)^2} \quad (87)$$

$$\frac{F_2}{m_2 G} = \frac{m_1}{a^2} + \frac{m_3}{b^2} - \frac{m_4}{(b+c)^2} \quad (88)$$

$$\frac{F_3}{m_3 G} = \frac{m_1}{(a+b)^2} - \frac{m_2}{b^2} - \frac{m_4}{c^2} \quad (89)$$

$$\frac{F_4}{m_4 G} = -\frac{m_1}{(a+b+c)^2} + \frac{m_2}{(b+c)^2} + \frac{m_3}{c^2}\,. \quad (90)$$

The interior product with the vector $\mathbf{k}_1 + \mathbf{k}_2$ is always positive definite. It can never been zero. □

The possibility of collinear central configuration of gravitational charges of both signs is discovered by numerical computation of several cases, by means of the algorithm to be published in [14]. Many central configuration with two particles of one charge and the other with different charge have been obtained when the sign of the charges alternates in the line: r_1 and r_3 of one sign, r_2 and r_4 of the opposite sign. Collinear central configurations with one charge different of the other has been computed in several cases when the different charge is located between two of the opposite charged particles. The presence of one different charge perturb the positions of the three others which expand when the value of the different charge increases.

Appendix

In the mathematical literature we have found some papers in the context of determining Four-Body central configurations that consider negative masses [15], [16], [17], [18], with a different equation of motion, a different definition of central configuration and without distinguishing between masses and charges. The difference with respect to the differential equation of motion used in those papers is stressed in this Appendix, where assuming the validity of Newton's second and third laws, the equation of motion used by those Authors is shown to produce, for a system of two masses of equal magnitude but different inertial and gravitational sign, a rigid body which is self-accelerated with no external force, violating Newton's first law.

In this paper we assume an equation of motion for the particles different from that used in references [15], [16], [17], [18]. We will show that their equation of motion applied to two particles of opposite charge, leads to a contradiction with Newton's first law of motion: under no external force, a body moves with a constant velocity vector.

To help distinguish clearly, we continue using the notation e_j to denote the charge with two possible choices of sign which these Authors denote with m_j. Their equation of motion is

$$\frac{d^2\mathbf{r}_1}{dt^2} = \frac{Ge_2}{r_{12}^3}(\mathbf{r}_2 - \mathbf{r}_1)\,, \quad \frac{d^2\mathbf{r}_2}{dt^2} = \frac{Ge_1}{r_{12}^3}(\mathbf{r}_1 - \mathbf{r}_2)\,, \tag{91}$$

which differ from equation (1) due to substitution of the mass m_j by the charge e_j, and that only two particles of opposite charge $e_1 = -e_2$ are considered.

The product of the first times e_1, of the second times e_2 and adding member by member we obtain instead of zero acceleration of the center of mass, zero acceleration of the relative position

$$e_1\frac{d^2(\mathbf{r}_2 - \mathbf{r}_1)}{dt^2} = \mathbf{0}\,, \tag{92}$$

which is trivially integrated in terms of two constant vectors of integration $\mathbf{A}$ and $\mathbf{B}$

$$\mathbf{r}_2 - \mathbf{r}_1 = \mathbf{A} + \mathbf{B}t\,. \tag{93}$$

Assume the initial condition $\mathbf{B} = \mathbf{0}$. Then the relative distance is constant

$$\mathbf{r}_2 - \mathbf{r}_1 = \mathbf{A}\,, \tag{94}$$

as if forming a rigid body. The two particles are however accelerated with the same constant acceleration

$$\frac{d^2\mathbf{r}_1}{dt^2} = \frac{d^2\mathbf{r}_2}{dt^2} = \frac{Ge_2}{|\mathbf{A}|^3}\mathbf{A}\,, \tag{95}$$

which contradicts Newton's first law since no external force is acting on this composite body.

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