Number of invariant straight lines for homogeneous polynomial vector fields of arbitrary degree and dimension

J. Llibre[1], V. Pilyugina[2]

[1] UAB, Barcelona, Catalonia; [2] SPSU, St.Petersburg, Russia — 2012

Introduction

For $d > 0$ we consider in $\mathbb{R}^d$ (respectively $\mathbb{C}^d$) the *real (respectively complex) polynomial differential system of degree* m:

$$\begin{aligned} \dot{x_1} &= P^1(x_1,\dots,x_d), \\ &\cdots\cdots \\ \dot{x_d} &= P^d(x_1,\dots,x_d), \end{aligned} \tag{1}$$

where $m \geq 1$ is the maximum of the degrees of the polynomials P_i for $i = 1,\dots,d$.

The *polynomial vector field* X *of degree* m in $\mathbb{F}^d$ associated to system (1) is

$$X(x_1,\dots,x_d) = \sum_{i=1}^{d} P^i(x_1,\dots,x_d)\frac{\partial}{\partial x_i}, \tag{2}$$

where $\mathbb{F}$ is either $\mathbb{R}$, or $\mathbb{C}$.

In the particular case that all polynomials P^i are homogeneous of degree m, then we say that the polynomial differential system (1) or its associated polynomial vector field X is *homogeneous* of degree m.

The main objective of our paper is to study the number of invariant straight lines through the origin taking in account their multiplicities that a homogeneous polynomial vector field X can exhibit, of course when this number is finite. The main result in this direction is stated in Section Main Theorem. As an application of this main result we get information on the number of infinite singular points of the polynomial vector field of $\mathbb{R}^d$, see Propositions 3 and 4.

Let X be a homogeneous polynomial vector field of degree $m \geq 1$ in $\mathbb{F}^d$ with $d \geq 1$. We say that the complex straight line through the origin of $\mathbb{C}^d$ with director vector $v \in \mathbb{C}^d \setminus \{0\}$ is *invariant* by the flow of X, if $X(v) = \lambda v$ for some $\lambda \in \mathbb{F}$. Let $(x_1,\dots,x_d) \in \mathbb{C}^d \setminus \{0\}$ be a director vector of an invariant straight line through the origin of X. Then, either there exists $\lambda \neq 0$ such that

$$P^i(x_1,\dots,x_d) = \lambda x_i, \quad \text{for } i = 1,\dots,d, \tag{3}$$

if the straight line is not formed by singular points; or

$$P^i(x_1,\dots,x_d) = 0, \quad \text{for } i = 1,\dots,d, \tag{4}$$

otherwise. We shall work with vector fields X such that the number of solutions of systems (3) and (4) is finite, of course proportional solutions are counted as one.

This definition generalizes the notion of invariant straight line associated to an eigenvector of a homogeneous linear differential system (i.e. a homogeneous polynomial differential system of degree 1) in $\mathbb{F}^d$ to a homogeneous polynomial differential system of degree m in $\mathbb{F}^d$. Note that we always work with complex invariant straight lines independently if the homogeneous polynomial differential system (1) is real or complex.

1. Main Theorem

Let $\mathcal{X}_{d,m}$ be the space of all homogeneous polynomial vector fields X of degree $m \geq 1$ in $\mathbb{F}^d$ with $d \geq 1$. We consider in $\mathcal{X}_{d,m}$ the *topology of the coefficients*; i.e. we identify the space $\mathcal{X}_{d,m}$ with the space $\mathbb{F}^N$ where N is the maximum number of coefficients that a homogeneous polynomial vector field of degree m in $\mathbb{F}^d$ can have. We define subspace $\mathcal{X}^{**}_{d,m}$ of $\mathcal{X}_{d,m}$ with induced topology by $\mathcal{X}_{d,m}$ which is an open and dense subset of $\mathcal{X}_{d,m}$ (see for more details [2]). So the vector field of $\mathcal{X}^{**}_{d,m}$ are generic in the set of all vector fields $\mathcal{X}_{d,m}$.

Theorem 1. *Let* X *be a vector field of* $\mathcal{X}^{**}_{d,m}$ *, and let* $N_{d,m}$ *be the number of its invariant straight lines through the origin, counted with multiplicities . Then*

$$N_{d,m} = \sum_{k=0}^{d-1} m^k = 1 + m + m^2 + ... + m^{d-1}$$

Consider as examples well known systems:

Example 1. *For linear systems* $N_{d,1} = \sum_{k=0}^{d-1} 1^k = d$

Example 2. *For one equation* $N_{1,m} = \sum_{k=0}^{0} m^k = 1$

Example 3. *For two equations* $N_{2,m} = \sum_{k=0}^{1} m^k = 1 + m$

In [2] we stated the following open question: the Theorem must also hold under the non-generic conditions.

Later on this open question has been answered in positive by Feng Rong [6].

Since $N_{m,d} = 1 + m + m^2 + \cdots + m^{d-1}$ is odd when m is even or d is odd, and the complex (non–real) solutions of a system of real polynomials appear by pairs (one solution and its conjugated), from Theorem 1 it follows immediately the following result.

Corollary 2. *Let* X *be a real homogeneous polynomial vector field (2) of degree* $m \geq 1$ *in* $\mathbb{R}^d$ *with* $d \geq 1$ *having finitely many invariant straight lines* $N_{m,d}$ *through the origin taking into account their multiplicities. Then generically* X *has some real invariant straight line passing through the origin of* $\mathbb{R}^d$ *when* m *is even or* d *is odd.*

2. Proof of Main Theorem

The key tool in the proof of Theorem 1 is the *Bezout Theorem* in the complex projective space $\mathbb{C}P^d$, see [3] and mainly [1] p. 198. More precisely, the number of solutions of a system

$$F_1(x_0,\dots,x_d) = 0, \quad \dots , \quad F_d(x_0,\dots,x_d) = 0,$$

of d homogeneous polynomial equations in $d+1$ unknowns is either infinite or equal to the product of the degrees, provided that their solutions are counted with their intersection numbers or multiplicities. Of course only non–zero solutions are considered, and proportional solutions are counted as one.

So without loss of generality we can assume that all the solutions $(x_1,\dots,x_d) \in \mathbb{C}^d \setminus \{0\}$ of systems (3) and (4) satisfy that $x_i \neq 0$ for all $i = 1,\dots,d$; otherwise we do a rotation of $SO(d)$ to the homogeneous polynomial vector field X. Therefore, the director vectors $(x_1,\dots,x_d)$ of the invariant straight lines satisfying systems (3) and (4) must verify

$$\frac{P^1(x_1,\dots,x_d)}{x_1} = \cdots = \frac{P^d(x_1,\dots,x_d)}{x_d}. \tag{5}$$

Hence, we can assume that the director vectors $(x_1,\dots,x_d) \in \mathbb{C}^d \setminus \{0\}$ of the invariant straight lines of the homogeneous polynomial vector field (2) must satisfy the system

$$\begin{aligned} x_iP^j(x_1,\dots,x_d) - x_jP^i(x_1,\dots,x_d) = 0, &\ \text{for } 1 \leq i < j \leq d, \\ x_i \neq 0, &\ \text{for } i = 1,\dots,d, \end{aligned} \tag{6}$$

in both cases (3) and (4). Since we are interested in the solutions of system (6) having all their coordinates different from zero, we shall put special attention to the solutions of the subsystem

$$\begin{aligned} x_iP^{i-1}(x_1,\dots,x_d) - x_{i-1}P^i(x_1,\dots,x_d) = 0, &\ \text{for } i = 2,\dots,d, \\ x_i \neq 0, &\ \text{for } i = 1,\dots,d. \end{aligned} \tag{7}$$

We assume that X is a homogeneous polynomial vector field of degree $m \geq 1$ in $\mathbb{F}^d$ with $d \geq 1$ having finitely many invariant straight lines all of them with director vector without any zero component. In other words, for such vector fields we want to study the solutions of the homogeneous polynomial system (7) having all their components non–zero taking into account their multiplicities, for a definition of multiplicity see [1]. For doing this study a key point will be to control the solutions of system (7) with their multiplicities having some zero component.

For studying the solutions of system (7) we will apply Bezout Theorem, considering the first set of equations of system (7) defined in $\mathbb{C}P^{d-1}$, and we want to compute its solutions in $\mathbb{C}P^{d-1}$ having all their components different from zero.

Of course, for $d = 1$ Theorem 1 is trivial because the phase space of these systems is the straight line $\mathbb{F}$ through the origin.

For $d = 2$ the result is not new, see for instance [4], nevertheless we give here the proof in *the case* $d = 2$. For $d = 2$ the first system of (7) reduces to

$$x_2P^1(x_1, x_2) - x_1P^2(x_1, x_2) = 0. \tag{8}$$

So, by the Fundamental Theorem of Algebra this homogeneous polynomial equation of degree $m+1$ has $m+1$ solutions in $\mathbb{C}P^1$.

When $x_1 = 0$ equation (8) becomes $x_2P^1(0, x_2) = 0$. We shall see that this equation has no solutions in $\mathbb{C}P^1$. Indeed x_2 cannot be zero because we are in $\mathbb{C}P^1$, so $P^1(0, x_2) = 0$. This means that x_1 is a factor of $P^1(x_1, x_2)$, and consequently the polynomial of the left hand side of (8) is identically zero when $x_1 = 0$. In other words the straight line with director vector $(0, 1)$ is invariant, but we do not allow that there are invariant straight lines having director vectors with some zero component. In short, we have proved that equation has no solutions with $x_1 = 0$.

In a similar way we can prove that equation (8) has no solutions in $\mathbb{C}P^1$ with $x_2 = 0$. Therefore, it follows that the $m+1$ solutions of (8) have their two components different from zero. In short, statement of Theorem 1 holds for $d = 2$.

Also we prove Theorem 1 for $d = 3, 4, 5$ and provide the steps for proving Theorem 1 for arbitrary $d > 5$. For more details of the proof see [2].

3. Applications

A polynomial vector field X in $\mathbb{R}^d$ can be extended to an analytic vector field $p(X)$ on the closed unit ball of $\mathbb{R}^{d+1}$, in such a way that the interior of this ball is diffeomorphic to $\mathbb{R}^d$ and the boundary of the ball plays the role of the infinity of $\mathbb{R}^d$. This extension is called the Poincaré compactification, for more details see [5]. Using this extension we can talk about the singular points of X at infinity, or simply about the infinite singular points of X. That is, the singular points of $p(X)$ which are in the boundary of the closed ball.

Under the assumptions of Theorem 1 the number $2N_{m,d}$ is equal to the number of infinite singular points of X . More precisely, we have the following two results on the infinite singular points for a polynomial vector field of $\mathbb{R}^d$, the first is for homogeneous polynomial vector field and the second for arbitrary polynomial vector fields.

Proposition 3. *Let* X *be a homogeneous polynomial vector field (2) of degree* $m \geq 1$ *in* $\mathbb{R}^d$ *with* $d \geq 1$ *belonging to* $\mathcal{X}^{**}_{d,m}$*. Then the number* $2N_{m,d}$ *is an upper bound for the number of infinite singular points of* X*. This upper bound is reached if all the invariant straight lines through the origin of* X *are real.*

The results of Proposition 3 can be extended to an arbitrary polynomial vector fields of $\mathbb{R}^d$, not necessarily homogeneous. More precisely, we have the following result.

Proposition 4. *Let* X *be a real polynomial vector field (2) of degree* m *in* $\mathbb{R}^d$*. Let* $P^i_m(x_1,\dots,x_d)$ *be the homogeneous part of degree* m *of the polynomial* $P^i(x_1,\dots,x_d)$*, for* $i = 1,\dots,d$*. Assume that the homogeneous vector field*

$$\sum_{i=1}^{d} P^i_m(x_1,\dots,x_d)\frac{\partial}{\partial x_i}.$$

belongs to $\mathcal{X}^{**}_{d,m}$*. Then the number* $2N_{m,d}$ *for the homogeneous polynomial vector field*

$$\sum_{i=1}^{d} P^i_m(x_1,\dots,x_d)\frac{\partial}{\partial x_i}.$$

is an upper bound for the number of infinite singular points of X*. This upper bound is reached if all the invariant straight lines through the origin of homogeneous part of* X *are real.*

Thank you for your attention

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