On the reversible quadratic polynomial vector fields on $\mathbb{S}^2$

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Abstract

We study a class of quadratic reversible polynomial vector fields on $\mathbb{S}^2$ with $(3,2)$-type reversibility. We classify all isolated singularities and we prove the nonexistence of limit cycles for this class. Our study provides tools to determine the phase portrait for these vector fields.

Introduction

A *polynomial vector field* X in $\mathbb{R}^3$ is a vector field of the form

$$X = P(x,y,z)\frac{\partial}{\partial x} + Q(x,y,z)\frac{\partial}{\partial y} + R(x,y,z)\frac{\partial}{\partial z}, \qquad (1)$$

where P, Q and R are polynomials in the variables x, y and z with real coefficients. We denote $m = \max\{\deg P, \deg Q, \deg R\}$ the *degree* of the polynomial vector field X. In what follows, X will denote the above polynomial vector field.

Let $\mathbb{S}^2$ be the 2–dimensional sphere $\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$. A *polynomial vector field X on* $\mathbb{S}^2$ is a polynomial vector field in $\mathbb{R}^3$ such that restricted to the sphere $\mathbb{S}^2$ defines a vector field on $\mathbb{S}^2$; i.e. it must satisfy the equality

$$xP(x,y,z) + yQ(x,y,z) + zR(x,y,z) = 0, \qquad (2)$$

for all points (x,y,z) of the sphere $\mathbb{S}^2$.

The vector field (1) is called *time-reversible* if there is a smooth involution $\phi : \mathbb{R}^3 \to \mathbb{R}^3$, i.e. $\phi \circ \phi = id$, satisfying the relation

$$X(\phi(x,y,z)) = -d\phi(x,y,z)X(x,y,z), \qquad (x,y,z) \in \mathbb{R}^3. \qquad (3)$$

In particular, if the dimension of the fixed point set of ϕ, Fix$\{\phi\}$, is equal to k, then (1) is said to be of $(3,k)$-*type reversibility*. It is clear that $0 \le k < 3$ (see [2]).

Various types of reversible systems have been investigated for many authors. For example, in [12] all $(2,1)$-type reversible systems are classified, in [3] $(2,0)$-type, and in [9] $(3,2)$-type. In [4], there is an exploration on $(3,1)$-type reversible vector fields having a nilpotent linear part.

By (3), if a quadratic polynomial vector field on $\mathbb{R}^3$ is of $(3,0)$-type reversibility and has a linear involution, then it is a homogeneous polynomial vector field on $\mathbb{R}^3$. The quadratic homogeneous polynomial vector fields on $\mathbb{S}^2$ have been studied in [6, 7, 8, 11]. In these papers, the main problems in the qualitative theory of ordinary differential equations, like determination of limit cycles, bifurcations and center–focus problem, are studied. More precisely, the authors solve the center-focus problem, study the Hopf bifurcation and give a topological classification of all global phase portraits to this kind of vector fields modulo limit cycles. The next step in this direction is study the quadratic polynomial vector fields on $\mathbb{S}^2$ of $(3,k)$-*type reversibility* with a linear involution and $k = 1, 2$. In [10] the case $(3,1)$-type was studied, the authors determined the global phase portraits of this class and the main question in qualitative theory was resolved.

In this paper we study quadratic polynomial vector fields on $\mathbb{S}^2$ of $(3,2)$-type reversibility with a linear involution.

Main results

The main results of this paper are the following ones.

The first theorem give us the general expression of the quadratic polynomial vector fields on $\mathbb{S}^2$ of $(3,2)$-type reversibility with a linear involution.

Theorem 1. *Let X be a quadratic polynomial vector field on $\mathbb{R}^3$. Then X is a polynomial vector field on $\mathbb{S}^2$ of $(3,2)$-type reversibility with a linear involution if and only if the system associate to X can be written as*

$$\begin{aligned}
\dot{x} = P(x,y,z) &= a_1 z + a_2 xz + a_3 yz,\\
\dot{y} = Q(x,y,z) &= b_1 z + b_2 xz + b_3 yz,\\
\dot{z} = R(x,y,z) &= c_0 - a_1 x - b_1 y - (a_2 + c_0)x^2 - (a_3 + b_2)xy\\
&\quad -(b_3 + c_0)y^2 - c_0 z^2.
\end{aligned} \qquad (4)$$

We call the singularities of system (4) on equator $\mathbb{S}^1 = \mathbb{S}^2 \cap \{z = 0\}$ of *nonsymmetric singularities* and the singularities which do not belongs to $\mathbb{S}^1$ of *symmetric singularities*.

The next two theorems characterizes the symmetric and nonsymmetric isolated singularities of system (4), respectively.

Theorem 2. *Let X be the vector field associated to system* (4) *and let p be a symmetric singularity of X. We can assume that $b_1 = 0$. If p is isolated, then we have $a_2b_3 - a_3b_2 \neq 0$ and $(a_2b_3 - a_3b_2)^2 - a_1^2(b_2^2 + b_3^2) > 0$. Moreover p can be either a node, a focus, a saddle or a center (see Figures 1, 2, 3, 4).*

Theorem 3. *Let X be the vector field associated to system* (4) *and let p be a nonsymmetric isolated singularity of X. We can assume that $p = (1,0,0)$, i.e. $a_1 = -a_2$.*

1. If $(a_3 + b_1 + b_2)(b_1 + b_2) < 0$, then p is a saddle.

2. If $(a_3 + b_1 + b_2)(b_1 + b_2) > 0$, then p is a center.

3. If $b_1 = -b_2$ and $a_3 \neq 0$, we denote $\lambda = (b_3 - a_2)^2 + 4b_2a_3$ and $\beta = b_3^2 + a_3b_2$, then
- *p is a saddle when $a_3b_2 - a_2b_3 > 0$;*
- *when $a_3b_2 - a_2b_3 < 0$ and $a_2 + b_3 \neq 0$, p is either a center if $\lambda < 0$, or a singularity with a elliptic sector and a hyperbolic sector if $\lambda \ge 0$ (see Figure 5);*
- *when $b_3 = -a_2$, p is either a center if $a_2^2 + a_3b_2 < 0$ or a saddle if $a_2^2 + a_3b_2 > 0$.*

4. If $a_3 = -(b_1 + b_2) \neq 0$, we have that
- *p is a cusp when $a_2 - 2b_3 \neq 0$ (see Figure 6);*
- *to $a_2 = 2b_3$, p is either a saddle when $\dfrac{b_1}{2b_1 + b_2} < 0$ or a center when $\dfrac{b_1}{2b_1 + b_2} > 0$, or a cusp when $b_1 = 0$ and $b_3 \neq 0$.*

5. If $a_3 = 0$, $b_2 = -b_1$, we have that
- *p is a cusp when $a_2(2b_3 - a_2) < 0$;*
- *p is a singularity with two elliptic sector when $a_2(2b_3 - a_2) > 0$ (see Figure 7);*
- *p is a center when $a_2 = 0$, $b_1 \neq 0$ and $b_3 = 0$;*
- *p is a center when $a_2 = 2b_3$ and $b_1^2 + b_3^2 \neq 0$.*

The next result give us upper bound for the number of singularities of system (4).

Proposition 1. *Let $X = (P,Q,R)$ be a vector field associate to system* (4)*. Suppose that X has isolated singularities, then it has at most six singularities. Moreover, X has at most two symmetric isolated singularities.*

Let U be an open subset of $\mathbb{R}^2$. Here a nonconstant analytic function $H : U \to \mathbb{R}$ is called a *first integral* of a vector field Y on U if it is constant on all solutions curves $(x(t), y(t))$ of Y on U; i.e. $H(x(t), y(t))$ is constant for all values of t for which the solution $(x(t), y(t))$ is defined in U. Clearly H is a first integral of the vector field Y on U if and only if $YH \equiv 0$ on U.

Consider a polynomial vector field Y on $\mathbb{S}^2$, through the stereographic projection, the vector field Y induces a polynomial vector field on the plane denoted by $\mathcal{P}(Y)$. We say that Y is *integrable on* $\mathbb{S}^2$ if $\mathcal{P}(Y)$ has a first integral.

Theorem 4. *Let X be the vector field associated to system* (4)*, then it is integrable on $\mathbb{S}^2$.*

Theorem 5. *Let X be the vector field associated to system* (4)*, then it does not have a limit cycle on $\mathbb{S}^2$.*

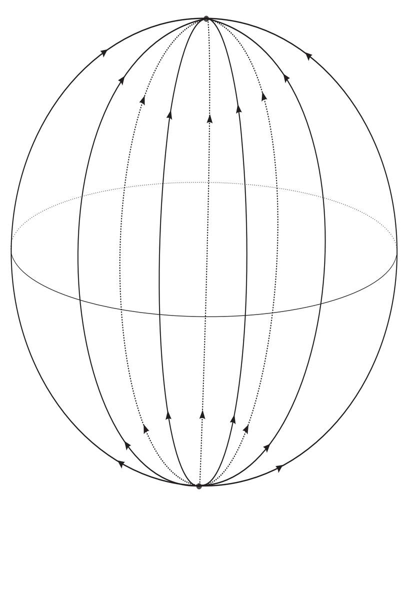

Figure 1: Phase portrait to system (4) with $a_1 = a_3 = 0$, $a_2 = 1$, $b_1 = b_2 = 0$ and $b_3 = 2$.

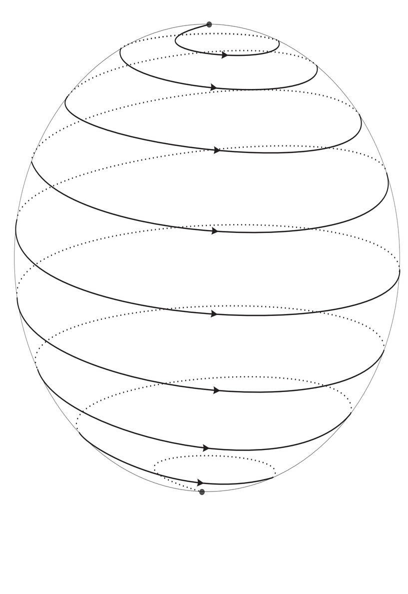

Figure 2: Phase portrait to system (4) with $a_1 = a_3 = 0$, $a_2 = 1$, $b_1 = b_2 = 0$ and $b_3 = 2$.

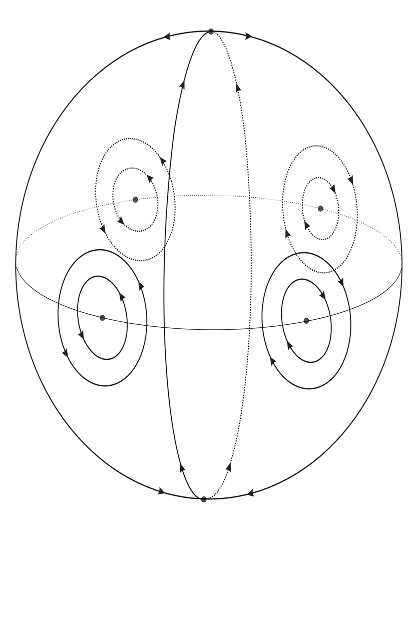

Figure 3: Phase portrait to system (4) with $a_1 = a_3 = 0$, $a_2 = 1$, $b_1 = b_2 = 0$ and $b_3 = 2$.

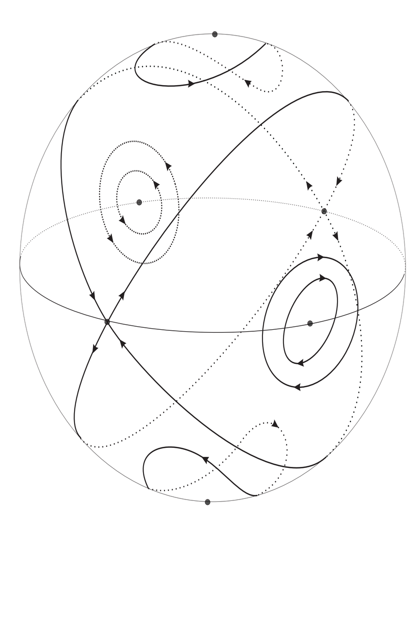

Figure 4: Phase portrait to system (4) with $a_1 = a_3 = 0$, $a_2 = 1$, $b_1 = b_2 = 0$ and $b_3 = 2$.

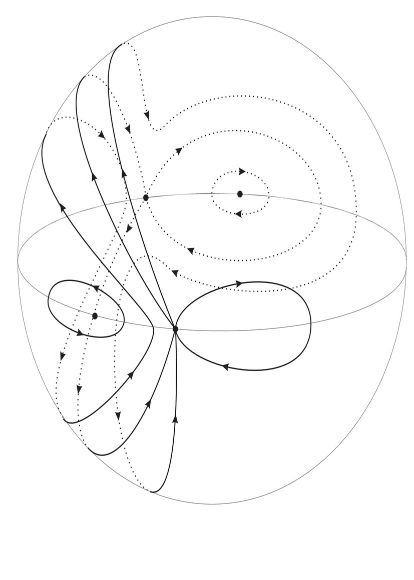

Figure 5: Phase portrait to system (4) with $a_1 = a_3 = 0$, $a_2 = 1$, $b_1 = b_2 = 0$ and $b_3 = 2$.

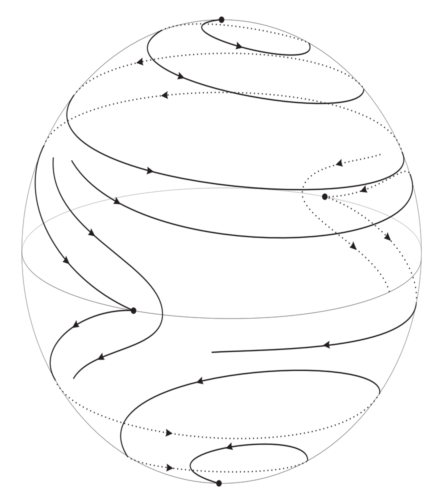

Figure 6: Phase portrait to system (4) with $a_1 = a_3 = 0$, $a_2 = 1$, $b_1 = b_2 = 0$ and $b_3 = 2$.

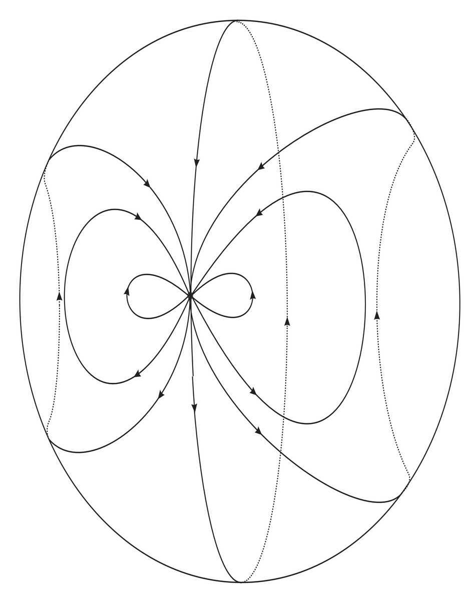

Figure 7: Phase portrait to system (4) with $a_1 = a_3 = 0$, $a_2 = 1$, $b_1 = b_2 = 0$ and $b_3 = 2$.

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This work was supported by: Fundação de Ampara à Pesquisa do Estado de São Paulo - Processo: 2011/13152-8