On the reversible quadratic polynomial vector fields on $S^2$

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Abstract
We study a class of quadratic reversible polynomial vector fields on $S^2$ with $(1,2)$-type reversibility. We classify all isolated singularities and we prove the nonexistence of limit cycles for this class. Our study provides tools to determine the phase portrait for these vector fields.

Introduction
A polynomial vector field $X$ in $\mathbb{R}^2$ is a vector field of the form

$$X = P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y},$$

where $P$ and $Q$ are polynomials in the variables $x$ and $y$ with real coefficients.

We denote $m = \max\{\deg P, \deg Q, \deg R\}$ the degree of the polynomial vector field $X$. In what follows, $X$ will denote the above polynomial vector field.

Let $S^2$ be the 2-dimensional sphere $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. A polynomial vector field $X$ on $S^2$ is a polynomial vector field in $\mathbb{R}^3$ such that restricted to the sphere $S^2$ defines a vector field on $S^2$ i.e. it must satisfy the equality

$$X(x, y, z) = (x, y, z) = 0.$$  

The next two theorems characterize the symmetric and non-symmetric singularities of the polynomial vector field $X$.

Theorem 1. Let $X$ be the vector field associated to system (4) and let $p$ be a non-symmetric isolated singularity of $X$. We can assume that $p = (1, 0, 0)$, i.e. $a_1 = 0$. 

1. If $a_1 + a_2 + b_1 + b_2 < 0$, then $p$ is a saddle.
2. If $a_1 + a_2 + b_1 + b_2 > 0$, then $p$ is a center.

3. If $a_1 - a_2 + b_1 + b_2 > 0$, then $p$ is a saddle when $a_2 - b_2 < 0$; 
   • when $a_2 - b_2 < 0$ and $a_1 + b_1 > 0$, $p$ is either a center if $a_2 - b_2 < 0$ or a saddle with a linear involution and a hyperbolic sector of $\beta > 0$ (see Figure 5); 
   • when $b_1 - a_2 < 0$, $p$ is either a center if $a_2 - b_2 < 0$ or a saddle when $a_2 > b_2$.

4. If $a_1 = (a_2 = b_2 = 0)$, we have that
   • $p$ is a cusp when $a_2 - b_2 > 0$ (see Figure 6); 
   • $p$ is a singularity with two elliptic sectors when $a_2(b_2 - a_2) > 0$ (see Figure 7). 
   • $p$ is a singularity with two elliptic sectors when $a_2(b_2 - a_2) > 0$.

5. If $a_1 = a_2 = b_2 = 0$, we have that
   • $p$ is a cusp when $a_2(b_2 - a_2) < 0$; 
   • $p$ is a singularity with two elliptic sectors when $a_2(b_2 - a_2) > 0$ (see Figure 8).

The next result gives us upper bound for the number of singularities of system (4).

Proposition 1. Let $X = (P(x, y), Q(x, y), R(x, y))$ be a vector field associated to system (4). Suppose that $X$ has isolated singularities, then it has at most six singularities. Moreover, $X$ has at most two symmetric isolated singularities.

We classify all isolated singularities of system (4) and we prove the nonexistence of limit cycles for this class.

Main results
The main results of this paper are the following ones:

The first theorem gives the general expression of the quadratic polynomial vector fields on $S^2$ of $(1, 2)$-type reversibility with a linear involution.

Theorem 2. Let $X$ be a quadratic polynomial vector field on $S^2$. Then $X$ is a polynomial vector field on $S^2$ of $(1, 2)$-type reversibility with a linear involution if and only if the system associated to $X$ can be written as

$$x = P(x, y, z) = a_1 x + a_2 y + a_3 z + a_4 x y + a_5 y z + a_6 z x + a_7 x y z - b_1 x y z - b_2 x y z - b_3 y z - b_4 x + b_5 y + b_6 z$$

$$y = Q(x, y, z) = b_1 x y z + b_2 x y z + b_3 y z + b_4 x + b_5 y + b_6 z + b_7 x y z - b_8 x y - b_9 x y z - b_{10} y z - b_{11} y z - b_{12} x y - b_{13} x y z - b_{14} x y z - b_{15} x y z$$

$$z = R(x, y, z) = c_1 x y z + c_2 x y z + c_3 y z + c_4 x + c_5 y + c_6 z + c_7 x y z - c_8 x y - c_9 x y z - c_{10} y z - c_{11} y z - c_{12} x y - c_{13} x y z - c_{14} x y z - c_{15} x y z$$

We call the singularities of system (4) on equator $S^2 = S^2 \{z = 0\}$ of non-symmetric singularities and the singularities which do not belong to $S^2$ of symmetric singularities.

The next two theorems characterize the symmetric and non-symmetric isolated singularities of (4), respectively.

Theorem 3. Let $X$ be the vector field associated to system (4) and let $p$ be a symmetric isolated singularity of $X$. We can assume that $b_1 = 0$. If $p$ is isolated, then we have $a_2 b_2 - a_2 b_3 \neq 0$ and $a_1 b_1 - a_1 b_2 > 0$. Moreover $p$ can be either a node, a focus, a saddle or a center (see Figures 1, 2, 3, 4).

References

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