Distributional chaos for linear operators

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Theorem (G. D. Birkhoff, 1929)

There is an entire function $f : \mathbb{C} \to \mathbb{C}$ such that, for any entire function $g : \mathbb{C} \to \mathbb{C}$ and for every $a \in \mathbb{C} \setminus \{0\}$, there is a sequence $(n_k)_k$ in $\mathbb{N}$ such that

$$\lim_k f(z + an_k) = g(z) \text{ uniformly on compact sets of } \mathbb{C}.$$

Birkhoff's result, in terms of dynamics

- $\mathcal{H}(\mathbb{C}) := \{f : \mathbb{C} \to \mathbb{C} \; ; \; f \text{ is entire}\}$.
- Endow $\mathcal{H}(\mathbb{C})$ with the compact-open topology τ_0 (topology of uniform convergence on compact sets of $\mathbb{C}$).
- Consider the (continuous and linear!) map

$$T_a : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C}), \quad f(z) \mapsto f(z+a).$$

- Then there are $f \in \mathcal{H}(\mathbb{C})$ so that the orbit under T_1:

$$Orb(T_a, f) := \{f, T_a f, T_a^2 f, \dots\}$$

  is dense in $\mathcal{H}(\mathbb{C})$.

Framework and definitions

- From now on X will be a **separable Fréchet space** and $T : X \to X$ an **operator**.
- Given $x \in X$, its **orbit** under an operator $T : X \to X$ is:

$$Orb(T,x) := \{x, Tx, T^2x, \dots\}.$$

- An operator $T : X \to X$ on a Fréchet space X is **hypercyclic** if there are $x \in X$ such that $\overline{Orb(T,x)} = X$.

Rolewicz, 1969

No finite dimensional space admits a hypercyclic operator

Birkhoff transitivity theorem, 1920

The following are equivalent:

a) T is hypercyclic;

b) T is **topologically transitive**:

$$\forall U, V \subset X \text{ open and non-empty, } \exists n \in \mathbb{N} : \quad T^n(U) \cap V \neq \emptyset.$$

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Li-Yorke and distributional chaos

Definitions

- (Li-Yorke, 1975) An uncountable subset $S \subset X$ of a metric space (X, d) is called a **scrambled set** for a dynamical system $f : X \to X$ if for any $x, y \in S$ with $x \neq y$ we have $\liminf_n d(f^n(x), f^n(y)) = 0$ and $\limsup_n d(f^n(x), f^n(y)) > 0$. f is called **Li-Yorke chaotic** if it admits an scrambled set.
- (Schweizer-Smítal, 1994) A dynamical system $f : X \to X$ with a scrambled set S is **distributionally chaotic** on S if, additionally, there is $\delta > 0$ so that for each $\varepsilon > 0$ and each pair $x, y \in S$ of distinct points we have

$$(1) \quad \liminf_n \frac{\mathrm{card}(\{k \leq n \ : \ d(f^k(x), f^k(y)) < \delta\})}{n} = 0$$

and

$$(2) \quad \limsup_n \frac{\mathrm{card}(\{k \leq n \ : \ d(f^k(x), f^k(y)) < \varepsilon\})}{n} = 1.$$

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- We recall that **the upper density** $\overline{\text{dens}}(A)$ of a set $A \subset \mathbb{N}$ is defined by:

$$\overline{\text{dens}}(A) = \limsup_n \frac{\text{card}(A \cap \{1, \ldots, n\})}{n}$$

- Equivalent definition of distributional chaos: A dynamical system $f : X \to X$ with a scrambled set S is distributionally chaotic on S if there is $\delta > 0$ so that for each $\varepsilon > 0$ and each pair $x, y \in S$ of distinct points we have

$$(1) \quad \overline{\text{dens}}(\{k \in \mathbb{N} \ : \ d(f^k(x), f^k(y)) \geq \delta\}) = 1$$

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- Given a sequence $v = (v_n)_n$ of positive weights, we will consider the weighted ℓ^p-space $(1 \leq p < \infty)$:

$$X = \ell^p(v) := \{x \in \mathbb{K}^{\mathbb{N}} : \| x \| := \left(\sum_{j=1}^{\infty} | x_j |^p \; v_j \right)^{1/p} < \infty\}$$

- The **backward shift** $T = B : \ell^p(v) \to \ell^p(v)$

$$B(x_1, x_2, x_3, \ldots) := (x_2, x_3, x_4, \ldots)$$

is well-defined (equivalently, continuous) iff $\sup_n \frac{v_n}{v_{n+1}} < \infty$.

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(Salas, 1995)

The backward shift B is hypercyclic on $X = \ell^p(v)$ if and only if $\liminf_k v_k = 0$.

(Godefroy, Shapiro, 1991)

The backward shift B is Devaney chaotic on $X = \ell^p(v)$ if and only if $\sum_{i \in \mathbb{N}} v_i < \infty$.

(Martínez-Giménez, Oprocha, Peris, 2009)

If the backward shift B is Devaney chaotic on $X = \ell^p(v)$, then it is distributionally chaotic.

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Let $v_j := 1/k$, $n_k \leq j < n_{k+1}$, where $n_k := (k!)^3$, $k \in \mathbb{N}$. Then $T := B$ is hypercyclic on $X = \ell^p(v)$, but it is not distributionally chaotic.

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Definitions

- (Beauzamy, 1988) A vector $x \in X$ is called **irregular** for an operator $T : X \to X$ on a Banach space X provided that $\sup_n \|T^n x\| = \infty$ and $\inf_n \|T^n x\| = 0$. In particular, the line $S := \{\lambda x \ : \ \lambda \in \mathbb{K}\}$ is a scrambled set for T.
- (Prajitura, 2009) An operator $T : X \to X$ is **completely irregular** if every $x \in X \setminus \{0\}$ is irregular. In particular, the full space $S = X$ is a scrambled set for T.

(Bermúdez, Bonilla, Martínez-Giménez, Peris, 2011)

An operator $T : X \to X$ on a Banach space is Li-Yorke chaotic if and only if it admits irregular vectors.

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Definition

An operator T on a Fréchet space X with a fundamental sequence of seminorms $(\|\cdot\|_k)_k$, and a vector $x \in X$, we say that x is a **distributionally irregular** vector for T if there are subsets $A, B \subset \mathbb{N}$ with $\overline{\mathrm{dens}}(A) = \overline{\mathrm{dens}}(B) = 1$, such that $\lim_{n \in A} T^n x = 0$, and there exists $m \in \mathbb{N}$ such that $\lim_{n \in B} \|T^n x\|_m = \infty$.

Definition

Let X be a Fréchet space with a fundamental system of seminorms $\|\cdot\|_k \quad (k \in \mathbb{N})$. Let $T \in B(X)$. We say that T satisfies the distributional chaotic criterion (DCC) if there exist sequences $(x_m)_m, (y_m)_m \subset X$ such that:

(a) there exists a subset $A \subset \mathbb{N}$ with $\overline{\mathrm{dens}}(A) = 1$ such that $\lim_{n \in A} T^n x_m = 0$ for all m;

(b) $y_m \in \overline{\mathrm{span}\{x_k : k \in \mathbb{N}\}}$, $\lim_{m \to \infty} y_m = 0$ and there exist $\varepsilon > 0$ and a sequence of positive integers $\{N_m\}_m$ with $\mathrm{card}\{j \leq N_m \ : \ d(T^j y_m, 0) > \varepsilon\} \geq N_m(1 - m^{-1})$ for all $m \in \mathbb{N}$.

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Theorem

Let X be a Fréchet space with a fundamental system $\|\cdot\|_k \quad (k \in \mathbb{N})$ of seminorms. Let $T \in B(X)$. The following statements are equivalent:

(i) T satisfies (DCC);

(ii) T has a distributionally irregular vector;

(iii) T is distributionally chaotic;

(iv) T admits a distributionally chaotic pair.

Theorem

Let T be a linear and continuous operator on X. If

1. there exists a dense set X_0 such that $\lim_{n\to\infty} T^n x = 0$, for all $x \in X_0$ and
2. one of the following conditions is true:
   a) X is a Fréchet space and there exists a eigenvalue λ with $|\lambda| > 1$.
   b) X is a Banach space and $\sum \frac{1}{\|T^n\|} < \infty$ (in particular if $r(T) > 1$).
   c) X is a Hilbert space and $\sum \frac{1}{\|T^n\|^2} < \infty$(in particular if $\sigma_p(T) \cap \mathbb{T}$ has positive Lebesgue measure).

then T is densely distributionally chaotic.

Example

Let Ω be a simply connected domain and ϕ is an automorphism on Ω and let $C_\phi : H(\Omega) \to H(\Omega)$ be the composition operator $C_\phi(f)(z) = f(\phi(z))$. Then the following statements are equivalent:

(i) C_ϕ is chaotic;

(ii) C_ϕ is mixing;

(iii) C_ϕ is hypercyclic;

(iv) $(\phi)^n)_n$ is a run-away sequence;

(v) ϕ has no a fixed point;

(vi) C_ϕ is densely distributionally chaotic.