# Existence and Uniqueness of Limit Cycles for Generalized $\varphi$-Laplacian Liénard Equations

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## Introduction

- Liénard equation,

$$x'' + f(x)x' + g(x) = 0, \qquad (1)$$

appears as simplified model in many domains in science and engineering. One of the first models where this equation appears was introduced by Balthasar van der Pol. Considering the equation modeling the oscillations of a triode vacuum tube

$$x'' + \mu(x^2 - 1)x' + x = 0.$$

- In [1], we extend some of these results for the case of the generalized $\varphi$-laplacian Liénard equation

$$(\varphi(x'))' + f(x)\psi(x') + g(x) = 0. \qquad (\mathrm{G}\varphi\mathrm{L})$$

This generalization appears when other derivations, different from the classic one, are considered, such as the relativistic one. Our results apply, for example, to the relativistic van der Pol equation

$$\left(\frac{x'}{\sqrt{1-\frac{x^2}{c^2}}}\right)' + \mu(x^2-1)x' + x = 0.$$

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## First Assumptions

- In the case of Liénard equation (1), it is usual to apply some change of variables to express the equation as the planar system

$$\begin{cases} \dot{x} = y - F(x), \\ \dot{y} = -g(x), \end{cases} \quad \text{or} \quad \begin{cases} \dot{x} = y, \\ \dot{y} = -g(x) - f(x)y. \end{cases}$$

In this work a variation of this approach is considered. Our results apply, after a time rescaling, to system

$$\begin{cases} \dot{x} = y\varphi'(y), \\ \dot{y} = -g(x) - f(x)\psi(y). \end{cases} \qquad (2)$$

- Based on the results of the classical Liénard equation (1), the following conditions, denoted by $(H)$, are established.

$(H_0)$ $f(x)$, $g(x)$ and $\psi(y)$ are of class $\mathcal{C}^{0,1}(\mathbb{R})$ and $\varphi(y)$ is of class $\mathcal{C}^{1,1}(\mathbb{R})$.
$(H_1)$ $xg(x) > 0$ for all $x \in (x_1, x_2) \setminus \{0\}$ and $g(0) = 0$.
$(H_2)$ $f(0) \neq 0$.
$(H_3)$ $\mathrm{Dom}(\varphi) \subseteq \mathrm{Dom}(\psi)$.
$(H_4)$ $\psi(0) = 0$.
$(H_5)$ $\varphi'(y) \in \mathbb{R}^+ \setminus \{0\}$ for all $y \in (y_1, y_2)$ and $\varphi(0) = 0$.

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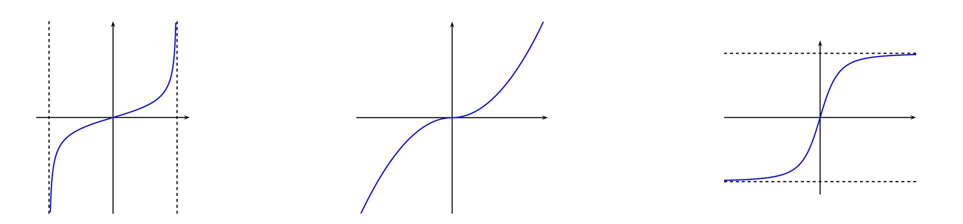
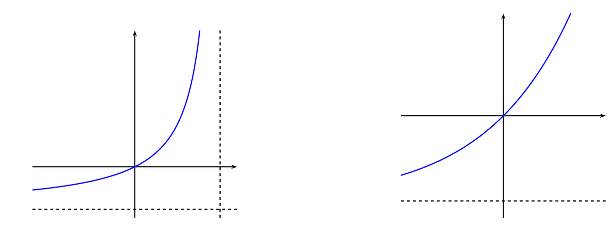
## Function Families

- In this work we consider three basic different behaviors of $\varphi(y)$ over $y_i$ for $i = 1, 2$.



(a) Singular $\varphi(s) = \frac{s}{\sqrt{1-s^2}}$ (b) Non-bounded Regular $\varphi(s) = |s|^{p-1}s$ (c) Bounded Regular $\varphi(s) = \frac{s}{\sqrt{1+s^2}}$

- Although the previous examples are all symmetric we do not ask for any symmetry to the function $\varphi(y)$, nor a symmetric behavior at the boundary of the domain. Therefore some mixed cases can also be considered. Hence, the results of this chapter apply also for functions like $\varphi(s) = s/(1-s)$ or $\varphi(s) = e^s - 1$.



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## Existence Theorem

**Theorem 1** *Consider system (2) under the hypotheses $(H)$. Additionally, next properties hold.*

*(i)* $y\psi(y)f(x) \leq 0$ *in a neighborhood of the origin,* $I_x \times I_y = [x_-, x_+] \times [y_-, y_+] \subset \mathcal{D}$, *except for a finite number of points where it vanishes.*

*(ii) There exist $\delta$ and $\eta$ in $\mathbb{R}$, with $x_1 < \eta < 0 < \delta < x_2$, such that $f(x) > 0$ for all $x \in (x_1, x_2) \setminus [\eta, \delta]$.*

*(iii) For each $i = 1, 2$ there exists $\lambda_i$ in $\mathbb{R}^+ \cup \{+\infty\}$ such that, if $|x_i| = +\infty$, then $\liminf_{x \to x_i} x(|g(x)| + f(x)) = \lambda_i$, and if $x_i \in \mathbb{R}$, then $\liminf_{x \to x_i} |x - x_i|(|g(x)| + f(x)) = \lambda_i$.*

*(iv) $y\psi(y) > 0$ for all $y \neq 0$.*

*(v) For $i = 1, 2$, $\lim_{y \to y_i} \psi(y)/(y\varphi'(y)) \in \mathbb{R}$.*

*(vi) The integral $\int_\eta^\delta f(x)dx$ is positive or, alternatively, there exists $y_0 \in (y_1, y_2)$ such that $-\psi(y_0) \in \left[\liminf_{x \to x_i} g(x)/f(x), \limsup_{x \to x_i} g(x)/f(x)\right]$ for at least one of the $x_i$ and there exists $U$, neighborhood of $y_0$, such that $\mathrm{sign}(\psi'(y))$ is constant almost for every $y \in U$.*

*Then system (2) has at least a periodic orbit contained in $\mathcal{D}$.*

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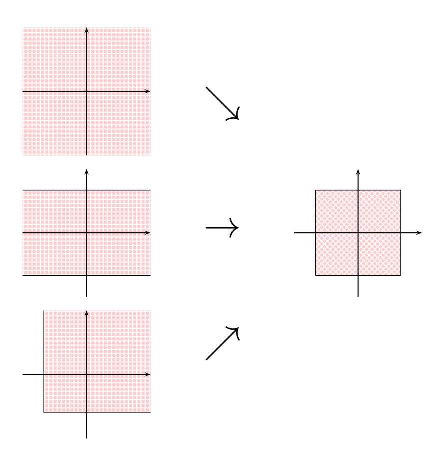
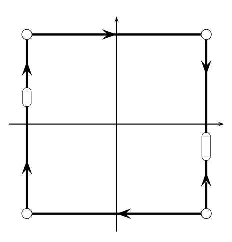
## Squaring the Domain

- The main tool in order to proof the existence is a transformation of the domain of definition of (2), $\mathcal{D} = (x_1, x_2) \times (y_1, y_2)$. So, we introduce a polygonal compactification of the domain. The domain of the transformed system is $\widetilde{\mathcal{D}} = (-1, 1) \times (-1, 1)$.



- We can extend the dynamical behavior of the transformed system in $\widetilde{\mathcal{D}}$ to every point of its closure, with the exception of the vertex. And, in this way we should distinguish between regular and singular points of the boundary. The vertex and singular points (or continuum sets of singular points) are represented in the following graphics as rounded regions on the boundary.



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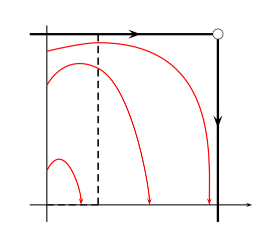
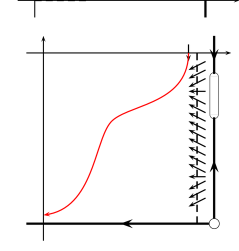
## The Boundary Cannot Be Reached

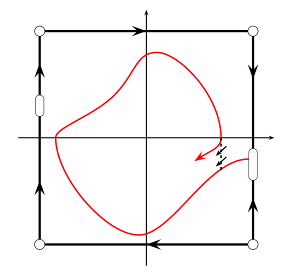
- The positive orbit of every point in $[0, 1) \times (0, 1)$ (resp. in $(-1, 0] \times (-1, 0)$) cuts transversally, in finite time, the segment $(0, 1) \times \{0\}$ (resp. $(-1, 0) \times \{0\}$).



- The positive orbit of every point in $(0, 1) \times (-1, 0]$ (resp. in $(-1, 0) \times [0, 1)$) cuts transversally, in finite time, the segment $\{0\} \times (-1, 0)$ (resp. $\{0\} \times (0, 1)$).



- Any orbit which $\alpha$-limit set is on $\{1\} \times (-1, 0)$ touches the positive horizontal axis, in finite time, passing through all quadrants in counterclockwise direction. And, as can be checked in the diagram, we can use it to construct a positively invariant set around the origin.



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## Unicity Theorem

**Theorem 2** *Consider system (2) under the hypotheses $(H)$. Additionally, next properties hold.*

*(i) $f, g \in \mathcal{C}^{0,1}((x_1, x_2))$ and $\varphi, \psi \in \mathcal{C}^{1,1}((y_1, y_2))$ with $x_1, y_1 \in \mathbb{R}^- \cup \{-\infty\}$ and $x_2, y_2 \in \mathbb{R}^+ \cup \{+\infty\}$.*

*(ii) There exist $a < 0 < b$ such that $f(x) < 0$ when $x \in (a, b)$ and $f(x) > 0$ when $x \in (x_1, x_2) \setminus [a, b]$,*

*(iii) $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) > 0$, for all $x \in (x_1, x_2) \setminus I_0$ where $I_0 \subset [a, b]$ such that $I_0$ contains the origin and $I_0 = (a, x_0)$ or $I_0 = (x_0, b)$ with $x_0$ satisfying that $\int_0^{x_0} g(s)ds = \min\left\{\int_0^a g(s)ds, \int_0^b g(s)ds\right\}$.*

*(iv) $\psi'(y) > 0$ and $\frac{d}{dy}\left(\frac{\psi'(y)}{y\varphi'(y)}\right) < 0$, for all $y$ in $(y_1, y_2) \setminus \{0\}$.*

*Then system (2) has at most one limit cycle. Moreover, when it exists, it is stable.*

- For the proof of the Unicity Theorem we need to introduce the function $E(x, y) = G(x) + \Phi(y)$, where $G(x) = \int_0^x g(u)du$ and $\Phi(y) = \int_0^y v\varphi'(v)dv$. It is a first integral of system (2) in $\mathcal{D}$ in the case of having a nule friction term.

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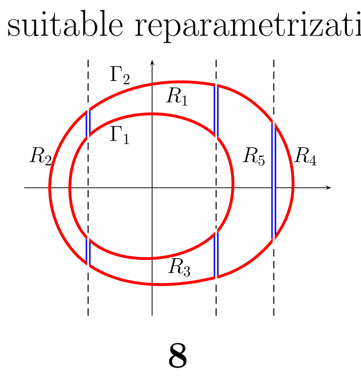
## Method of Comparison

- Under our hypotheses the origin is the unique singular point, and it is a repellor. Additionally, there is no periodic orbits entirely contained in $(a, b) \times (y_1, y_2)$ because $\dot{E} = -f(x)y\varphi'(y)\psi(y) > 0$, for all $(x, y) \in (a, b) \times (y_1, y_2) \setminus \{(0, 0)\}$. Moreover, all the periodic orbits contain the region $\{(x, y) \in \mathcal{D} : 0 \leq E(x, y) \leq \min(G(a), G(b))\}$, because it is negatively invariant.
- The proof is done by the method of comparison. Let us suppose that we have two different limit cycles, $\Gamma_1$ and $\Gamma_2$. Then we prove that the integral of the divergence of the vector field (2), between them, is different from zero, in fact, it is negative. This contradicts the existence of two limit cycles because it implies that both orbits have the same stability.
- The integral of the divergence of equation (2) between both periodic orbits is computed decomposing the region in five different regions $R_i$, $i = 1, \ldots, 5$. And the integrals are computed in each one considering the suitable reparametrization.

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## Summary

- Our results generalize some of the most classic results on Liénard equation. In fact, models like Van der Pol's equation (relativistic case also) satisfies the hypotheses of both theorems. So we can combine both in order to obtain an existence and uniqueness theorem.
- This work involve an ad hoc compactification designed with two objectives. First, to unify the different behaviors of the functions satisfying our hypothesis. And second, to make possible the comprehension of the global phase portrait.
- The way we have apply the comparison method allow us to avoid the classic hypothesis in Liénard uniqueness results that ask that the segment $(a, b) \times \{0\}$ should be completely contained in any limit cycle of the system.

## Bibliography

(1) S. Pérez-González, J. Torregrosa and P. J. Torres. *Existence and Uniqueness of limit cycles for generalized $\varphi$-laplacian Liénard equations.* Prepublicacions del Departament de Matemàtiques, Núm. 24/2012, 2012.

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