Planar quasi-homogeneous polynomial differential systems and their integrability

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ABSTRACT

The quasi–homogeneous polynomial differential systems have been studied from many different points of view, mainly for their integrability (see, for example, [3],[5],[6]), for their centers (see [4]), for their normal forms (see [1]), for their limit cycles (see [2]). But so far there has not been (in what we know) an algorithm for constructing all the quasi-homogeneous polynomial differential systems of a given degree. We remark that this goal is very different from the search for the quasi homogeneous systems with a fixed weight degree.

In our work some properties are studied of the quasi-homogeneous polynomial differential systems that allow us to obtain the looked for algorithm. As an application of this algorithm all quasi–homogeneous vector fields of degree 2 and 3 are obtained.

It is known that the quasi–homogeneous polynomial differential systems are Liouvillian integrable. In our work we used an adequate inverse of integrating factor and, taking into account the classification given by the algorithm, we characterized all the quasi–homogeneous vector fields of degree 2 and 3 having a polynomial, rational or global analytical first integral.

1. BASIC DEFINITIONS

A polynomial differential system

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y),$$

of degree $n = \max\{deg(P), deg(Q)\}$ is called ***quasi–homogeneous*** (in short, QH-system) if there exist $s_1,\ s_2, d \in \mathbb{N}$ such that for arbitrary $\alpha \in \mathbb{R}^+ = \{a \in \mathbb{R},\ a > 0\}$, one has that

$$P(\alpha^{s_1}x, \alpha^{s_2}y) = \alpha^{s_1-1+d}P(x,y),$$

$$Q(\alpha^{s_1}x, \alpha^{s_2}y) = \alpha^{s_2-1+d}Q(x,y).$$

We call ***weight vector*** of the system to the vector $\mathrm{w} = (s_1, s_2, d)$, where s_1 and s_2 are the *weight exponents* of the system, and d the *weight degree* with respect to s_1 and s_2.

We say that the weight vector $\mathrm{w}_m = (s_1^*, s_2^*, d^*)$ is the ***minimal weight vector*** of the system if any $\mathrm{w} = (s_1, s_2, d)$ verifies $s_1^* \leq s_1$, $s_2^* \leq s_2$ and $d^* \leq d$.

3. PROPERTIES OF THE QHNH-SYSTEMS

Case 1: d=1. We have proved that the general expression of the system in this case is

$$\dot{x} = a_{1,0}x + a_{0,n}y^n, \quad \dot{y} = b_{0,1}y.$$

Furthermore one has that $\mathrm{w}_m = (n, 1, 1)$.

Case 2: d>1. If X is a vector field associated to a QHNH-system of degree n, then there exist p, t, k such that $X_n^p X_{n-t}^{p,k} \neq 0$. Therefore $e_p^0[0]$ and $e_p^t[k]$ holds and, by solving this linear system, it can be obtained that

$$s_1 = (t+k)(d-1)/D, s_2 = k(d-1)/D,$$

where $D = (p-1)t + (n-1)k$.

Moreover, the minimal weight vector of X is

$$\mathrm{w}_m = ((t+k)/s, k/s, 1 + D/s),$$

where s is the greatest common divisor of t and k.

In order for X to have other nonzero homogeneous part we must consider other equation $e_p^{t^*}[k^*]$. We have proved that this third equation is compatible with the two previous equations if and only if

$$kt^* = k^*t.$$

5. CANONICAL FORMS OF DEGREE 2 AND 3

A quadratic QHNH polynomial differential system without common factors can be written (after changes of the variables and the time) as any of the following ones:

- $x' = y^2$, $y' = x$, with $\mathrm{w}_m = (3, 2, 2)$.
- $x' = axy$, $y' = x + y^2$, with $a \neq 0$ and $\mathrm{w}_m = (2, 1, 2)$.
- $x' = x + y^2$, $y' = ay$, with $a \neq 0$ $\mathrm{w}_m = (2, 1, 1)$.

A cubic QHNH polynomial differential system without common factors can be written (after changes of the variables and the time) as any of the following ones:

- $x' = y(ax + by^2)$, $y' = x + y^2$, with $a \neq b$, or $x' = y(ax \pm y^2)$, $y' = x$, and both with $\mathrm{w}_m = (2, 1, 2)$.
- $x' = x^2 + y^3$, $y' = axy$, with $a \neq 0$ and $\mathrm{w}_m = (3, 2, 4)$.
- $x' = y^3$, $y' = x^2$, with minimal weight vector $\mathrm{w}_m = (4, 3, 6)$.
- $x' = x(x + ay^2)$, $y' = y(bx + y^2)$, with $ab \neq 1$ and $\mathrm{w}_m = (2, 1, 3)$.
- $x' = axy^2$, $y' = \pm x^2 + y^3$, with $a \neq 0$ and $\mathrm{w}_m = (3, 2, 5)$.
- $x' = axy^2$, $y' = x + y^3$, with $a \neq 0$ and $\mathrm{w}_m = (3, 1, 3)$.
- $x' = ax + y^3$, $y' = y$, with $a \neq 0$ and $\mathrm{w}_m = (3, 1, 1)$.

2. FIRST RESULTS ABOUT QH-SYSTEMS

We consider a vector field $X = (P, Q)$ associated to a QH-system of degree n with weight vector $\mathrm{w} = (s_1, s_2, d)$. We have proved that:

- If $s_1 = s_2$, then X is a homogeneous system with $\mathrm{w}_m = (1, 1, n)$.
- If $s_1 \neq s_2$, there is $p \in \{0, \ldots, n\}$ such that the homogeneous part of highest degree of X is
$$X_n^p = (a_{p,n-p}x^p y^{n-p}, b_{p-1,n-p+1}x^{p-1}y^{n-p+1})$$
and the following equation holds:
$$e_p^0[0] \equiv (p-1)s_1 + (n-p)s_2 + 1 - d = 0.$$
Moreover, if the homogeneous part of X of degree $n-t$ is nonzero, then there is $k \in \{1, \ldots, n-t-p+1\}$ such that
$$X_{n-t}^{p,k} = (a_{p+k,n-p-k-t}x^{p+k}y^{n-p-k-t}, b_{p+k-1,n-p-k-t+1}x^{p+k-1}y^{n-p-k-t+1})$$
and the next equation is verified:
$$e_p^t[k] \equiv (p+k-1)s_1 + (n-t-p-k)s_2 + 1 - d = 0.$$

Since the homogeneous systems are well known, in order to obtain all the QH-systems of a fixed degree, we need to obtain all the quasi-homogeneous non homogeneous systems (in the following QHNH-systems) and hence we consider only the case $s_1 \neq s_2$. Moreover, through an adequate change of variables, we can restrict to the case $s_1 > s_2$.

4. THE ALGORITHM

This algorithm allow us to obtain all the QHNH-systems of a fixed degree n with $d > 1$. We remark that, taking into account the above results, each QHNH-system is associated to a linear system formed by the equations that correspond to all its nonzero homogeneous parts.

- **Step 1**. We choose $p \in \{0, \ldots, n\}$ such that X_n^p is the homogeneous part of degree n of the vector field X, that is, the associated equation is $e_p^0[0]$.
- **Step 2**. We choose a value of $t \in \{1, \ldots, n-p\}$ and a value of $k \in \{1, \ldots, n-t-p+1\}$ such that the homogeneous part $X_{n-t}^{p,k}$ is also nonzero and hence the equation $e_p^t[k]$ holds. The resolution of the linear system defined by $e_p^0[0]$ and $e_p^t[k]$, allow us to obtain the values of s_1 and s_2, and furthermore the minimal weight vector w_m.
- **Step 3**. We determine all the homogeneous parts that can be added to X_n^p and $X_{n-t}^{p,k}$. In order to do this we determine, for each $t^* \in \{1, \ldots, n-p\}$ with $t \neq t^*$, the value $k_{t^*} \in \{1, \ldots, n-t^*-p+1\}$, (if it exists), such that the equation $e_p^{t^*}[k_{t^*}]$ satisfies the compatibility condition with the equations of Step 1 and Step 2.
- **Step 4**. We obtain the QHNH-vector field of degree n formed by all the homogeneous parts of the above steps, that is
$$X = X_n^p + X_{n-t}^{p,k} + \sum_{t^* \in \{1,\ldots,n-p\} \setminus \{t\} \text{ and } k_{t^*}t = kt^*} X_{n-t^*}^{p,k_{t^*}},$$
where the explicit expression of each homogeneous part is defined in Section 2.
- **Step 5**. We go back to Step 2 and consider other choices of t and k. When this is not possible, we change in Step 1 the value of p and repeat the process.

6. INTEGRABILITY

- All QH-systems are integrable (see, for instance, [3]).
- One can prove (for example, by using a generalization of the Euler's formula) that if a QH-system has weight exponents s_1 and s_2, then $V = s_1xQ - s_2yP$ is an inverse of integrating factor of the system.
- From the results of [5] it is easy to deduce that a QH-system has a global analytic first integral if and only if it has a polynomial first integral.
- Using the above results we can obtain all the polynomial, rational and global analytic first integrals of the QH-systems of degree $\mathbf{2}$ and $\mathbf{3}$.

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