On the Darboux theory of integrability of non-autonomous polynomial differential systems

D. Blázquez and Ch. Pantazi
Speaker: Chara Pantazi

Dept.Matemàtica Aplicada I, (MA1)
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Salou, 02 of Octumber of 2012

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We consider the complex polynomial differential systems in $\mathbb{C}^n$ of the form

$$\dot{x}_j = \frac{dx_j}{dt} = P_j(x_1, \cdots, x_n) = \sum_{0 \le i_1 + \cdots + i_n \le m} a_{j i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n},$$

$$j = 1, \cdots, n$$

$a_{j i_1 \cdots i_n} \in \mathbb{C}$

Degree of the system $m = \max\{\deg P_1, \cdots, \deg P_n\}$.

The associated vector field is

$$X = \sum_{i=1}^{n} P_i \frac{\partial}{\partial x_i}.$$

$f \in \mathbb{C}[x_1, \cdots, x_n]$ be irreducible. The hypersurface $f = 0$ is an invariant hypersurface of X if

$$\frac{X(f)}{f} = X(\log(f)) = K,$$

with $K \in \mathbb{C}_{m-1}[x_1, \cdots, x_n]$ the cofactor of $f = 0$ of degree $\leq m - 1$.

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with $K \in \mathbb{C}_{m-1}[x_1, \cdots, x_n]$ the cofactor of $f = 0$ of degree $\leq m - 1$.
$h, g \in \mathbb{C}[x_1, \cdots, x_n]$ coprimes.

$$F(x_1, \cdots, x_n) = \exp\left(g/h\right)$$

is called an exponential factor

$$X\left(\frac{g}{h}\right) = K,$$

and we say that K is the cofactor
traditionally taken of bounded degree.

$f_1, \cdots, f_p$ irreducible + coprime in $\mathbb{C}[x_1, \cdots, x_n]$

and $F_1 = \exp(g_1/h_1), \cdots, F_q = \exp(g_q/h_q)$ exponential factors.

$$\text{Darboux function} := f_1^{\lambda_1} \cdots f_p^{\lambda_p} \left(\exp\left(\frac{g_1}{h_1}\right)\right)^{\mu_1} \cdots \left(\exp\left(\frac{g_q}{h_q}\right)\right)^{\mu_q},$$

$\lambda_i, \mu_i \in \mathbb{C}$

$$f_1^{\lambda_1} \cdots f_p^{\lambda_p} \exp\left(\frac{g}{h}\right).$$

Let W be an open subset of $\mathbb{C}^n$.
A non constant analytic function $I : W \longrightarrow \mathbb{C}$ is a first integral of the vector field X on W if

$$X(I) = 0 \quad \text{on } W$$

$M : W \longrightarrow \mathbb{C}$ is a Jacobi multiplier if

$$X(M) = -(\text{div} X)M$$

where

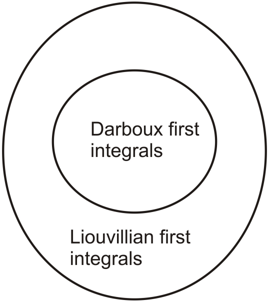
$$\text{div} X = \sum_{i=1}^{n} \partial P_i / \partial x_i$$

$\exists M \Longrightarrow MX = (MP_1, \cdots, MP_n)$.

Here, Darboux first integral
and
Darboux Jacobi multiplier.

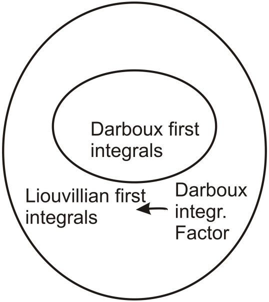
Particular case: $n = 2$

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y).$$



Darboux function $:= f_1^{\lambda_1} \cdots f_p^{\lambda_p} \left(\exp\left(\frac{g_1}{h_1}\right)\right)^{\mu_1} \cdots \left(\exp\left(\frac{g_q}{h_q}\right)\right)^{\mu_q},$

Singer (and also Christopher)



$$f_1^{\lambda_1} \cdots f_p^{\lambda_p} \exp\left(\frac{g}{h}\right).$$

The main Problems

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(1) Find all invariant algebraic curves of X.

(2) Decide whether a Darboux integrating factor exists (and if does exist, find it!).

- Are classical problems (start with Poincaré and later with Darboux).
- Poincaré (1891): For a given X give an effective algorithm to calculate the maximum degree of $f = 0$.
- Jouanolou (1979): For a given X the maximum degree is bounded: X as either a finite number of inv.alg. curves or has a rational first integral.

  Some bounds: (Carnicer and Campillo, Cerveau and Lins Neto...)

Bassically Llibre and Zhang

Theorem

X is v.f. in $\mathbb{C}^n$ of degree m that admits

(i) *p invariant algebraic hypersurfaces $f_1 = 0, \cdots, f_p = 0$ with cofactors $K_1, \cdots, K_p$. Here f_i irreducible+coprimes.*

(ii) *q exponential factors $F_{p+1}, \cdots, F_{p+q}$ with cofactors $K_{p+1}, \cdots, K_{p+q}$ of degree $\le m-1$.*

Then the following statements hold.

(a)

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{i=p+1}^{q} \mu_j K_j + \rho \, div(X) = 0,$$

iff *DFI or a DJM:*

$$f_1^{\lambda_1} \cdots f_p^{\lambda_p} \left(\exp\left(\frac{g_1}{h_1}\right)\right)^{\mu_1} \cdots \left(\exp\left(\frac{g_q}{h_q}\right)\right)^{\mu_q}.$$

(b) **If** $p+q=\begin{pmatrix} n+m-1 \\ n \end{pmatrix}$, **then**

$$\sum_{i=1}^{p}\lambda_i K_i+\sum_{i=1}^{q}\mu_j K_j+\rho \mathrm{div}(X)=0,$$

the vector field X has either a DFI or a DJM.

$$f_1^{\lambda_1}\cdots f_p^{\lambda_p}\left(\exp\left(\frac{g_1}{h_1}\right)\right)^{\mu_1}\cdots\left(\exp\left(\frac{g_q}{h_q}\right)\right)^{\mu_q}$$

(c) **If** $p+q=\begin{pmatrix} n+m-1 \\ n \end{pmatrix}+1$, **then**

$$\sum_{i=1}^{p}\lambda_i K_i+\sum_{i=1}^{q}\mu_j L_j=0,$$

X has a DFI.

(d) **If** $p+q=\begin{pmatrix} n+m-1 \\ n \end{pmatrix}+n$ **then** X has a rational first integral.

Exemple

$$\begin{aligned}\dot{x} &= xz(bx^{n_1}y^{n_2}z^{n_3} - n_2)\\ \dot{y} &= -y^{n_2+1}x^{n_1}z^{n_3}(\gamma cx + az) + y(n_3\gamma x + n_1 z)\\ \dot{z} &= \gamma xz(bx^{n_1}y^{n_2}z^{n_3} - n_2),\end{aligned}$$

$n_1, n_2, n_3 \in \mathbb{Z}_+$ and $a, b, c \in \mathbb{C}$ degree $m = n_1 + n_2 + n_3 + 2$.

$$f_1 = x \;\; f_2 = y \;\; f_3 = z, \;\; F = \exp\left(\frac{1}{x^{n_1}y^{n_2}z^{n_3}}\right)$$

$$K_1 = bx^{n_1}y^{n_2}z^{1+n_3} - n_2 z, K_2 = -y^n 2x^{n_1}z^{n_3}(\gamma cx + az) + n_1 z + n_3\gamma x$$

$$K_3 = \gamma x(bx^{n_1}y^{n_2}z^{n_3} - n_2), \;\; K = -n_1 bz + n_2\gamma cx + n_2 az - \gamma n_3 bx.$$

Exemple

$$\begin{array}{rcl}\dot{x} &=& xz(bx^{n_1}y^{n_2}z^{n_3}-n_2)\\ \dot{y} &=& -y^{n_2+1}x^{n_1}z^{n_3}(\gamma cx+az)+y(n_3\gamma x+n_1 z)\\ \dot{z} &=& \gamma xz(bx^{n_1}y^{n_2}z^{n_3}-n_2),\end{array}$$

$n_1, n_2, n_3 \in \mathbb{Z}_+$ and $a, b, c \in \mathbb{C}$ degree $m = n_1 + n_2 + n_3 + 2$.

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$$K_3 = \gamma x(bx^{n_1}y^{n_2}z^{n_3} - n_2), \; K = -n_1 bz + n_2\gamma cx + n_2 az - \gamma n_3 bx.$$

Here, $n = 3$, $m = n_1 + n_2 + n_3 + 2$ and so

$$\binom{n+m-1}{n} = \frac{(n_1+n_2+n_3+2)(n_1+n_2+n_3+3)}{6}.$$

is arbitrary high.

$K_1, \cdots, K_r$ cofactors of hypersurfaces or of exponential factors of X.
Wronskian matrix

$$\mathcal{W} = \mathcal{W}(K_1, \cdots, K_r) = \begin{pmatrix} K_1 & \cdots & K_r \\ X(K_1) & \cdots & X(K_r) \\ & \cdots & \\ X^{(r-1)}(K_1) & \cdots & X^{(r-1)}(K_r) \end{pmatrix},$$

where $X^{(l+1)}(K_j) = X(X^{(l)}(K_j))$.

Blázquez and Pantazi

Theorem

X in $\mathbb{C}^n$ of degree m admits

(i) *p inv.hyp. $f_1 = 0, \cdots, f_p = 0$ with cofactors $K_1, \cdots, K_p$. $f_1, \cdots, f_p$ irreducible+coprimes.*

(ii) *q exponential factors $F_{p+1}, \cdots, F_{p+q}$ with cofactors $K_{p+1}, \cdots, K_{p+q}$.*

We further assume that the 1-forms $d \log f_1, \ldots, d \log f_p, d \log F_{p+1}, \cdots, d \log F_{p+q}$ are linearly independent over $\mathbb{C}$.

Theorem

(a) $p+q > \operatorname{rank}(\mathcal{W}(K_1, \cdots, K_{p+q}))$ ***iff*** X *admits a FI which is either rational or Darboux.*

(b) $p+q > \operatorname{rank}(\mathcal{W}(K_1, \cdots, K_{p+q}, \operatorname{div} X)) + 1$ ***iff*** *the vector field* X *admits either a rational FI or a Darboux Jacobi multiplier DJM.*

(c) ***if*** $p+q > \operatorname{rank}(\mathcal{W}(K_1, \cdots, K_{p+q})) + n$ ***then*** *the vector field* X *has at least one rational FI.*

Lemma

Let us consider elements $f_1, \cdots, f_r, \psi \in \mathcal{F}$ and denote by $a_i = (\partial f_i) f_i^{-1} = \partial \log f_i$. The following conditions are equivalent:

(i) *The rank of the Wronskian matrices $\mathcal{W}(a_1, \ldots, a_r, \psi)$ and $\mathcal{W}(a_1, \ldots, a_r)$ coincide.*

(ii) *ψ is a non-trivial $C(\mathcal{F})$-linear combination of $a_1, \ldots, a_r$*

$$\psi = \lambda_1 a_1 + \cdots + \lambda_r a_r. \tag{1}$$

(iii) *There exist constants $\lambda_1, \cdots, \lambda_r$ in $C(\mathcal{F})$ such that for any differential extension $\mathcal{F} \subset \mathcal{F}(b_1, \cdots, b_r)$ by elements b_i satisfying $\partial b_i = \lambda_i a_i b_i$ the element $M = b_1 \cdots b_r$ verifies $\partial M = \psi \cdot M$.*

in the previous example...

$$\begin{array}{rcl}\dot{x} & = & xz(bx^{n_1}y^{n_2}z^{n_3}-n_2)\\ \dot{y} & = & -y^{n_2+1}x^{n_1}z^{n_3}(\gamma cx+az)+y(n_3\gamma x+n_1 z)\\ \dot{z} & = & \gamma xz(bx^{n_1}y^{n_2}z^{n_3}-n_2),\end{array}$$

$$f_1=x \quad f_2=y \quad f_3=z, \quad F=\exp\left(\frac{1}{x^{n_1}y^{n_2}z^{n_3}}\right)$$

$$K_1=bx^{n_1}y^{n_2}z^{1+n_3}-n_2 z, K_2=-y^n 2x^{n_1}z^{n_3}(\gamma cx+az)+n_1 z+n_3\gamma x$$

$$K_3=\gamma x(bx^{n_1}y^{n_2}z^{n_3}-n_2),\ K=-n_1 bz+n_2\gamma cx+n_2 az-\gamma n_3 bx.$$

We have

$$p+q=4>\operatorname{rank}(\mathcal{W}(K_1,K_2,K_3,K))=3$$

$$H(x,y,z)=x^a y^b z^c\exp\left(\frac{1}{x^{n_1}y^{n_2}z^{n_3}}\right).$$

Another example

$$X = (\mu x + \beta xy)\frac{\partial}{\partial x} + (\lambda y + \alpha xy)\frac{\partial}{\partial y}.$$

$$f_1 = x,\ f_2 = y,\ F_3 = e^x, \quad F_4 = e^y$$

with cofactors

$$K_1 = \mu + \beta y, K_2 = \lambda + \alpha x$$

$$K_3 = xK_1, \quad K_4 = yK_2$$

of degree $2 > m - 1$.

$$4 = p + q > \mathrm{rank}\mathcal{W}(K_1, K_2, K_3, K_4) = 3$$

$$I = \frac{x^\lambda}{y^\mu}\frac{e^{\alpha x}}{e^{\beta y}} = x^\lambda y^{-\mu} e^{\alpha x - \beta y}.$$

ofcourse $e^{\alpha x - \beta y}$ cofactor $K = \alpha\mu x - \beta\lambda y$

$$3 = p + q > \mathrm{rank}(\mathcal{W}(K_1, K_2, K)) = 2,\ \lambda K_1 - \mu K_2 + K = 0.$$

Definitions

We consider non autonomous polynomial differential systems of the form

$$\dot{x_j} = \frac{dx_j}{dt} = P_j(t,x) = \sum_{0 \le i_1+\cdots+i_n \le m} a_{ji_1\cdots i_n}(t) x_1^{i_1}\cdots x_n^{i_n}, \quad j = 1,\cdots,n \tag{2}$$

$a_{ji_1\cdots i_n} \in M(U,\mathbb{C})$

$P_j \in M(U,\mathbb{C})[x_1,\cdots,x_n]$ for $j = 1,\cdots,n$.

degree $m = \max\{\deg P_1,\cdots,\deg P_n\}$.

We associate the v.f.

$$X = \frac{\partial}{\partial t} + \sum_{j=1}^{n} P_j \frac{\partial}{\partial x_j}.$$

$x = (x_1, \cdots, x_n)$
X is a derivation of the ring $M(U, \mathbb{C})[x]$
and so of the quotient field $M(U, \mathbb{C})(x)$.

Differential field extension $(M(U, \mathbb{C}), d/dt) \subset (M(U, \mathbb{C})(x), X)$.

$x = (x_1, \cdots, x_n)$
$f_1, \cdots, f_p \in M(U, \mathbb{C})[x]$ be irreducible and coprime.

$F = \exp(g/h)$ with $g, h \in M(U, \mathbb{C})[x]$ coprimes.

$$f_1^{\lambda_1} \cdots f_p^{\lambda_p} \left(\exp\left(\frac{g_1}{h_1}\right)\right)^{\mu_1} \cdots \left(\exp\left(\frac{g_q}{h_q}\right)\right)^{\mu_q} = f_1^{\lambda_1} \cdots f_p^{\lambda_p} \exp\left(\frac{g}{h}\right),$$

will be called a generalized Darboux function.

Theorem

Consider X in $U \times \mathbb{C}^n$ of degree m having

(i) p *invariant hypersurfaces* $f_1 = 0, \cdots, f_p = 0$ *with cofactors* $K_1, \cdots, K_p$.

$f_1, \cdots, f_p \in M(U, \mathbb{C})[x]$ *are irreducible + coprimes.*

(ii) q *exponential factors* $F_{p+1}, \cdots, F_{p+q}$ *with cofactors* $K_{p+1}, \cdots, K_{p+q}$.
Here $F_i = \exp(g_i/h_i)$
with $g_i, h_i \in M(U, \mathbb{C})[x]$ *are irreducible +coprimes.*

We further assume that $d \log f_1, \ldots, d \log f_p, d \log F_{p+1}, \ldots, d \log F_{p+q}$ *are* $\mathbb{C}$*-linearly independent as meromorphic* 1*-forms in* $U \times \mathbb{C}^n$.

Theorem

Then the following statements hold.

(a) $p + q > \mathrm{rank}(\mathcal{W}(K_1, \cdots, K_r))$ ***iff*** *there is a FI. This first integral is either rational or Darboux.*

(b) $p + q > \mathrm{rank}(\mathcal{W}(K_1, \cdots, K_r, \mathit{div}\, X)) + 1$ ***iff*** *there is a rational FI or a Darboux JM*

(c) ***If*** $p + q > \mathrm{rank}(\mathcal{W}(K_1, \cdots, K_r)) + n$ *for* $j = 1, \cdots, n$ ***then*** *the vector field X has at least one rational first integral.*

Lemma

Let W be an open subset of $U \times \mathbb{C}^n$. Let $J_1, \cdots, J_n$ be functionally independent analytic first integrals of X on W.

(a) *If J_{n+1} is another first integral then there exists analytic functions $C_1, \cdots, C_n$ such that:*

$$dJ_{n+1} = C_1 dJ_1 + \cdots + C_n dJ_n. \tag{3}$$

(b) *The functions $C_1, \ldots, C_n$ are first integrals of X.*

(c) *Assume $J_k = \log(I_k)$ for $k = 1, \ldots, n+1$ where $I_1, \ldots, I_{n+1}$ are Darboux first integrals in which the factors $f_1, \ldots, f_p, F_{p+1}, \ldots, F_{p+q}$ appear, being $d\log f_1, \ldots,$ $d\log f_p, d\log F_{p+1}, \ldots, d\log F_{p+q}$ $\mathbb{C}$-linearly independent. Assume that at least one factor appearing in I_{n+1} does not appear in the $I_1, \ldots, I_n$. Then, the functions C_i are in $M(U, \mathbb{C})(x)$ and at least one of them is not a constant.*

3-D non-autonomous Lotka-Volterra

Example

$$
\begin{aligned}
\dot{x} &= x\left(\frac{ta_1(t)-1}{t} - \frac{\lambda_2 b_{21} + b_{31}}{\lambda_1} tx + b_{12} y + b_{13} z\right) \\
\dot{y} &= y\left(a_2(t) + b_{21} tx - \frac{\lambda_1 b_{12} + b_{32}}{\lambda_2} y + b_{23} z\right) \\
\dot{z} &= z(-\lambda_1 a_1(t) - \lambda_2 a_2(t) + tb_{31} x + b_{32} y - (\lambda_1 b_{13} + \lambda_2 b_{23}) z).
\end{aligned}
\tag{4}
$$

3-D non-autonomous Lotka-Volterra

Example

$$\begin{aligned}
\dot{x} &= x\left(\frac{ta_1(t)-1}{t} - \frac{\lambda_2 b_{21}+b_{31}}{\lambda_1}tx + b_{12}y + b_{13}z\right)\\
\dot{y} &= y\left(a_2(t) + b_{21}tx - \frac{\lambda_1 b_{12}+b_{32}}{\lambda_2}y + b_{23}z\right)\\
\dot{z} &= z(-\lambda_1 a_1(t) - \lambda_2 a_2(t) + tb_{31}x + b_{32}y - (\lambda_1 b_{13}+\lambda_2 b_{23})z).
\end{aligned} \tag{4}$$

System (4) admits the invariant hypersurfaces $f_1 = x$, $f_2 = y$ and $f_3 = z$ with cofactors

$$\begin{aligned}
K_1 &= -\frac{(\lambda_2 b_{21}+b_{31})}{\lambda_1}tx + b_{12}y + b_{13}z + \frac{ta_1(t)-1}{t}\\
K_2 &= a_2(t) + b_{21}tx - \frac{\lambda_1 b_{12}+b_{32}}{\lambda_2}y + b_{23}z\\
K_3 &= -\lambda_1 a_1(t) - \lambda_2 a_2(t) + tb_{31}x + b_{32}y - (\lambda_1 b_{13}+\lambda_2 b_{23})z.
\end{aligned}$$

we continue with the example...

Example

$$p + q = 3 = rank(\mathcal{W}(K_1, K_2, K_3)) = 3.$$

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$$F = \exp(\lambda_1 \ln(t)) \qquad \text{with} \quad K = \lambda_1 / t$$

we continue with the example...

Example

$$p + q = 3 = rank(\mathcal{W}(K_1, K_2, K_3)) = 3.$$

$$F = \exp(\lambda_1 \ln(t)) \qquad \text{with} \quad K = \lambda_1/t$$

$$p + q = 4 > rank(\mathcal{W}(K_1, K_2, K_3, K)) = 3.$$

The system admits the Darboux first integral

$$H = x^{\lambda_1} y^{\lambda_2} z t^{\lambda_1}.$$

Some literature far of be...complete...

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D. BLÀZQUEZ AND CH. PANTAZI, *A note on the Darboux theory of integrability of Non–autonomous polynomial differential systems*, Nonlinearity, **25** 2615, 2012

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Thank you for your attention!